QUASIANALYTIC ULTRADIFFERENTIABILITY CANNOT BE TESTED IN LOWER DIMENSIONS

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Abstract. We show that, in contrast to the real analytic case, quasianalytic ultradifferentiability can never be tested in lower dimensions. Our results are based on a construction due to Jaffe.

1. Introduction

In a recent paper [5] Bochnak and Kucharz proved that a function on a compact real analytic manifold is real analytic if and only if its restriction to every closed real analytic submanifold of dimension two is real analytic. A local version of this theorem can be found in [6]. It is natural to ask if a similar statements holds in quasianalytic classes of smooth functions $\mathcal{C}$ which are strictly bigger than the real analytic class, but share the property of analytic continuation:

> *Is a function defined on a $\mathcal{C}$-manifold of class $\mathcal{C}$ provided that all its restrictions to $\mathcal{C}$-submanifolds of lower dimension are of class $\mathcal{C}$?*

We will show in this paper that the answer to this question is negative for all standard quasianalytic *ultradifferentiable* classes defined by growth estimates for the iterated derivatives, even if we already know that the function is smooth. We shall always assume that the classes $\mathcal{C}$ are stable under composition and admit an inverse function theorem, consequently, manifolds of class $\mathcal{C}$ are well-defined.

This article is partly motivated by the development of the *convenient setting* for ultradifferentiable function classes in [13, 14, 15] which provides an (ultra)differential calculus for mappings between infinite dimensional locally convex spaces with a mild completeness property. Typically, the convenient calculus is based on Osgood–Hartogs type theorems which describe objects by "restrictions" to certain better understood test objects (cf. [20]). While many non-quasianalytic classes can be tested along non-quasianalytic *curves* in the same class [13], the analogous statement is false for quasianalytic classes even if the function in question is smooth. This was shown by Jaffe [10] for quasianalytic Denjoy–Carleman classes of Roumieu type. In [15] we overcame this problem by testing along all *Banach plots* in the class (i.e. mappings defined in arbitrary Banach spaces) which raised the question if there is a subclass of plots sufficient for recognizing the class.

The results of this paper show that in finite dimensions quasianalytic $\mathcal{C}$-plots with lower dimensional domain are never enough for testing $\mathcal{C}$-regularity (even if smoothness is already known). In particular, restrictions to $\mathcal{C}$-submanifolds of lower

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dimensions cannot recognize $\mathcal{C}$-regularity. Actually, we will prove more: For any $n \ge 2$, any regular quasianalytic class $\mathcal{C}$, and any positive sequence $N = (N_k)$ there exists a function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f \circ p \in \mathcal{C}$ for all $\mathcal{C}$-plots $p : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ with $m < n$, but

$$\sup_{\substack{x \in K \\ \alpha \in \mathbb{N}^n}} \frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|}|\alpha|! N_{|\alpha|}} = \infty$$

for all neighborhoods K of 0 in $\mathbb{R}^n$ and all $\rho > 0$. It will be specified in the next two subsections what we mean here by a regular quasianalytic class.

All our results follow from slight modifications of Jaffe's construction.

1.1. **Denjoy–Carleman classes.** Let $U \subseteq \mathbb{R}^n$ be open. Let $M = (M_k)$ be a positive sequence. For $\rho > 0$ and $K \subseteq U$ compact consider the seminorm

$$\|f\|^M_{K,\rho} := \sup_{\substack{x \in K \\ \alpha \in \mathbb{N}^n}} \frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|}|\alpha|! M_{|\alpha|}}, \quad f \in \mathcal{C}^\infty(U).$$

The *Denjoy–Carleman class of Roumieu type* is defined by

$$\mathcal{E}^{\{M\}}(U) := \{f \in \mathcal{C}^\infty(U) : \forall \text{ compact } K \subseteq U \ \exists \rho > 0 : \|f\|^M_{K,\rho} < \infty\},$$

and the *Denjoy–Carleman class of Beurling type* by

$$\mathcal{E}^{(M)}(U) := \{f \in \mathcal{C}^\infty(U) : \forall \text{ compact } K \subseteq U \ \forall \rho > 0 : \|f\|^M_{K,\rho} < \infty\},$$

We shall assume that $M = (M_k)$ is

1. logarithmically convex, i.e. $M_k^2 \le M_{k-1} M_{k+1}$ for all k, and satisfies
2. $M_0 = 1 \le M_1$ and
3. $M_k^{1/k} \to \infty$.

A positive sequence $M = (M_k)$ having these properties 1.–3. is called a *regular weight sequence*. The Denjoy–Carleman classes $\mathcal{E}^{\{M\}}$ and $\mathcal{E}^{(M)}$ associated with a regular weight sequence M are stable under composition and admit a version of the inverse function theorem (cf. [18]).

Let $M = (M_k)$ and $N = (N_k)$ be positive sequences. Then boundedness of the sequence $(M_k/N_k)^{1/k}$ is a sufficient condition for the inclusions $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{\{N\}}$ and $\mathcal{E}^{(M)} \subseteq \mathcal{E}^{(N)}$ (this means that the inclusions hold on all open sets). The condition is also necessary provided that $k!M_k$ is logarithmically convex, see [21] and [8], (so in particular if M is a regular weight sequence). For instance, stability of the classes $\mathcal{E}^{\{M\}}$ and $\mathcal{E}^{(M)}$ by derivation is equivalent to boundedness of $(M_{k+1}/M_k)^{1/k}$ (for the necessity we assume that $k!M_k$ is logarithmically convex). If $(M_k/N_k)^{1/k} \to 0$ then $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)}$, and conversely provided that $k!M_k$ is logarithmically convex. Hence regular weight sequences M and N are called *equivalent* if there is a constant $C > 0$ such that $C^{-1} \le (M_k/N_k)^{1/k} \le C$.

For the constant sequence $\mathbf{1} = (1, 1, 1, \ldots)$ we get the class of real analytic functions $\mathcal{E}^{\{\mathbf{1}\}} = \mathcal{C}^\omega$ in the Roumieu case and the restrictions of entire functions $\mathcal{E}^{(\mathbf{1})}$ in the Beurling case. Note that the conditions 1. and 2. imply that the sequence $M_k^{1/k}$ is increasing. Thus, if M satisfies 1. and 2. then the strict inclusions $\mathcal{C}^\omega \subsetneq \mathcal{E}^{\{M\}}$ and $\mathcal{C}^\omega \subsetneq \mathcal{E}^{(M)}$ are both equivalent to 3. (for the latter observe that 3. and $\mathcal{C}^\omega = \mathcal{E}^{(M)}$ would imply that all classes $\mathcal{C}^\omega \subseteq \mathcal{E}^{(\sqrt{M})} \subseteq \mathcal{E}^{\{\sqrt{M}\}} \subseteq \mathcal{E}^{(M)}$ actually coincide, a contradiction).

A regular weight sequence $M = (M_k)$ is called *quasianalytic* if

$$\sum_k \frac{M_k}{(k+1)M_{k+1}} = \infty. \tag{1}$$

By the Denjoy–Carleman theorem, this is the case if and only if the class $\mathcal{E}^{\{M\}}$ is quasianalytic, or equivalently $\mathcal{E}^{(M)}$ is quasianalytic. See e.g. [9, Theorem 1.3.8] and [11, Theorem 4.2].

A class $\mathcal{C}$ of $\mathcal{C}^\infty$-functions is called *quasianalytic* if the restriction to $\mathcal{C}(U)$ of the map $\mathcal{C}^\infty(U) \ni f \mapsto T_a f$ which takes f to its infinite Taylor series at a is injective for any connected open $U \ni a$. For example, the real analytic class $\mathcal{C}^\omega$ has this property and indeed (1) reduces to $\sum_k \frac{1}{k+1} = \infty$ in this case. Further examples of quasianalytic classes $\mathcal{E}^{\{M\}}$ and $\mathcal{E}^{(M)}$ that strictly contain $\mathcal{C}^\omega$ are given by $M_k := (\log(k+e))^{\delta k}$ for any $0 < \delta \le 1$.

Let $V \subseteq \mathbb{R}^m$ be open. A mapping $p : V \to U$ of class $\mathcal{E}^{\{M\}}$ (which means that the component functions p_j are of class $\mathcal{E}^{\{M\}}$) is called a *$\mathcal{E}^{\{M\}}$-plot in U of dimension m*. If $m < n$ we say that p is *lower dimensional*.

Now we are ready to state our first results.

Theorem 1. *Let $M = (M_k)$ be a quasianalytic regular weight sequence. For any $n \ge 2$ and any positive sequence $N = (N_k)$ there exists a $\mathcal{C}^\infty$-function f on $\mathbb{R}^n$ of class $\mathcal{E}^{\{M\}}$ on $\mathbb{R}^n \setminus \{0\}$ which does not belong to $\mathcal{E}^{\{N\}}(\mathbb{R}^n)$, but $f \circ p \in \mathcal{E}^{\{M\}}$ for all lower dimensional $\mathcal{E}^{\{M\}}$-plots p in $\mathbb{R}^n$.*

The following Beurling version is an easy consequence; $\mathcal{E}^{(M)}$-plots are defined in analogy to $\mathcal{E}^{\{M\}}$-plots.

Theorem 2. *Let $M = (M_k)$ be a quasianalytic regular weight sequence. For any $n \ge 2$ and any positive sequence $N = (N_k)$ there exists a $\mathcal{C}^\infty$-function f on $\mathbb{R}^n$ of class $\mathcal{E}^{(M)}$ on $\mathbb{R}^n \setminus \{0\}$ which does not belong to $\mathcal{E}^{\{N\}}(\mathbb{R}^n)$, but $f \circ p \in \mathcal{E}^{(M)}$ for all lower dimensional $\mathcal{E}^{(M)}$-plots p in $\mathbb{R}^n$.*

The proofs can be found in Section 2.

Remark. The theorems also show that *non-quasianalytic* ultradifferentiability cannot be tested on lower dimensional quasianalytic plots: Suppose that L is a non-quasianalytic regular weight sequence, $M \le L$ is a quasianalytic regular weight sequence, and N is an arbitrary positive sequence. By Theorem 1 there is a $\mathcal{C}^\infty$-function f on $\mathbb{R}^n$ of class $\mathcal{E}^{\{M\}}$ off 0 not in $\mathcal{E}^{\{N\}}(\mathbb{R}^n)$, but of class $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{\{L\}}$ along every $\mathcal{E}^{\{M\}}$-plot.

1.2. **Braun–Meise–Taylor classes.** Another way to define ultradifferentiable classes which goes back to Beurling [2] and Björck [4] and was generalized by Braun, Meise, and Taylor [7] is to use weight functions instead of weight sequences. By a *weight function* we mean a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$ that satisfies

1. $\omega(2t) = O(\omega(t))$ as $t \to \infty$,
2. $\omega(t) = O(t)$ as $t \to \infty$,
3. $\log t = o(\omega(t))$ as $t \to \infty$, and
4. $\varphi(t) := \omega(e^t)$ is convex.

Consider the *Young conjugate* $\varphi^*(t) := \sup_{s\ge 0}\big(st - \varphi(s)\big)$, for $t > 0$, of φ. For compact $K \subseteq U$ and $\rho > 0$ consider the seminorm

$$\|f\|^\omega_{K,\rho} := \sup_{x\in K,\, \alpha\in\mathbb{N}^n} |\partial^\alpha f(x)| \exp(-\tfrac{1}{\rho}\varphi^*(\rho|\alpha|)), \quad f \in \mathcal{C}^\infty(U),$$

and the ultradifferentiable classes of *Roumieu type*

$$\mathcal{E}^{\{\omega\}}(U) := \{f \in \mathcal{C}^\infty(U) : \ \forall \text{ compact } K \subseteq U \ \exists \rho > 0 : \|f\|^\omega_{K,\rho} < \infty\},$$

and of *Beurling type*

$$\mathcal{E}^{(\omega)}(U) := \{f \in \mathcal{C}^\infty(U) : \ \forall \text{ compact } K \subseteq U \ \forall \rho > 0 : \|f\|^\omega_{K,\rho} < \infty\}.$$

The classes $\mathcal{E}^{\{\omega\}}$ and $\mathcal{E}^{(\omega)}$ are in general not representable by any Denjoy–Carleman class, but they are representable (algebraically and topologically) by unions and intersections of Denjoy–Carleman classes defined by 1-parameter families of positive sequences associated with ω [17]. The classes $\mathcal{E}^{\{\omega\}}$ and $\mathcal{E}^{(\omega)}$ are quasianalytic if and only if

$$\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt = \infty.$$

If σ is another weight sequence then $\mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{\{\sigma\}}$ and $\mathcal{E}^{(\omega)} \subseteq \mathcal{E}^{(\sigma)}$ if and only if $\sigma(t) = O(\omega(t))$ as $t \to \infty$. The inclusion $\mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{(\sigma)}$ holds if and only if $\sigma(t) = o(\omega(t))$ as $t \to \infty$. For details see e.g. [17]. Thus ω and σ are called *equivalent* if $\sigma(t) = O(\omega(t))$ and $\omega(t) = O(\sigma(t))$ as $t \to \infty$.

We will assume that the weight function ω satisfies $\omega(t) = o(t)$ as $t \to \infty$ which is equivalent to the strict inclusion $\mathcal{C}^\omega = \mathcal{E}^{\{t\}} \subsetneq \mathcal{E}^{(\omega)}$. If ω is equivalent to a concave weight function, then the classes $\mathcal{E}^{\{\omega\}}$ and $\mathcal{E}^{(\omega)}$ are stable under composition and admit a version of the inverse function theorem (and conversely, see [16, Theorem 11]). They are always stable by derivation.

We shall prove in Section 2:

Theorem 3. *Let ω be a quasianalytic concave weight function such that $\omega(t) = o(t)$ as $t \to \infty$. For any $n \ge 2$ and any positive sequence $N = (N_k)$ there exists a $\mathcal{C}^\infty$-function f on $\mathbb{R}^n$ of class $\mathcal{E}^{\{\omega\}}$ on $\mathbb{R}^n \setminus \{0\}$ which does not belong to $\mathcal{E}^{\{N\}}(\mathbb{R}^n)$, but $f \circ p \in \mathcal{E}^{\{\omega\}}$ for all lower dimensional $\mathcal{E}^{\{\omega\}}$-plots p in $\mathbb{R}^n$.*

Theorem 4. *Let ω be a quasianalytic concave weight function such that $\omega(t) = o(t)$ as $t \to \infty$. For any $n \ge 2$ and any positive sequence $N = (N_k)$ there exists a $\mathcal{C}^\infty$-function f on $\mathbb{R}^n$ of class $\mathcal{E}^{(\omega)}$ on $\mathbb{R}^n \setminus \{0\}$ which does not belong to $\mathcal{E}^{\{N\}}(\mathbb{R}^n)$, but $f \circ p \in \mathcal{E}^{(\omega)}$ for all lower dimensional $\mathcal{E}^{(\omega)}$-plots p in $\mathbb{R}^n$.*

$\mathcal{E}^{\{\omega\}}$- and $\mathcal{E}^{(\omega)}$-plots are defined in analogy to $\mathcal{E}^{\{M\}}$-plots.

1.3. **New quasianalytic classes.** Let us turn the conditions of the theorems into a definition.

Let $M = (M_k)$ be any quasianalytic regular weight sequence and let ω be any quasianalytic concave weight function with $\omega(t) = o(t)$ as $t \to \infty$. In the following $\star$ stands for either $\{M\}$, (M), $\{\omega\}$, or (ω).

Let $\bar{\mathcal{A}}^\star_1(\mathbb{R}^n)$ be the set of all $\mathcal{C}^\infty$-functions f on $\mathbb{R}^n$ such that f is of class $\mathcal{E}^\star$ along all affine lines in $\mathbb{R}^n$. Then $\bar{\mathcal{A}}^\star_1(\mathbb{R}^n)$ is quasianalytic in the sense that $T_a f = 0$ implies $f = 0$ for any $a \in \mathbb{R}^n$. Indeed, if f is infinitely flat at a, then so is the restriction of f to any line ℓ through a. Since the class $\mathcal{E}^\star$ is quasianalytic, $f|_\ell = 0$ for every line ℓ through a and thus $f = 0$ on $\mathbb{R}^n$. On the other hand $\bar{\mathcal{A}}^\star_1(\mathbb{R}^n)$

contains $\mathcal{E}^\star(\mathbb{R}^n)$ but is not contained in any Denjoy–Carleman class whatsoever, by Theorems 1 to 4.

There are many ways to modify the definition: Let U be an open subset of an Euclidean space. If $\mathcal{A}^\star_m(U)$ is the set of all $\mathcal{C}^\infty$-functions f on U such that f is of class $\mathcal{E}^\star$ along all $\mathcal{E}^\star$-plots in U of dimension m, then $\mathcal{A}^\star_m(U)$ is quasianalytic and stable under composition. Thus $\mathcal{A}^\star_m$-mappings between open subsets of Euclidean spaces form a quasianalytic category askew to all Denjoy–Carleman classes. We have strict inclusions

$$\mathcal{E}^\star(\mathbb{R}^n) = \mathcal{A}^\star_n(\mathbb{R}^n) \subsetneq \mathcal{A}^\star_{n-1}(\mathbb{R}^n) \subsetneq \cdots \subsetneq \mathcal{A}^\star_1(\mathbb{R}^n).$$

Indeed the first inclusion is strict by the theorems proved in this paper. That the other inclusions are strict follows immediately: if $f \in \mathcal{A}^\star_{n-1}(\mathbb{R}^n) \setminus \mathcal{A}^\star_n(\mathbb{R}^n)$ then $\tilde f(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k}) := f(x_1,\ldots,x_n) \in \mathcal{A}^\star_{n-1}(\mathbb{R}^{n+k}) \setminus \mathcal{A}^\star_n(\mathbb{R}^{n+k})$ for all $k \ge 1$.

None of the categories $\mathcal{A}^\star_m$ is cartesian closed:

$$\mathcal{A}^\star_m(\mathbb{R}^m, \mathcal{A}^\star_m(\mathbb{R}^m)) \ne \mathcal{A}^\star_m(\mathbb{R}^m \times \mathbb{R}^m) \quad (\text{via } f(x)(y) \mapsto f^\wedge(x,y)).$$

In fact, the left-hand side equals $\mathcal{E}^\star(\mathbb{R}^m, \mathcal{E}^\star(\mathbb{R}^m))$ and is contained in $\mathcal{E}^\star(\mathbb{R}^m \times \mathbb{R}^m)$, by [15, Theorem 5.2] and [19], which in turn is strictly included in the right-hand side.

Each $\mathcal{A}^\star_m$ is closed under reciprocals: if $f \in \mathcal{A}^\star_m$ and $f(0) \ne 0$ then $1/f \in \mathcal{A}^\star_m$ on a neighborhood of 0. This follows from stability under composition and the fact that $x \mapsto 1/x$ is real analytic off 0.

Suppose that $\mathcal{E}^\star$ is stable under differentiation. If $f \in \mathcal{A}^\star_m$ then $d^k_v f \in \mathcal{A}^\star_{m-1}$ for all $m \ge 2$, all vectors v, and all k, thus also $\partial^\alpha f \in \mathcal{A}^\star_{m-1}$ for all multi-indices α. Indeed, if p is a $\mathcal{E}^*$-plot of dimension $m-1$, then

$$d^k_v f(p(s)+tv) = \partial^k_t\big(f(p(s)+tv)\big)$$

is of class $\mathcal{E}^\star$ in s for all t, since $(s,t) \mapsto p(s)+tv$ is an $\mathcal{E}^\star$-plot of dimension m and $\mathcal{E}^\star$ is stable under differentiation.

Another interesting stability property of $\mathcal{A}^\star_1$ and $\bar{\mathcal{A}}^\star_1$, under the assumption that $\mathcal{E}^\star$ is stable under differentiation, is the following: Assume that the coefficients of a polynomial

$$\varphi(x,y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$$

are germs of $\mathcal{A}^\star_1$ (resp. $\bar{\mathcal{A}}^\star_1$) functions at 0 in $\mathbb{R}^n$ and h is germ of a $\mathcal{C}^\infty$-function at 0 such that $\varphi(x,h(x)) = 0$. Then h is actually also a germ of a $\mathcal{A}^\star_1$ (resp. $\bar{\mathcal{A}}^\star_1$) function. This follows immediately from the case $n=1$ due to [22]; in this reference only the case $\star = \{M\}$ was treated, but the arguments apply to all cases. It seems to be unknown whether a similar result holds for $\mathcal{E}^\star$ and $n > 1$, but see [1].

2. Proofs

2.1. **Proof of Theorem 1.** The proof is based on a construction due to Jaffe [10].

Lemma 5 ([10, Proposition 5.2])**.** *Let M be a regular weight sequence. For any integer $n \ge 2$ there exists a function $f \in \mathcal{E}^{\{M\}}(\mathbb{R}^n)$ with the following properties: there is a constant $B = B(n)$ such that for all compact $K \subseteq \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n$*

$$|\partial^\alpha f(x)| \le B^{|\alpha|}(|K|+1)^{|\alpha|}|\alpha|!M_{|\alpha|} \quad \textit{for all } x \in K,$$

$$|\partial^\alpha f(x)| \le B^{|\alpha|}(|K|+1)^{|\alpha|}|\alpha|!\big(1+|x|^{-2(|\alpha|+1)}\big) \quad \textit{for all } x \in K \setminus \{0\},$$

and for all $k \geq 1$ *and* $i = 1, \dots, n$

$$\Big|\frac{\partial^{2k} f}{\partial x_i^{2k}}(0)\Big| \geq \frac{(2k)! M_k}{2^k}.$$

Here $|K| := \sup_{x \in K} |x|$.

It is not hard to see that the fact that M is logarithmically convex, or equivalently, $m_k := M_{k+1}/M_k$ is increasing, implies that

$$M_k = \frac{m_k^{k+1}}{\varphi(m_k)}, \qquad \text{where } \varphi(t) := \sup_{k \geq 0} \frac{t^{k+1}}{M_k}.$$

This can be used to see that

$$f(x) := \sum_{k=1}^{\infty} 2^{-k} \varphi(m_k)^{-1} \big(x - i/m_k\big)^{-1}$$

defines a smooth function on $\mathbb{R}$ with $\|f^{(k)}\|_{L^\infty} \leq k! M_k$, $|f^{(k)}(x)| \leq k!/|x|^{k+1}$ if $x \neq 0$ and $|f^{(k)}(0)| \geq k! M_k / 2^k$ for all k. Composing f with the squared Euclidean norm in $\mathbb{R}^n$ gives a function with the properties in the lemma. For details see [10].

Let $\varphi : [0,1] \to [0,1]$ be a strictly monotone infinitely flat smooth surjective function with $\varphi(t) \leq t$ for all $t \in [0,1]$. Let $\varphi_{[n]} := \varphi \circ \varphi_{[n-1]}$, $n \geq 1$, with $\varphi_{[0]} := \mathrm{Id}$ denote the iterates of φ. Consider the arc

$$A := \big\{\Phi(t) := (t, \varphi(t), \varphi_{[2]}(t), \cdots, \varphi_{[n-1]}(t)) : t \in (0,1)\big\} \subseteq \mathbb{R}^n.$$

Note that $t \geq \varphi(t) \geq \cdots \geq \varphi_{[n-1]}(t)$ for all t.

Without loss of generality we may assume that the sequence $M_k^{1/k}$ is *strictly* increasing [10, Lemma 4.3]. We define a sequence of points a_k in A by fixing the n-th coordinate of a_k to

$$(a_k)_n := M_k^{-1/(4k)}.$$

For each $\ell \in \mathbb{N}_{\geq 1}$ define a sequence $M^{(\ell)} = (M_k^{(\ell)})$ by

$$M_k^{(\ell)} := \begin{cases} 1 & \text{if } 0 \leq k < \ell, \\ c_\ell^{2k-2\ell+1} M_k & \text{if } k \geq \ell, \end{cases}$$

where $c_\ell \geq M_\ell$ are constants to be determined below. Notice that each $M^{(\ell)}$ is a regular weight sequence equivalent to M.

By Lemma 5, for each $\ell \in \mathbb{N}_{\geq 1}$ there is a function $f_\ell \in \mathcal{E}^{\{M^{(\ell)}\}}(\mathbb{R}^n) = \mathcal{E}^{\{M\}}(\mathbb{R}^n)$ such that for all compact $K \subseteq \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n$ we have (for $a := 1 + \sup_\ell |a_\ell|$)

$$|\partial^\alpha f_\ell(x)| \leq B^{|\alpha|}(|K| + a)^{|\alpha|} |\alpha|! M_{|\alpha|}^{(\ell)} \quad \text{for all } x \in K, \tag{2}$$

$$|\partial^\alpha f_\ell(x)| \leq B^{|\alpha|}(|K| + a)^{|\alpha|} |\alpha|! \big(1 + |x - a_\ell|^{-2(|\alpha|+1)}\big) \quad \text{for all } x \in K \setminus \{a_\ell\}, \tag{3}$$

where $B = B(n)$, and for all $k \geq 1$

$$\Big|\frac{\partial^{2k} f_\ell}{\partial x_1^{2k}}(a_\ell)\Big| \geq \frac{(2k)! M_k^{(\ell)}}{2^k}. \tag{4}$$

Define

$$f := \sum_{\ell=1}^{\infty} 2^{-\ell} f_\ell.$$

It is easy to check that f is $\mathcal{C}^\infty$ on $\mathbb{R}^n$ and of class $\mathcal{E}^{\{M\}}$ on $\mathbb{R}^n \setminus \{0\}$.

Note that f depends on the choice of the coefficients c_ℓ. Next we will show that, given any positive sequence $N = (N_k)$, we may choose the constants c_ℓ and hence f in such a way that f does not belong to $\mathcal{E}^{\{N\}}$ in any neighborhood of the origin.

Lemma 6. *The constants $c_\ell \ge M_\ell$ can be chosen such that for all $k \ge 1$*

$$\Big|\frac{\partial^{2k} f}{\partial x_1^{2k}}(a_k)\Big| \ge (2k)! M_{2k} N_{2k}.$$

Proof. Since $M_k^{(k)} = c_k M_k$, (3) and (4) give

$$\Big|\frac{\partial^{2k} f}{\partial x_1^{2k}}(a_k)\Big| \ge 4^{-k}(2k)! c_k M_k - \sum_{\ell \ne k} 2^{-\ell} B^{2k} (|K| + a)^{2k} (2k)! \big(1 + |a_k - a_\ell|^{-2(2k+1)}\big).$$

The sum on the right-hand side is bounded by a constant (depending on k) since the sequence $M_k^{1/k}$ is strictly increasing and hence $\inf_{\ell \ne k} |a_k - a_\ell| > 0$. The assertion follows easily. □

Lemma 6 implies that f cannot be of class $\mathcal{E}^{\{N\}}$ in any neighborhood of the origin. Otherwise there would be constants $C, \rho > 0$ such that, for large k,

$$(2k)! M_{2k} N_{2k} \le \Big|\frac{\partial^{2k} f}{\partial x_1^{2k}}(a_k)\Big| \le C \rho^{2k} (2k)! N_{2k}$$

which leads to a contradiction as $M_k^{1/k} \to \infty$.

It remains to show that $f \circ p \in \mathcal{E}^{\{M\}}(V)$ for any $\mathcal{E}^{\{M\}}$-plot $p : V \to \mathbb{R}^n$, where $V \subseteq \mathbb{R}^m$ with $m < n$. We will use the following lemma.

Lemma 7. *Let $K \subseteq \mathbb{R}^n \setminus \{a_k\}_k$ be a compact set such that*

$$\operatorname{dist}(a_k, K) \ge M_k^{-1/(4k)} \qquad \text{for all } k > k_0.$$

Then there exists $\rho > 0$ such that $\|f\|^M_{K,\rho} < \infty$. Neither ρ nor $\|f\|^M_{K,\rho}$ depend on the choice of the constants c_ℓ.

Proof. For $x \in K$ and $|\alpha| \ge 1$,

$$|\partial^\alpha f(x)| \le \sum_{\ell=1}^{\infty} 2^{-\ell} |\partial^\alpha f_\ell(x)| = \sum_{\ell=1}^{|\alpha|} 2^{-\ell} |\partial^\alpha f_\ell(x)| + \sum_{\ell=|\alpha|+1}^{\infty} 2^{-\ell} |\partial^\alpha f_\ell(x)|.$$

By (2) and the definition of $M^{(\ell)}$, the second sum is bounded by $B^{|\alpha|}(|K|+a)^{|\alpha|}|\alpha|!$. For the first sum we have, by (3),

$$\begin{aligned}\sum_{\ell=k_0+1}^{|\alpha|} 2^{-\ell} |\partial^\alpha f_\ell(x)| &\le B^{|\alpha|}(|K| + a)^{|\alpha|} |\alpha|! \sum_{\ell=k_0+1}^{|\alpha|} 2^{-\ell} \big(1 + |x - a_\ell|^{-2(|\alpha|+1)}\big) \\ &\le B^{|\alpha|}(|K| + a)^{|\alpha|} |\alpha|! M_{|\alpha|} \sum_{\ell=k_0+1}^{|\alpha|} 1 \\ &\le (eB)^{|\alpha|}(|K| + a)^{|\alpha|} |\alpha|! M_{|\alpha|}.\end{aligned}$$

A similar estimate holds for $\sum_{\ell=1}^{k_0} 2^{-\ell} |\partial^\alpha f_\ell(x)|$ since $\operatorname{dist}(a_k, K) \ge \epsilon > 0$ for all $k \le k_0$. □

Let $p = (p_1, \dots, p_n) : V \to \mathbb{R}^n$ be an $\mathcal{E}^{\{M\}}$-plot, where $V \subseteq \mathbb{R}^m$ is a neighborhood of the origin and $m < n$.

Lemma 8. *There is a compact neighborhood $L \subseteq V$ of 0 such that $K := p(L)$ satisfies*

$$\operatorname{dist}(\Phi(t), K) \geq \varphi_{[n-1]}(t) \quad \textit{for all small } t > 0. \tag{5}$$

Proof. We may assume that no component p_j vanishes identically; indeed, if $p_j \equiv 0$ then K is contained in the coordinate plane $y_j = 0$ and hence $\operatorname{dist}(\Phi(t), K) \geq \varphi_{[j-1]}(t) \geq \varphi_{[n-1]}(t)$ for all t.

Suppose that $p(0) \neq 0$. Then there exists a compact neighborhood L of 0 such that $\operatorname{dist}(0, K) =: \epsilon > 0$, where $K = p(L)$. For sufficiently small $t > 0$ we have $|\Phi(t)| \leq \epsilon/2$. For such t,

$$\operatorname{dist}(\Phi(t), K) \geq \operatorname{dist}(0, K) - |\Phi(t)| \geq \epsilon/2 \geq |\Phi(t)| \geq \varphi_{[n-1]}(t).$$

Assume that $p(0) = 0$ and that $p_j(x) = x^{\alpha_j} u_j(x)$ for $j = 1, \ldots, n$, where $x = (x_1, \ldots, x_m)$, all u_j are non-vanishing and the set of exponents $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{N}^m$ is totally ordered with respect to the natural partial order of multiindices (that is, for all $1 \leq i, j \leq n$ we have $\alpha_i \leq \alpha_j$ or $\alpha_j \leq \alpha_i$). Let $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ be an ordered arrangement of $\{\alpha_1, \ldots, \alpha_n\}$. Let m_i be the number of zero components of β_i, for $i = 1, \ldots, n$. Since $p(0) = 0$, we have $m_1 \leq m - 1$. On the other hand $m_i \geq m_{i+1}$ for all $i = 1, \ldots, n-1$. Since $m < n$, we must have $m_{i_0} = m_{i_0+1}$ for some i_0. That means there exist two distinct numbers $i, j \in \{1, \ldots, n\}$ with $\alpha_i \leq \alpha_j$ such that α_i and α_j have the same number of zero components. Thus we may find a positive integer d such that $d \cdot \alpha_i \geq \alpha_j$. Consequently, there is a constant $C > 0$ such that for all x in a neighborhood L of $0 \in \mathbb{R}^m$,

$$|p_j(x)| \leq C\,|p_i(x)| \quad \text{and} \quad |p_i(x)|^d \leq C\,|p_j(x)|.$$

This implies that $K = p(L)$ satisfies (5). In fact, the i-th component of $\Phi(t)$ is $\varphi_{[i-1]}(t)$ and the j-th component is $\varphi_{[j-1]}(t) = \varphi_{[j-i]}\big(\varphi_{[i-1]}(t)\big)$. Since $\varphi_{[j-i]}$ is an infinitely flat function while K is contained in the set $\{C^{-1}|y_i|^d \leq |y_j| \leq C|y_i|\}$, $\operatorname{dist}(\Phi(t), K)$ is larger than $\varphi_{[j-1]}(t)$ for all sufficiently small $t > 0$.

The general situation can be reduced to these special cases by the desingularization theorem [3, Theorem 5.12] using [3, Lemma 7.7] in order to get the exponents totally ordered. Indeed, applying [3, Theorem 5.12] to the product of all nonzero p_j and all nonzero differences of any two p_i, p_j we may assume that after pullback by a suitable mapping σ the components p_j are locally a monomial times a non-vanishing factor (in suitable coordinates), and the collection of exponents of the monomials is totally ordered. Here we apply the desingularization theorem to the quasianalytic class $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \mathcal{E}^{\{M^{+k}\}}$, where M^{+k} is the regular weight sequence defined by $M^{+k}_j := M_{j+k}$, which has all required properties. This is necessary since the class $\mathcal{E}^{\{M\}}$ might not be closed under differentiation. □

Remark 9. For later reference we note that Lemma 8 holds for all lower dimensional $\mathcal{C}$-plots, where $\mathcal{C}$ is any quasianalytic class of smooth functions which contains the restrictions of polynomials, is stable by composition, differentiation, division by coordinates, and admits an inverse function theorem; cf. [3].

Now we can prove that $f \circ p \in \mathcal{E}^{\{M\}}(V)$ for any lower dimensional $\mathcal{E}^{\{M\}}$-plot $p : V \to \mathbb{R}^n$. To be of class $\mathcal{E}^{\{M\}}$ is a local condition. So we may assume without loss of generality that V is a neighborhood of 0. By Lemma 8, we may further assume that (after shrinking) $V = L$ is a compact neighborhood of 0 such that $K = p(L)$ satisfies (5). By Lemma 7, there exists $\rho > 0$ such that $\|f\|^M_{K,\rho} =: C < \infty$. Since

$p \in \mathcal{E}^{\{M\}}$, there exists $\sigma > 0$ such that $\|p\|^M_{L,\sigma} =: D < \infty$. Logarithmic convexity of M implies $M_1^k M_k \ge M_j M_{\alpha_1} \cdots M_{\alpha_j}$ for all $\alpha_i \in \mathbb{N}_{>0}$ with $\alpha_1 + \cdots + \alpha_j = k$ (cf. [13, Lemma 2.9]). Consequently, in view of the Faá di Bruno formula, for $k > 0$ and $x \in L$,

$$\begin{aligned}
\frac{\|(f \circ p)^{(k)}(x)\|_{L^k(\mathbb{R}^m,\mathbb{R})}}{k!} &\le \sum_{j \ge 1} \sum_{\alpha_i} \frac{\|f^{(j)}(p(x))\|_{L^j(\mathbb{R}^n,\mathbb{R})}}{j!} \prod_{i=1}^{j} \frac{\|p^{(\alpha_i)}(x)\|_{L^{\alpha_i}(\mathbb{R}^m,\mathbb{R}^n)}}{\alpha_i!} \\
&\le \sum_{j \ge 1} \sum_{\alpha_i} C \rho^j M_j \prod_{i=1}^{j} D \sigma^{\alpha_i} M_{\alpha_i} \\
&\le C (M_1 \sigma)^k M_k \sum_{j \ge 1} \binom{k-1}{j-1} (D\rho)^j \\
&\le C D \rho (M_1 \sigma)^k (1 + D\rho)^{k-1} M_k,
\end{aligned}$$

that is, there exists $\tau > 0$ such that $\|f \circ p\|^M_{L,\tau} < \infty$. This ends the proof of Theorem 1.

2.2. **Proof of Theorem 2.** Set $L_k := M_k^{1/2}$. Then $L = (L_k)$ is a quasianalytic regular weight sequence satisfying $(L_k/M_k)^{1/k} \to 0$. Theorem 1 associates a function f with L which is as required. Indeed, f is of class $\mathcal{E}^{\{L\}} \subseteq \mathcal{E}^{(M)}$ along the image of lower dimensional $\mathcal{E}^{(M)}$-plots p, by Lemma 8 and Remark 9, and thus $f \circ p$ is $\mathcal{E}^{(M)}$, since the class is stable by composition.

2.3. **Proof of Theorem 3.** By [16, Theorem 11], there is a family $\mathfrak{M}$ of quasianalytic regular weight sequences $M = (M_k)$ such that

$$\mathcal{E}^{\{\omega\}}(U) = \{f \in \mathcal{C}^\infty(U) : \forall \text{ compact } K \subseteq U \ \exists M \in \mathfrak{M} \ \exists \rho > 0 : \|f\|^M_{K,\rho} < \infty\}.$$

Fix $M \in \mathfrak{M}$ and a positive sequence $N = (N_k)$. Let f be the $\mathcal{C}^\infty$-function associated with M and N provided by Theorem 1. Then f is not of class $\mathcal{E}^{\{N\}}$. Let p be any lower dimensional $\mathcal{E}^{\{\omega\}}$-plot. Then $f \circ p$ is of class $\mathcal{E}^{\{\omega\}}$, by Lemma 8 and Remark 9, since $\mathcal{E}^{\{\omega\}}$ is stable under composition as ω is concave.

2.4. **Proof of Theorem 4.** By [17], there is a one-parameter family $\mathfrak{M} = \{M^x\}_{x>0}$ of quasianalytic positive sequences with $(M_k^x)^{1/k} \to \infty$ for all x, $M^x \le M^y$ if $x \le y$, and

$$\mathcal{E}^{(\omega)}(U) = \mathcal{E}^{(\mathfrak{M})}(U) := \bigcap_{x>0} \mathcal{E}^{(M^x)}(U).$$

The next lemma is inspired by [12, Lemma 6].

Lemma 10. *There is a quasianalytic regular weight sequence L such that $(L_k/M_k^x)^{1/k} \to 0$ for all $x > 0$.*

Proof. Choose a positive sequence x_p which is strictly decreasing to 0. For every $p \ge 1$ we know that $(M_k^{x_p})^{1/k} \to \infty$ as $k \to \infty$. Thus for every p there is a constant $C_p > 0$ such that

$$\frac{1}{(M_k^{x_p})^{1/k}} \le \frac{C_p^{1/k}}{p} \quad \text{for all } k.$$

Choose a strictly increasing sequence j_p of positive integers such that $C_p \le 2^{j_p}$ for all p. Consider the sequence L defined by $L_j := 1$ if $j < j_1$ and

$$L_j := \sqrt{M_j^{x_p}} \quad \text{if } j_p \le j < j_{p+1}.$$

First, for $j_p \le j < j_{p+1}$,

$$L_j^{1/j} = \sqrt{(M_j^{x_p})^{1/j}} \ge \sqrt{\frac{p}{C_p^{1/j}}} \ge \sqrt{\frac{p}{2}}$$

which tends to infinity as $j \to \infty$. On the other hand, for $j_p \le j < j_{p+1}$ and $x_p \le x$,

$$\Big(\frac{L_j}{M_j^x}\Big)^{1/j} = \Big(\frac{\sqrt{M_j^{x_p}}}{M_j^x}\Big)^{1/j} \le \frac{1}{\sqrt{(M_j^x)^{1/j}}}$$

which tends to 0 as $j \to \infty$.

Let $\underline{L}$ be the log-convex minorant of L. Since $L_k^{1/k} \to \infty$, there exists a sequence $k_j \to \infty$ of integers such that $\underline{L}_{k_j} = L_{k_j}$ for all j. It follows that $\underline{L}_k^{1/k} \to \infty$, since $\underline{L}_k^{1/k}$ is increasing by logarithmic convexity. Thus $\underline{L}$ has all required properties. □

The proof of Theorem 4 now follows the arguments in the proof of Theorem 2. Theorem 1 associates a function f with the sequence L from Lemma 10 which is of class $\mathcal{E}^{\{L\}}$ along the image of any lower dimensional $\mathcal{E}^{(\omega)}$-plots p (by Lemma 8 and Remark 9). Since $(L_k/M_k^x)^{1/k} \to 0$ for all $x > 0$, we have an inclusion of classes $\mathcal{E}^{\{L\}} \subseteq \mathcal{E}^{(\omega)}$. Since ω is concave, the class $\mathcal{E}^{(\omega)}$ is stable under composition, whence $f \circ p$ is of class $\mathcal{E}^{(\omega)}$. The proof of Theorem 4 is complete.

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