Introduction to

Riemann Surfaces

Lecture Notes

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$$\dim H^0(X, \mathscr{L}_{-D}) - \dim H^0(X, \mathscr{L}^1_D) = 1 - g - \deg D$$

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Preface

These are lecture notes for the course *Riemann surfaces* held in Vienna in Spring 2018 (three semester hours). The presentation is primarily based on the book [**4**] which is followed quite closely. Also [**13**] had some influence. Apart from some familiarity with basic complex analysis, general topology, and basic algebra no other prerequisites are demanded. All necessary tools will be developed when needed.

Riemann surfaces were originally conceived in complex analysis in order to deal with multivalued functions. The analytic continuation of a given holomorphic function element along different paths leads in general to different branches of that function. Riemann replaced the domain of the function by a multiple-sheeted covering of the complex plane to get a single valued function on the covering space.

Abstract Riemann surfaces are by definition connected complex one-dimensional manifolds. They are the natural domains of definitions of holomorphic functions in one variable.

In chapter 1 we introduce Riemann surfaces and discuss basic properties. We develop the fundamentals of the theory of topological covering spaces including the fundamental group, the universal covering, and deck transformations. It will turn out that non-constant holomorphic maps between Riemann surfaces are covering maps, possibly with branch points.

In chapter 2 we get acquainted with the language of sheaves. It proves very useful in the construction of Riemann surfaces which arise from the analytic continuation of germs of holomorphic functions. Some attention is devoted to the Riemann surfaces of algebraic functions, i.e., functions which satisfy a polynomial equation with meromorphic coefficients.

For the further study of Riemann surfaces we need the calculus of differential forms which is introduced in chapter 3. We also briefly discuss periods and summands of automorphy.

Another important tool for the investigation of the geometry of Riemann surfaces is Čech cohomology. We develop the basics of this theory in chapter 4. We shall only need the cohomology groups of zeroth and first order. The long exact cohomology sequence will prove useful for the computation of cohomology groups. On Riemann surfaces we prove versions of Dolbeault's and deRham's theorem.

The next chapter 5 is devoted to compact Riemann surfaces. We present and prove the main classical results, like the Riemann–Roch theorem, Abel's theorem, and the Jacobi inversion problem. Following Serre, all the main theorems are derived from the fact that the first cohomology group with coefficients in the sheaf of holomorphic functions is a finite dimensional complex vector space. The proof of this fact is based on a functional-analytic result due to Schwartz. Its dimension is the genus of the Riemann surface. By means of the Serre duality theorem we will see that the genus equals the maximal number of linearly independent holomorphic one-forms on the compact Riemann surface. Eventually, it will turn out that the genus is a topological invariant. Much of this chapter is concerned with the existence of meromorphic functions on compact Riemann surfaces with prescribed principal parts or divisors.

Non-compact Riemann surfaces are at the focus of chapter 6. The function theory of non-compact Riemann surfaces has many similarities with the one on regions in the complex plane. In contrast to compact Riemann surfaces, there are analogues of Runge's theorem, the Mittag–Leffler theorem, and the Weierstrass theorem. The solution of the Dirichlet problem, based on Perron's method, will provide a further existence theorem. It will lead to a proof of Radó's theorem that every Riemann surface has a countable topology. We shall also prove the uniformization theorem for Riemann surfaces: any simply connected Riemann surface is isomorphic to one of three normal forms, i.e, the Riemann sphere, the complex plane, or the unit disk. Evidently, this is a generalization of the Riemann mapping theorem. As a consequence we get the classification of Riemann surfaces: every Riemann surface is isomorphic to the quotient of one of the three normal forms by a group of Möbius transformations isomorphic to the fundamental group of the Riemann surface which acts discretely and fixed point freely.

Notation. A domain is a nonempty open subset $U \subseteq \mathbb{C}$. A connected domain is called a region. We denote by $D_r(c) = \{z \in \mathbb{C} : |z - c| < r\}$ the open disk of radius r and center c. $\overline{D}_r(c)$ denotes the closed disk and $\partial D_r(c)$ its boundary; if not stated otherwise, it is always assumed to be oriented counterclockwise. By $\mathbb{D}$ we denote the unit disk $\mathbb{D} = D_1(0)$, by $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the upper half plane. The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is denoted by $\widehat{\mathbb{C}}$. We use $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^*_a = \mathbb{C} \setminus \{a\}$, for $a \in \mathbb{C}$, as well as $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ and $D^*_r(a) := D_r(a) \setminus \{a\}$. If V is a relatively compact open subset of U we write $V \Subset U$.

Contents

CHAPTER 1

Covering spaces

1. Riemann surfaces

1.1. Complex manifolds. An n-dimensional (topological) manifold is a Hausdorff topological space which is locally euclidean, i.e., every point has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

Let X be a $2n$ dimensional manifold. A **complex chart** on X is a homeomorphism $\varphi : U \to V$ of an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}^n$. A **complex atlas** on X is an open cover $\mathfrak{A} = \{\varphi_i : U_i \to V_i\}_{i \in I}$ of X by complex charts such that the transition maps

$$\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i \cap U_j)} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are holomorphic for all $i, j \in I$. We say that the charts are **holomorphically compatible**.

Two complex atlases on X are said to be equivalent if their union is again a complex atlas. A **complex structure** on X is an equivalence class of equivalent complex atlases on X.

A **complex manifold** is a $2n$ dimensional manifold X equipped with a complex structure. Then n is the **complex dimension** of X.

A complex structure on X can be given by the choice of a complex atlas $\mathfrak{A}$. Every complex structure on X contains a unique maximal atlas. Indeed, if the complex structure on X is represented by an atlas $\mathfrak{A}$, then the maximal atlas consists of all complex charts on X which are holomorphically compatible with $\mathfrak{A}$.

1.2. Riemann surfaces. A **Riemann surface** is a connected complex manifold X of complex dimension 1. We shall see below, Theorem 23.3, that every Riemann surface has a countable base of its topology, by a theorem of Radó.

Henceforth, by a chart on X we always mean a complex chart in the maximal atlas of the complex structure on X.

Example 1.1 (complex plane)**.** The complex structure is defined by the atlas $\{\mathrm{id} : \mathbb{C} \to \mathbb{C}\}$.

Example 1.2 (open connected subsets of Riemann surfaces)**.** Let X be a Riemann surface. Let $Y \subseteq X$ be an open connected subset. Then Y is a Riemann surface in a natural way. An atlas is formed by all complex charts $\varphi : U \to V$ on X with $U \subseteq Y$.

Example 1.3 (Riemann sphere)**.** Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and we introduce the following topology. A subset of $\widehat{\mathbb{C}}$ is open if it is either an open subset of $\mathbb{C}$ or it is of the form $U \cup \{\infty\}$, where $U \subseteq \mathbb{C}$ is the complement of a compact subset of $\mathbb{C}$. With this topology $\widehat{\mathbb{C}}$ is a compact Hausdorff topological space, homeomorphic to the 2-sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ via the stereographic projection. Let $U_1 := \mathbb{C}$ and $U_2 := \mathbb{C}^* \cup \{\infty\}$. Let $\varphi_1 := \mathrm{id} : U_1 \to \mathbb{C}$ and let $\varphi_2 : U_2 \to \mathbb{C}$ be defined by $\varphi_2(z) = 1/z$ if $z \in \mathbb{C}^*$ and $\varphi_2(\infty) = 0$. Then φ_1, φ_2 are homeomorphisms.

Thus $\widehat{\mathbb{C}}$ is a two-dimensional manifold. Since U_1, U_2 are connected and have nonempty intersection, $\widehat{\mathbb{C}}$ is connected. The transition map $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$ is the mapping $z \to 1/z$ from $\mathbb{C}^*$ to itself. This complex structure makes the **Riemann sphere** $\widehat{\mathbb{C}}$ to a compact Riemann surface. It is also called the **complex projective line** and denoted by $\mathbb{P}^1$; cf. Section 17.1.

Example 1.4 (complex tori, I)**.** Let $w_1, w_2 \in \mathbb{C}^*$ be linearly independent over $\mathbb{R}$. The set

$$\Lambda = \mathbb{Z}w_1 + Zw_2$$

is called the **lattice** spanned by w_1 and w_2. Then Λ is a subgroup of $\mathbb{C}$ and acts on $\mathbb{C}$ by $\lambda(z) = z + \lambda$, $\lambda \in \Lambda$, $z \in \mathbb{C}$. Consider the equivalence relation on $\mathbb{C}$ defined by $z \sim w$ if $z - w \in \Lambda$. Let $\mathbb{C}/\Lambda$ be the quotient space and $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ the canonical projection. With the quotient topology (i.e., $U \subseteq \mathbb{C}/\Lambda$ is open if $\pi^{-1}(U) \subseteq \mathbb{C}$ is open) $\mathbb{C}/\Lambda$ is a Hausdorff topological space and π is continuous. Since $\mathbb{C}$ is connected, so is $\mathbb{C}/\Lambda$. Moreover, $\mathbb{C}/\Lambda$ is compact, since it is the image under π of the compact parallelogram $\{sw_1 + tw_2 : s, t \in [0, 1]\}$. For an open set $V \subseteq \mathbb{C}$ we have

$$\pi^{-1}(\pi(V)) = \bigcup_{w \in \Lambda} (V + w)$$

which shows that π is open.

Let us define a complex structure on $\mathbb{C}/\Lambda$. Let $V \subseteq \mathbb{C}$ be an open set no two points of which are equivalent under Λ. Then $U = \pi(V)$ is open and $\pi|_V : V \to U$ is a homeomorphism. Its inverse $\varphi : U \to V$ is a complex chart on $\mathbb{C}/\Lambda$. Let $\mathfrak{A}$ be the set of all charts obtained in this way. Any two charts in $\mathfrak{A}$ are holomorphically compatible. For, if $z \in \varphi_1(U_1 \cap U_2)$ then

$$\pi(\varphi_2(\varphi_1^{-1}(z))) = \varphi_1^{-1}(z) = \pi(z).$$

Thus $\varphi_2(\varphi_1^{-1}(z)) - z \in \Lambda$. Since Λ is discrete and $\varphi_2 \circ \varphi_1^{-1}$ is continuous, $\varphi_2(\varphi_1^{-1}(z)) = z + \lambda$ for some constant $\lambda \in \Lambda$ on every connected component of $\varphi_1(U_1 \cap U_2)$, and hence is holomorphic.

The Riemann surface $\mathbb{C}/\Lambda$ defined by this complex structure is said to be a **complex torus**. A model of $\mathbb{C}/\Lambda$ is obtained by identifying opposite sides of the parallelogram with vertices 0, w_1, w_2, and $w_1 + w_2$.

Example 1.5 (orbit spaces $\mathbb{H}/\Gamma$)**.** Let Γ be a discrete fixed point free subgroup of

$$\mathrm{Aut}(\mathbb{H}) = \{z \mapsto (az + b)/(cz + d) : a, b, c, d \in \mathbb{R},\ ad - bc = 1\}.$$

With the quotient topology the orbit space $\mathbb{H}/\Gamma$ is a Hausdorff topological space and the quotient map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is continuous. In analogy to Example 1.4, there is a natural complex structure on $\mathbb{H}/\Gamma$ (with $z \mapsto \gamma(z)$, $\gamma \in \Gamma$, as transition maps) which makes $\mathbb{H}/\Gamma$ to a Riemann surface.

1.3. Holomorphic functions and maps. Let X be a Riemann surface, and let $Y \subseteq X$ be an open subset. A function $f : Y \to \mathbb{C}$ is **holomorphic** if for every chart $\varphi : U \to V$ on X the function $f \circ \varphi^{-1} : \varphi(U \cap Y) \to \mathbb{C}$ is holomorphic. We denote by $\mathcal{O}(Y)$ the set of all holomorphic functions on Y. Clearly, $\mathcal{O}(Y)$ is a $\mathbb{C}$-algebra.

It is enough to verify the condition on any atlas. Note that every chart $\varphi : U \to V$ is trivially holomorphic. We call (U, φ) a **coordinate neighborhood** of any point $a \in U$ and φ a **local coordinate**.

Let X, Y be Riemann surfaces. A continuous map $f : X \to Y$ is called **holomorphic** if for every pair of charts $\varphi_1 : U_1 \to V_1$ on X and $\varphi_2 : U_2 \to V_2$ on Y with $f(U_1) \subseteq U_2$,

$$\varphi_2 \circ f \circ \varphi_1^{-1} : V_1 \to V_2$$

is holomorphic. A map $f : X \to Y$ is a **biholomorphism** if there is a holomorphic map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Two Riemann surfaces are called **isomorphic** if there is a biholomorphism between them.

In the case that $Y = \mathbb{C}$, holomorphic maps $f : X \to \mathbb{C}$ clearly are just holomorphic functions.

Holomorphic maps behave well under composition. In fact, Riemann surfaces with holomorphic maps between them form a category.

Let $f : X \to Y$ be a continuous map between Riemann surfaces. Then f is holomorphic if and only if for each open $V \subseteq Y$ and every $\varphi \in \mathcal{O}(V)$ the function $f^*(\varphi) = \varphi \circ f : f^{-1}(V) \to \mathbb{C}$ belongs to $\mathcal{O}(f^{-1}(V))$. The **pullback**

$$f^* : \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V)), \quad f^*(\varphi) = \varphi \circ f,$$

is a ring homomorphism. It satisfies $(g \circ f)^* = f^* \circ g^*$.

1.4. Elementary properties of holomorphic maps. Many results for holomorphic functions defined on domains (i.e., non-empty open subsets) in $\mathbb{C}$ persist on Riemann surfaces.

Theorem 1.6 (identity theorem)**.** *Let X, Y be Riemann surfaces. Let $f_1, f_2 : X \to Y$ be holomorphic maps which coincide on a set $A \subseteq X$ with a limit point a in X. Then $f_1 = f_2$.*

Proof. Let $Z := \{x \in X : f_1 = f_2 \text{ near } x\}$. Then Z is open. We claim that Z is also closed. Let $b \in \overline{Z}$. Then $f_1(b) = f_2(b)$ by continuity. Choose charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with $b \in U$ and such that $f_i(U) \subseteq U'$, $i = 1, 2$. Assume that U is connected. Then the maps $g_i = \psi \circ f_i \circ \varphi^{-1} : V \to V'$, $i = 1, 2$, are holomorphic. By the identity theorem for domains in $\mathbb{C}$, $g_1 = g_2$, since $U \cap Z \neq \emptyset$. It follows that f_1 and f_2 coincide on U, and Z is closed. The set Z is non-empty, in fact, $a \in Z$, again by the identity theorem for domains in $\mathbb{C}$. Since X is connected, we may conclude $X = Z$ which gives the assertion. □

Theorem 1.7 (local normal form of holomorphic maps)**.** *Let X, Y be Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic map. Let $a \in X$ and $b = f(a)$. Then there is an integer $k \geq 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y such that $a \in U$, $\varphi(a) = 0$, $b \in U'$, $\psi(b) = 0$, $f(U) \subseteq U'$ and*

$$\psi \circ f \circ \varphi^{-1} : V \to V' : z \mapsto z^k.$$

Proof. Let $F := \psi \circ f \circ \varphi^{-1}$. Then $F(0) = 0$ and so there is a positive integer k such that $F(z) = z^k g(z)$, where $g(0) \neq 0$. Thus, there is a neighborhood of 0 and a holomorphic function h on this neighborhood such that $h^k = g$. The mapping $\alpha(z) := zh(z)$ is a biholomorphism from an open neighborhood of 0 onto an open neighborhood of 0. Replacing the chart φ by $\alpha \circ \varphi$ implies the statement. □

The number k is called the **multiplicity** of f at a and denoted by $m_a(f)$. The multiplicity is independent of the choice of the charts.

Theorem 1.8 (open mapping theorem)**.** *Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is open.*

Proof. This an immediate consequence of Theorem 1.7: if U is a neighborhood of $a \in X$ then $f(U)$ is a neighborhood of $f(a)$ in Y. □

Corollary 1.9. *Let $f : X \to Y$ be an injective holomorphic map between Riemann surfaces. Then f is a biholomorphism from X to $f(X)$.*

Proof. Since f is injective the multiplicity is always 1, so the inverse map is holomorphic. □

Corollary 1.10. *Let X, Y be Riemann surfaces, where X is compact. Let $f : X \to Y$ be a non-constant holomorphic map. Then f is surjective and Y is compact.*

Proof. By the open mapping theorem, $f(X)$ is open. Moreover, $f(X)$ is compact and thus closed. Since Y is connected, $Y = f(X)$. □

Remark 1.11. This implies the fundamental theorem of algebra. Exercise.

Corollary 1.12. *Every holomorphic function on a compact Riemann surface is constant.*

Proof. Apply the previous corollary. □

Remark 1.13. This implies Liouville's theorem. Exercise.

Theorem 1.14 (maximum principle)**.** *Let X be a Riemann surface and let $f : X \to \mathbb{C}$ be holomorphic. If there is $a \in X$ such that $|f(x)| \le |f(a)|$ for all $x \in X$, then f is constant.*

Proof. The condition means that $f(X) \subseteq D_{|f(a)|}(0)$. So $f(X)$ is not open, and the statement follows from the open mapping theorem. □

Theorem 1.15 (Riemann's theorem on removable singularities)**.** *Let U be an open subset of a Riemann surface. Let $a \in U$. If $f \in \mathcal{O}(U \setminus \{a\})$ is bounded in a neighborhood of a, then there is $F \in \mathcal{O}(U)$ such that $F|_{U\setminus\{a\}} = f$.*

Proof. This follows immediately from Riemann's theorem on removable singularities in the complex plane. □

1.5. Meromorphic functions. Let X be a Riemann surface and let Y be an open subset of X. A **meromorphic function** on Y is a holomorphic function $f : Y' \to \mathbb{C}$, where Y' is an open subset of Y such that $Y \setminus Y'$ contains only isolated points and

$$\lim_{x\to a} |f(x)| = \infty \quad \text{for all } a \in Y \setminus Y'.$$

The points of $Y \setminus Y'$ are called the **poles** of f. The set of all meromorphic functions on Y is denoted by $\mathscr{M}(Y)$. It is easy to see that $\mathscr{M}(Y)$ is a $\mathbb{C}$-algebra.

Example 1.16. Any non-constant polynomial is an element of $\mathscr{M}(\widehat{\mathbb{C}})$ with a pole at ∞.

Theorem 1.17. *Let X be a Riemann surface and let $f \in \mathscr{M}(X)$. For each pole a of f define $f(a) := \infty$. The resulting map $f : X \to \widehat{\mathbb{C}}$ is holomorphic. Conversely, let $f : X \to \widehat{\mathbb{C}}$ be holomorphic. Then f is either identically equal to ∞ or $f^{-1}(\infty)$ consists of isolated points and $f : X \setminus f^{-1}(\infty) \to \mathbb{C}$ is meromorphic on X.*

So we may identify meromorphic functions $f \in \mathscr{M}(X)$ with holomorphic maps $f : X \to \widehat{\mathbb{C}}$.

Proof. Let $f \in \mathscr{M}(X)$. The induced map $f : X \to \widehat{\mathbb{C}}$ is clearly continuous. Let $\varphi : U \to V$ be a chart on X and $\psi : U' \to V'$ a chart on $\widehat{\mathbb{C}}$ with $f(U) \subseteq U'$. It suffices to show that $\psi \circ f \circ \varphi^{-1} : V \to V'$ is holomorphic. This follows easily from Riemann's theorem on removable singularities.

The converse is a consequence of the identity theorem 1.6. □

Corollary 1.18. *Let X be a Riemann surface. Then $\mathscr{M}(X)$ is a field, the so-called* ***function field*** *of X.*

Proof. Any function $f \in \mathscr{M}(X)$ induces a holomorphic map $f : X \to \widehat{\mathbb{C}}$. By the identity theorem 1.6, f has only isolated zeros unless it vanishes identically. □

Theorem 1.19 (function field of the Riemann sphere)**.** *The function field $\mathscr{M}(\widehat{\mathbb{C}})$ consists precisely of the rational functions.*

Proof. Clearly, every rational function is in $\mathscr{M}(\widehat{\mathbb{C}})$. Let $f \in \mathscr{M}(\widehat{\mathbb{C}})$. We may assume that ∞ is not a pole of f; otherwise consider $1/f$ instead of f. Let $a_1, \ldots, a_n \in \mathbb{C}$ be the poles of f in $\mathbb{C}$; there are finitely many since $\widehat{\mathbb{C}}$ is compact. Let $h_i = \sum_{j=-k_i}^{-1} c_{ij}(z - a_i)^j$ be the principal part of f at a_i, $i = 1, \ldots, n$. Then $f - (h_1 + \cdots + h_n)$ is holomorphic on $\widehat{\mathbb{C}}$ and hence constant, by Corollary 1.12. This implies that f is rational. □

Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ be a lattice in $\mathbb{C}$. A meromorphic function $f \in \mathscr{M}(\mathbb{C})$ is called an **elliptic** or **doubly periodic** function with respect to Λ if $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$ and all $z \in \mathbb{C}$.

Theorem 1.20 (complex tori, II)**.** *Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ be a lattice. The function field $\mathscr{M}(\mathbb{C}/\Lambda)$ is in one-to-one correspondence with the elliptic functions with respect to Λ.*

Proof. Let $f \in \mathscr{M}(\mathbb{C}/\Lambda)$. We may assume that f is non-constant. Thus $f : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ is holomorphic and hence $\tilde{f} := f \circ \pi : \mathbb{C} \to \widehat{\mathbb{C}}$ is holomorphic, where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the quotient map. Thus $\tilde{f}$ is elliptic. Conversely, every elliptic function $\tilde{f}$ with respect to Λ induces a meromorphic function on $\mathbb{C}/\Lambda$. □

Corollary 1.21. *Every holomorphic elliptic function is constant. Every non-constant elliptic function attains every value in $\widehat{\mathbb{C}}$.*

Proof. Follows from Theorem 1.20, Corollary 1.10, and Corollary 1.12. □

2. The fundamental group

2.1. Homotopy of curves. Let X be a topological space and let $a, b \in X$. Two curves $\gamma_0, \gamma_1 : [0,1] \to X$ from a to b are **homotopic** if there exists a continuous map $H : [0,1]^2 \to X$ such that

(1) $H(0,t) = \gamma_0(t)$ for all $t \in [0,1]$,
(2) $H(1,t) = \gamma_1(t)$ for all $t \in [0,1]$,
(3) $H(s,0) = a$ and $H(s,1) = b$ for all $s \in [0,1]$.

We will set $\gamma_s(t) := H(s,t)$. Then $\{\gamma_s\}_{s\in[0,1]}$ is a continuous deformation of γ_0 into γ_1. Homotopy defines an equivalence relation on the set of all curves from a to b in X.

A closed curve $\gamma : [0,1] \to X$ (i.e. $\gamma(0) = \gamma(1) = a$) is **null-homotopic** if it is homotopic to the constant curve a.

Suppose γ_1 is a curve from a to b and γ_2 is a curve from b to c. Then we can define the **product curve** $\gamma_1 \cdot \gamma_2$ from a to c by

$$(\gamma_1 \cdot \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

It runs first through γ_1 then through γ_2 at twice the speed. The **inverse** γ^- of γ passes through γ in the opposite direction,

$$\gamma^-(t) := \gamma(1-t), \quad t \in [0,1].$$

If γ_1 and σ_1 are homotopic curves from a to b and γ_2 and σ_2 are homotopic curves from b to c, then $\gamma_1 \cdot \gamma_2$ and $\sigma_1 \cdot \sigma_2$ are homotopic. Moreover, γ_1^- and σ_1^- are homotopic.

2.2. The fundamental group.

Theorem 2.1. *Let X be a topological space and $a \in X$. The set $\pi_1(X,a)$ of homotopy classes of closed curves in X with initial and end point a forms a group under the operation induced by the product of curves. It is called the **fundamental group** of X with base point a.*

Proof. Exercise. □

If $[\gamma]$ denotes the homotopy class of a closed curve γ, then the group operation in $\pi_1(X,a)$ is by definition $[\gamma][\sigma] = [\gamma \cdot \sigma]$. The identity element is the class of null-homotopic curves. The inverse of $[\gamma]$ is given by $[\gamma]^{-1} = [\gamma^-]$.

For an path-connected space X the fundamental group is independent of the base point; in that case we write $\pi_1(X)$ instead of $\pi_1(X,a)$. Indeed, if $a, b \in X$ and σ is a curve in X joining a and b then the map $\pi_1(X,a) \to \pi_1(X,b)$, $[\gamma] \mapsto [\sigma^- \cdot \gamma \cdot \sigma]$, is an isomorphism. (The isomorphism depends on σ. One can show that there is a canonical isomorphism if $\pi_1(X,a)$ is abelian.)

A path-connected space X is called **simply connected** if its fundamental group is trivial, $\pi_1(X) = 0$.

Two closed curves $\gamma_0, \gamma_1 : [0,1] \to X$ which do not necessarily have the same initial point are called **free homotopic** as closed curves if there is a continuous map $H : [0,1]^2 \to X$ satisfying (1), (2), and

(3') $H(s,0) = H(s,1)$ for all $s \in [0,1]$.

Theorem 2.2. *A path-connected space X is simply connected if and only if any two closed curves in X are free homotopic as closed curves.*

Proof. Exercise. □

Example 2.3. (1) Star-shaped sets in $\mathbb{R}^n$ are simply connected.

(2) The Riemann sphere $\widehat{\mathbb{C}}$ is simply connected.

(3) The complex tori $\mathbb{C} \setminus \Lambda$ are not simply connected.

Let $f : X \to Y$ be a continuous map between topological spaces. If $\gamma_0, \gamma_1 : [0,1] \to X$ are homotopic curves in X, then $f \circ \gamma_1, f \circ \gamma_2$ are homotopic curves in Y. It follows that f induces a map

$$f_* : \pi_1(X,a) \to \pi_1(Y, f(a))$$

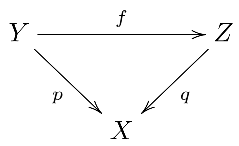
which is a group homomorphism since $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$. If $g : Y \to Z$ is another continuous map, then $(g \circ f)_* = g_* \circ f_*$.

3. Covering maps

We will see in this section that non-constant holomorphic maps between Riemann surfaces are covering maps, possibly with branch points. Let us recall some background on covering maps and covering spaces.

3.1. Discrete fibers. Let X and Y be topological spaces. Let $p : Y \to X$ be a continuous map. For $x \in X$, the preimage $p^{-1}(x)$ is called the **fiber** of p over x. Points $y \in p^{-1}(x)$ are said to **lie over** x.

Suppose that $p : Y \to X$ and $q : Z \to X$ are continuous. A map $f : Y \to Z$ is called **fiber-preserving** if the following diagram commutes.



It means that every point lying over x is mapped to a point also lying over x.

We say that $p : Y \to X$ is **discrete** if all fibers $p^{-1}(x)$, $x \in X$, are discrete subsets of Y (i.e., each point $y \in p^{-1}(x)$ has a neighborhood V in Y such that $V \cap p^{-1}(x) = \{y\}$).

Lemma 3.1. *Let $p : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. Then p is open and discrete.*

Proof. By the open mapping theorem 1.8, p is open. If there is a fiber which is not discrete, p is constant, by the identity theorem 1.6. □

Example 3.2 (multivalued functions)**.** Let $p : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. A holomorphic (resp. meromorphic) function $f : Y \to \mathbb{C}$ (resp. $f : Y \to \widehat{\mathbb{C}}$) can be considered as a holomorphic (resp. meromorphic) **multivalued function** on X. Indeed, this multivalued function takes $x \in X$ to the set $\{f(y) : y \in p^{-1}(x)\}$. Clearly, it might happen that $p^{-1}(x)$ is a single point or empty. For example, let $p = \exp : \mathbb{C} \to \mathbb{C}^*$. Then the identity $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ corresponds to the multivalued logarithm on $\mathbb{C}^*$.

3.2. Branch points. Let $p : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. A point $y \in Y$ is called a **branch point** of p if there is no neighborhood of y on which p is injective, or equivalently, if $m_y(p) \geq 2$. We say that p is **unbranched** if it has no branch points.

Proposition 3.3. *A non-constant holomorphic map $p : Y \to X$ between Riemann surfaces is unbranched if and only if it is a local homeomorphism.*

Proof. If p is unbranched and $y \in Y$, then p is injective on a neighborhood V of y. Since p is continuous and open, $p : V \to p(V)$ is a homeomorphism. Conversely, if p is a local homeomorphism, then p is locally injective and hence unbranched. □

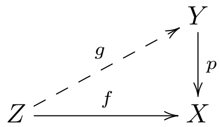
Example 3.4. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is an unbranched holomorphic map. The power map $p_k : \mathbb{C} \to \mathbb{C}$, $p_k(z) = z^k$, has a branch point at 0 if $k \geq 2$; off 0 it is a local homeomorphism. By Theorem 1.7, every holomorphic map has this form near a branch point.

Theorem 3.5. *Let X be a Riemann surface, let Y be a Hausdorff topological space, and let $p : Y \to X$ be a local homeomorphism. Then there is a unique complex structure on Y such that p is holomorphic.*

Proof. Let $\varphi : U \to V$ be a chart for the complex structure on X such that $p : p^{-1}(U) \to U$ is a homeomorphism. Then $\varphi \circ p : p^{-1}(U) \to V$ is a complex chart on Y. Let $\mathfrak{A}$ be the set of all complex charts on Y obtained in this way. Then the charts in $\mathfrak{A}$ cover Y and are all compatible. Equip Y with the complex structure defined by $\mathfrak{A}$. Then p is locally biholomorphic, hence holomorphic.

It remains to show uniqueness. Suppose there is another altas $\mathfrak{B}$ on Y such that $p : (Y, \mathfrak{B}) \to X$ is holomorphic, and thus locally biholomorphic (by Corollary 1.9). Then $\mathrm{id} : (Y, \mathfrak{A}) \to (Y, \mathfrak{B})$ is a local biholomorphism, and consequently a biholomorphism. □

3.3. Lifting of continuous maps. Let X, Y, Z be topological spaces and let $p : Y \to X$ and $f : Z \to X$ be continuous maps. A **lifting** of f over p is a continuous map $g : Z \to Y$ such that $f = p \circ g$.



Lemma 3.6 (uniqueness of liftings)**.** *Let X, Y be Hausdorff spaces and let $p : Y \to X$ be a local homeomorphism. Let Z be a connected topological space. Let $f : Z \to X$ be continuous and assume that g_1, g_2 are liftings of f. If there exists $z_0 \in Z$ such that $g_1(z_0) = g_2(z_0)$, then $g_1 = g_2$.*

Proof. Let $A = \{z \in Z : g_1(z) = g_2(z)\}$. Then $z_0 \in A$ and A is closed, since Y is Hausdorff (Y is Hausdorff if and only if the diagonal $\Delta \subseteq Y \times Y$ is closed, A is the preimage of Δ under (g_1, g_2)). We claim that A is also open. For, let $z \in A$ and $y = g_1(z) = g_2(z)$. There is an open neighborhood V of y such that $p(V) = U$ is open and $p|_V$ is a homeomorphism onto U. Since g_1, g_2 are continuous, there is a neighborhood W of z such that $g_1(W) \subseteq V$, $g_2(W) \subseteq V$. For every $w \in W$, $p(g_1(w)) = f(w) = p(g_2(w))$, and thus, since $p|_V$ is injective, $g_1 = g_2$ on W. That is $W \subseteq A$, and A is open. The statement follows, since Z is connected. □

Theorem 3.7 (holomorphic lifting)**.** *Let X, Y, Z be Riemann surfaces. Let $p : Y \to X$ be an unbranched holomorphic map and let $f : Z \to X$ be holomorphic. Then every lifting $g : Z \to Y$ of f is holomorphic.*

Proof. This follows from the fact that p is a local biholomorphism, by Corollary 1.9 and Proposition 3.3. □

Corollary 3.8. *Let X, Y, Z be Riemann surfaces. Let $p : Y \to X$ and $q : Z \to X$ be unbranched holomorphic maps. Then every continuous fiber-preserving map $f : Y \to Z$ is holomorphic.*

Proof. Apply Theorem 3.7. □

3.4. Lifting of homotopic curves.

Theorem 3.9 (monodromy theorem)**.** *Let X, Y be Hausdorff spaces and $p : Y \to X$ a local homeomorphism. Let $\tilde{a} \in Y$ and $a = p(\tilde{a})$. Let $H : [0,1]^2 \to X$ be a homotopy between γ_0 and γ_1 fixing the initial point $a = \gamma_0(0) = \gamma_1(0)$. Suppose that each curve $\gamma_s := H_s$, $s \in [0,1]$, has a lifting $\tilde{\gamma}_s$ over $p : Y \to X$ with initial point $\tilde{a}$. Then $\tilde{H}(s,t) := \tilde{\gamma}_s(t)$ is a homotopy between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.*

Proof. We must show continuity of $\tilde{H} : [0,1]^2 \to Y$. Let $I := [0,1]$.

Fix $(s_0, t_0) \in I^2$. Since $\tilde{\gamma}_{s_0}$ is continuous and hence $\tilde{\gamma}_{s_0}(I)$ is compact, we may choose open sets $V_0, \ldots, V_n$ in Y and points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1$ such that $p|_{V_j} =: p_j$ is a homeomorphism onto an open set U_j in X and $\tilde{\gamma}_{s_0}([\tau_j, \tau_{j+1}]) \subseteq V_j$, $j = 0, 1, \ldots, n-1$. We may assume without loss of generality that t_0 is an interior point of some $[\tau_{j_0}, \tau_{j_0+1}]$, unless t_0 is 0 or 1.

By the continuity of H, there exists $\epsilon > 0$ such that $\gamma_s(t) \in U_j$ for $|s - s_0| < \epsilon$, $s \in I$, $t \in [\tau_j, \tau_{j+1}]$, and $j = 0, 1, \ldots, n-1$. We will prove that, for $|s - s_0| < \epsilon$, $s \in I$, $t \in [\tau_j, \tau_{j+1}]$, and $j = 0, 1, \ldots, n-1$,

$$\tilde{\gamma}_s(t) = p_j^{-1}(\gamma_s(t)). \tag{3.1}$$

This implies that $\tilde{H}$ is continuous at (s_0, t_0), since (s_0, t_0) is an interior point (relative to I^2) of the set $\{s \in I : |s - s_0| < \epsilon\} \times [\tau_{j_0}, \tau_{j_0+1}]$.

We show (3.1) by induction on j. Let $j = 0$. Fix $s \in I$ with $|s - s_0| < \epsilon$. The curves $\tilde{\gamma}_s$ and $p_0^{-1} \circ \gamma_s$ are both liftings of γ_s on the interval $[\tau_0, \tau_1]$, and $\tilde{\gamma}_s(0) = \tilde{a} = (p_0^{-1} \circ \gamma_s)(0)$ (because $\tilde{a} = \tilde{\gamma}_{s_0}(0) \in V_0$). By uniqueness of liftings 3.6, (3.1) holds for $j = 0$.

Suppose that (3.1) has been proved for all $0 \le j < k$. For fixed s, the curves $\tilde{\gamma}_s$ and $p_k^{-1} \circ \gamma_s$ are both liftings of γ_s on the interval $[\tau_k, \tau_{k+1}]$. By Lemma 3.6, it is enough to prove

$$\tilde{\gamma}_s(\tau_k) = p_k^{-1}(\gamma_s(\tau_k)) \quad \text{for } |s - s_0| < \epsilon,\ s \in I. \tag{3.2}$$

By induction hypothesis, (3.1) for $j = k - 1$ and $t = \tau_k$ gives

$$\tilde{\gamma}_s(\tau_k) = p_{k-1}^{-1}(\gamma_s(\tau_k)) \quad \text{for } |s - s_0| < \epsilon,\ s \in I. \tag{3.3}$$

In particular, for $s = s_0$,

$$p_k^{-1}(\gamma_{s_0}(\tau_k)) = \tilde{\gamma}_{s_0}(\tau_k) = p_{k-1}^{-1}(\gamma_{s_0}(\tau_k)),$$

since $\tilde{\gamma}_{s_0}(\tau_k) \in V_{k-1} \cap V_k$. Thus, $s \mapsto p_{k-1}^{-1}(\gamma_s(\tau_k))$ and $s \mapsto p_k^{-1}(\gamma_s(\tau_k))$ are both liftings of $s \mapsto \gamma_s(\tau_k)$, for $|s - s_0| < \epsilon$, $s \in I$, and they coincide for $s = s_0$. By Lemma 3.6, $p_{k-1}^{-1}(\gamma_s(\tau_k)) = p_k^{-1}(\gamma_s(\tau_k))$ for all $|s - s_0| < \epsilon$, $s \in I$, which together with (3.3) implies (3.2) and hence (3.1) for $j = k$. □

Corollary 3.10. *Let X, Y be Hausdorff spaces and $p : Y \to X$ a local homeomorphism. Let $\tilde{a} \in Y$, $a = p(\tilde{a})$, and $b \in X$. Let $H : [0,1]^2 \to X$ be a homotopy between γ_0 and γ_1 fixing $a = \gamma_0(0) = \gamma_1(0)$ and $b = \gamma_0(1) = \gamma_1(1)$. Suppose that each curve $\gamma_s := H_s$, $s \in [0,1]$, has a lifting $\tilde{\gamma}_s$ over $p : Y \to X$ which starts at $\tilde{a}$. Then the endpoints of $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ coincide, and $\tilde{\gamma}_s(1)$ is independent of s.*

Proof. By Theorem 3.9, the mapping $s \mapsto \tilde{\gamma}_s(1)$ is continuous. Thus it is a lifting of the constant curve $s \mapsto \gamma_s(1) = b$, and so it is itself constant, by Lemma 3.6. □

3.5. Covering maps. We say that a continuous map $p : Y \to X$ has the **curve lifting property** if for every curve $\gamma : [0,1] \to X$ and every $y_0 \in p^{-1}(\gamma(0))$ there exists a lifting $\tilde{\gamma} : [0,1] \to Y$ of γ with $\tilde{\gamma}(0) = y_0$.

Generally, local homeomorphism do not have the curve lifting property. We shall see that they have this property if and only if they are covering maps.

A continuous map $p : Y \to X$ (between topological spaces Y, X) is called a **covering map** if every $x \in X$ has an open neighborhood U such that

$$p^{-1}(U) = \bigcup_{j \in J} V_j,$$

where the V_j, $j \in J$, are disjoint open subsets of Y and all maps $p|_{V_j} : V_j \to U$ are homeomorphisms. Evidently, every covering map is a local homeomorphism.

Example 3.11. (1) Let $\iota : U \to \mathbb{C}$ be the inclusion of a bounded domain U in $\mathbb{C}$. Then ι is a local homeomorphism, but not a covering map (the defining property fails at points on the boundary of U).

(2) The map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map.

(3) For $k \in \mathbb{N}_{\ge 1}$ the map $p_k : \mathbb{C}^* \to \mathbb{C}^*$, $p_k(z) = z^k$, is a covering map.

(4) Let Λ be a lattice in $\mathbb{C}$. Then the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is a covering map.

Lemma 3.12 (curve lifting property of coverings)**.** *Every covering map $p : Y \to X$ has the curve lifting property.*

Proof. Let $\gamma : [0,1] \to X$ be a curve in X with $\gamma(0) = a$, and let $\tilde{a} \in p^{-1}(a)$. Since $[0,1]$ is compact, there exist a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open sets $U_j \subseteq X$, $1 \le j \le n$, such that $\gamma([t_{j-1}, t_j]) \subseteq U_j$, $p^{-1}(U_j)$ is a disjoint union of open sets $V_{jk} \subseteq Y$, and $p|_{V_{jk}} : V_{jk} \to U_j$ is a homeomorphism. We show by induction on j the existence of a lifting $\tilde{\gamma}_j$ on $[0, t_j]$ with $\tilde{\gamma}_j(0) = \tilde{a}$. There is nothing to prove for $j = 0$. Suppose that $j \ge 1$ and that $\tilde{\gamma}_{j-1}$ is already constructed. Set $y_{j-1} := \tilde{\gamma}_{j-1}(t_{j-1})$. Then $p(y_{j-1}) = \gamma(t_{j-1}) \in U_j$ and y_{j-1} lies in V_{jk} for some k. Setting

$$\tilde{\gamma}_j(t) := \begin{cases} \tilde{\gamma}_{j-1}(t) & \text{if } t \in [0, t_{j-1}], \\ p|_{V_{jk}}^{-1}(\gamma(t)) & \text{if } t \in [t_{j-1}, t_j], \end{cases}$$

yields a lifting on $[0, t_j]$. □

Proposition 3.13 (number of sheets)**.** *Let X, Y be Hausdorff spaces with X path-connected. Let $p : Y \to X$ be a covering map. Then the fibers of p all have the same cardinality. In particular, if $Y \neq \emptyset$, then p is surjective.*

The cardinality of the fibers is called the **number of sheets** of the covering.

Proof. Let $x_0, x_1 \in X$ and choose a curve $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. For each $y \in p^{-1}(x_0)$ there is precisely one lifting $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = y$, by the uniqueness of liftings 3.6. The end point of $\tilde{\gamma}$ lies in $p^{-1}(x_1)$. This defines a bijective map between the two fibers. □

Theorem 3.14 (existence of liftings)**.** *Let X, Y be Hausdorff spaces and $p : Y \to X$ a covering map. Let Z be a simply connected, path-connected and locally path-connected topological space and $f : Z \to X$ continuous. For every $z_0 \in Z$ and every $y_0 \in Y$ with $p(y_0) = f(z_0)$ there exists a unique lifting $\tilde{f} : Z \to Y$ such that $\tilde{f}(z_0) = y_0$.*

Proof. For $z \in Z$ let $\gamma : [0,1] \to Z$ be a curve from z_0 to z. Then $\mu = f \circ \gamma$ is a curve in X with initial point $a = f(z_0)$ which admits a lifting $\tilde{\mu}$ to Y with $\tilde{\mu}(0) = y_0$, by the curve lifting property of coverings 3.12. We define

$$\tilde{f}(z) := \tilde{\mu}(1).$$

Let us prove that $\tilde{f}(z)$ is independent of γ. Set $\gamma_0 = \gamma$ and let γ_1 be another curve in Z from z_0 to z. Since Y is simply connected, there is a homotopy H between γ_0 and γ_1 fixing the endpoints. Then $f \circ H$ is a homotopy between μ and $\mu_1 := f \circ \gamma_1$ fixing the endpoints. If $\tilde{\mu}_1$ is the lifting of μ_1 to Y with $\tilde{\mu}_1(0) = y_0$, then $\tilde{\mu}$ and $\tilde{\mu}_1$ have the same endpoints, by Corollary 3.10. Thus $\tilde{f}(z)$ is independent of γ.

Clearly, $\tilde{f}$ satisfies $p \circ \tilde{f} = f$. It remains to show that $\tilde{f}$ is continuous. Let $z \in Z$, $y = \tilde{f}(z)$, and let V be a neighborhood of y. We must show that there is a neighborhood W of z such that $\tilde{f}(W) \subseteq V$. Shrinking V if necessary, we may assume that there is a neighborhood U of $p(y) = f(z)$ such that $p|_V : V \to U$ is a homeomorphism. Since Z is locally path-connected, there is a path-connected neighborhood W of z such that $f(W) \subseteq U$.

Let $z' \in W$ and let γ' be a curve in W from z to z'. Then $\mu' := f \circ \gamma'$ is a curve in U which has a lifting $\tilde{\mu}' = p|_V^{-1} \circ \mu'$ with initial point y. The product curve $\tilde{\mu} \cdot \tilde{\mu}'$

is a lifting of $\mu\cdot\mu' = f\circ(\gamma\cdot\gamma')$ with initial point y_0. So $\tilde f(z') = \tilde\mu\cdot\tilde\mu'(1) = \tilde\mu'(1) \in V$. This proves $\tilde f(W) \subseteq V$. □

Remark 3.15. The only properties of p used in the previous proof are that p is a local homeomorphism and has the curve lifting property.

Theorem 3.16. *Let X be a manifold, Y a Hausdorff space, and $p : Y \to X$ a local homeomorphism with the curve lifting property. Then p is a covering map.*

Proof. Let $x_0 \in X$ and $p^{-1}(x_0) = \{y_j : j \in J\}$. Let U be a neighborhood of x_0 homeomorphic to a ball (here we use that X is a manifold) and let $f : U \to X$ be the inclusion. By Remark 3.15, for each $j \in J$ there is a lifting $\tilde f_j : U \to Y$ of f with $\tilde f_j(x_0) = y_j$. It is easy to check that the sets $V_j := \tilde f_j(U)$ are pairwise disjoint, $p|_{V_j} : V_j \to U$ is a homeomorphism, and $p^{-1}(U) = \bigcup_{j\in J} V_j$ (exercise). □

3.6. Proper maps. Recall that a map between topological spaces is called **proper** if the preimage of every compact set is compact.

Lemma 3.17. *Let X, Y be locally compact Hausdorff spaces. A proper continuous map $p : Y \to X$ is closed.*

Proof. In a locally compact Hausdorff space a subset is closed if and only if its intersection with every compact set is compact. □

Proposition 3.18. *Let X, Y be locally compact Hausdorff spaces. A proper local homeomorphism $p : Y \to X$ is a covering map.*

Proof. Let $x \in X$. Since p is a local homeomorphism, the fiber $p^{-1}(x)$ is discrete. Since p is proper, the fiber is finite, $p^{-1}(x) = \{y_1, \ldots, y_n\}$. We find for each j an open neighborhood W_j of y_j and an open neighborhood U_j of x such that $p|_{W_j} : W_j \to U_j$ is a homeomorphism. We may assume that the W_j are pairwise disjoint. Then $W := W_1 \cup \cdots \cup W_n$ is an open neighborhood of $p^{-1}(x)$. We claim that there is an open neighborhood $U \subseteq U_1 \cap \cdots \cap U_n$ of x such that $p^{-1}(U) \subseteq W$. For, $Y \setminus W$ is closed, hence $p(Y \setminus W)$ is closed, by Lemma 3.17, and $U := (X \setminus p(Y \setminus W)) \cap U_1 \cap \cdots \cap U_n$ is as desired.

Letting $V_j := W_j \cap p^{-1}(U)$, the V_j are disjoint, $p^{-1}(U) = V_1 \cup \cdots \cup V_n$, and $p|_{V_j} : V_j \to U$ is a homeomorphism for all j. □

3.7. Proper holomorphic maps. Let X, Y be Riemann surfaces. Let $f : X \to Y$ be a proper non-constant holomorphic map. By the local normal form of holomorphic maps 1.7, the set A of branch points of f is closed and discrete. Since f is proper, also $B := f(A)$ is closed and discrete, by Lemma 3.17. We call B the set of **critical values** of f.

Let $Y' := Y \setminus B$ and $X' := X \setminus f^{-1}(B) \subseteq X \setminus A$. Then $f|_{X'} : X' \to Y'$ is a proper unbranched holomorphic covering map. By Proposition 3.18, Proposition 3.13, and by properness, it has a finite number m of sheets. That means that every value $y \in Y'$ is taken exactly m times. This statement extends also to the critical values if we count multiplicities: The map $f : X \to Y$ is said to **take the value** $y \in Y$**,** m **times** (counting multiplicities) if

$$m = \sum_{x\in f^{-1}(y)} m_x(f).$$

Theorem 3.19 (degree)**.** *Let X, Y be Riemann surfaces, and let $f : X \to Y$ be a proper non-constant holomorphic map. Then there is a positive integer n such that f takes every value $y \in Y$, n times. The number n is called the* ***degree*** *of f.*

Proof. Let n be the number of sheets of the unbranched covering $f|_{X'} : X' \to Y'$. Let $b \in B$, $f^{-1}(b) = \{x_1, \dots, x_k\}$ and $m_j = m_{x_j}(f)$. By the local normal form of holomorphic maps 1.7, there exist disjoint neighborhoods U_j of x_j and V_j of b such that for each $c \in V_j \setminus \{b\}$ the set $f^{-1}(c) \cap U_j$ consists of exactly m_j points. As in the proof of Proposition 3.18, there is a neighborhood $V \subseteq V_1 \cap \cdots \cap V_k$ of b such that $f^{-1}(V) \subseteq U_1 \cup \cdots \cup U_k$. Then, for every $c \in V \cap Y'$, the fiber $f^{-1}(c)$ consists of $m_1 + \cdots + m_k$ points. Thus $n = m_1 + \cdots + m_k$. □

Corollary 3.20. *Let X be a compact Riemann surface and let $f \in \mathscr{M}(X)$ be non-constant. Then f has as many zeros as poles (counted with multiplicities).*

Proof. The mapping $f : X \to \widehat{\mathbb{C}}$ is proper, since X is compact. Apply Theorem 3.19. □

Corollary 3.21 (fundamental theorem of algebra)**.** *Any polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z]$, $a_0 \neq 0$, has n roots (counted with multiplicities).*

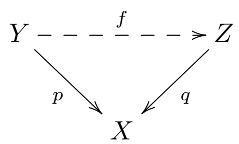
Proof. The meromorphic function $p \in \mathscr{M}(\widehat{\mathbb{C}})$ has a pole of order n at ∞. □

A proper non-constant holomorphic map between Riemann surfaces is sometimes called a **branched (holomorphic) covering**. It may have branch points and in that case it is not a covering map in the topological sense. By an **unbranched (holomorphic) covering** we mean a proper non-constant holomorphic map between Riemann surfaces without branch points. It is a covering map in the topological sense.

4. The universal covering

Every Riemann surface X admits a universal covering by a simply connected Riemann surface $\widetilde{X}$. The group of fiber-preserving homeomorphisms of $\widetilde{X}$ is isomorphic to the fundamental group $\pi_1(X)$.

4.1. Existence and uniqueness of the universal covering. Let X, Y be connected topological spaces. A covering map $p : Y \to X$ is called the **universal covering** of X if it satisfies the following universal property. For every covering map $q : Z \to X$, for connected Z, and every $y_0 \in Y$, $z_0 \in Z$ with $p(y_0) = q(z_0)$ there exists a unique continuous fiber-preserving map $f : Y \to Z$ such that $f(y_0) = z_0$.



Up to isomorphism there is at most one universal covering of a connected space X which follows easily from the universal property.

Proposition 4.1. *Let X, Y be connected manifolds, where Y is simply connected. Let $p : Y \to X$ be a covering map. Then p is the universal covering of X.*

Proof. This is immediate from Theorem 3.14. □

Theorem 4.2 (existence of the universal covering)**.** *Let X be a connected manifold. Then there exists a connected, simply connected manifold $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$.*

Proof. Fix a point $x_0 \in X$. For each $x \in X$ let $\pi(x_0, x)$ be the set of homotopy classes of curves with initial point x_0 and end point x. Define

$$\widetilde{X} := \{(x, \Gamma) : x \in X,\, \Gamma \in \pi(x_0, x)\}$$

and $p : \widetilde{X} \to X$ by $p(x, \Gamma) = x$.

Next we will define a topology on $\widetilde{X}$. Let $(x, \Gamma) \in \widetilde{X}$ and let U be an open, connected, simply connected neighborhood of x in X. Let (U, Γ) be the set of all points $(y, \Lambda) \in \widetilde{X}$ such that $y \in U$ and $\Lambda = [\gamma \cdot \sigma]$, where γ is a curve from x_0 to x such that $\Gamma = [\gamma]$ and σ is a curve from x to y in U. Since U is simply connected, Λ is independent of the choice of σ. We claim that the family $\mathfrak{B}$ of all such sets is a base of topology. Indeed, $\mathfrak{B}$ evidently covers $\widetilde{X}$. If $(z, \Sigma) \in (U, \Gamma) \cap (V, \Lambda)$, then $z \in U \cap V$ and there is an open, connected, simply connected neighborhood $W \subseteq U \cap V$ of z. Thus, $(z, \Sigma) \in (W, \Sigma) \subseteq (U, \Gamma) \cap (V, \Lambda)$.

We claim that $\widetilde{X}$ endowed with the topology generated by $\mathfrak{B}$ is Hausdorff. For, let $(x, \Gamma), (y, \Sigma) \in \widetilde{X}$ be distinct points. If $x \neq y$ then there are disjoint neighborhoods U, V of x, y, respectively, and so $(U, \Gamma), (V, \Sigma)$ are disjoint neighborhoods of $(x, \Gamma), (y, \Sigma)$, respectively. Let us assume that $x = y$ and $\Gamma \neq \Sigma$. Let U be an open, connected, simply connected neighborhood of x in X. We claim that $(U, \Gamma) \cap (U, \Sigma) = \emptyset$. Otherwise there exists $(z, \Lambda) \in (U, \Gamma) \cap (U, \Sigma)$. Suppose $\Gamma = [\gamma]$, $\Sigma = [\sigma]$ and let τ be a curve in U from x to z. Then $\Lambda = [\gamma \cdot \tau] = [\sigma \cdot \tau]$. This implies $\Gamma = [\gamma] = [\sigma] = \Sigma$, a contradiction.

To see that $p : \widetilde{X} \to X$ is a covering map it suffices to check that it is a local homeomorphism, has the curve lifting property, and $\widetilde{X}$ is connected. The first assertion follows from the fact that for every $(U, \Gamma) \in \mathfrak{B}$ the restriction $p|_{(U,\Gamma)} : (U, \Gamma) \to U$ is a homeomorphism. Next we show the curve lifting property. First let $\gamma : [0, 1] \to X$ be a curve with initial point x_0. For $s \in [0, 1]$ set $\gamma_s(t) := \gamma(st)$. Let $\sigma : [0, 1] \to X$ be a closed curve with $\sigma(0) = \sigma(1) = x_0$. Then

$$\widetilde{\gamma} : [0, 1] \to X, \quad t \mapsto (\gamma(t), [\sigma \cdot \gamma_t])$$

is a lifting of γ with initial point $\widetilde{\gamma}(0) = (x_0, [\sigma])$. This also shows that $\widetilde{X}$ is path-connected. Now let $\tau : [0, 1] \to X$ be a curve with arbitrary initial point $\tau(0) = x_1$, let $\Sigma \in \pi(x_0, x_1)$ and σ a curve from x_0 to x_1 with $\Sigma = [\sigma]$. Then the lifting $\widetilde{\gamma}$ of $\gamma := \sigma \cdot \tau$ with initial point $\widetilde{\gamma}(0) = (x_0, [x_0])$, where $[x_0]$ denotes the homotopy class of the constant curve $[x_0]$, gives rise to a lifting of τ with initial point (x_1, Σ).

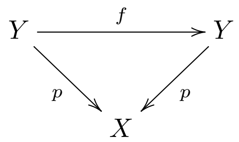
Finally, we show that $\widetilde{X}$ is simply connected. Let $\tau : [0, 1] \to \widetilde{X}$ be a closed curve with initial and end point $(x_0, [x_0])$. Then $\gamma := p \circ \tau$ is a closed curve in X with initial and end point x_0. Then γ has a lifting $\widetilde{\gamma}$ through $(x_0, [x_0])$, and by uniqueness of liftings $\widetilde{\gamma} = \tau$. It follows that $\widetilde{\gamma}(1) = (x_0, [\gamma]) = (x_0, [x_0])$ and so γ is null-homotopic. By the monodromy theorem 3.9, also τ is null-homotopic. Thus $\pi_1(\widetilde{X}, (x_0, [x_0]))$ is trivial. Since $\widetilde{X}$ is path-connected, we may conclude that $\widetilde{X}$ is simply connected. □

Corollary 4.3. *Every Riemann surface has a universal covering which is a Riemann surface in a natural way.*

Proof. This follows from Theorem 4.2 and Theorem 3.5. □

4.2. Deck transformations. Let X, Y be topological spaces and let $p : Y \to X$ be a covering map. A **deck transformation** is a fiber-preserving homeomorphism

$f : Y \to Y$, i.e., the following diagram is commutative.



The set of all deck transformations of $p : Y \to X$ forms a group with respect to composition of maps which is denoted $\mathrm{Deck}(p : Y \to X)$ or simply $\mathrm{Deck}(Y \to X)$.

Suppose that X, Y are connected Hausdorff spaces. The covering $p : Y \to X$ is called **normal** if for every pair of points $y_0, y_1 \in Y$ with $p(y_0) = p(y_1)$ there is a deck transformation $f : Y \to Y$ with $f(y_0) = y_1$. (There exists at most one $f : Y \to Y$ with $f(y_0) = y_1$ since f is a lifting of p.)

Theorem 4.4. *Let X be a connected manifold and let $p : \widetilde{X} \to X$ be its universal covering. Then p is a normal covering and* $\mathrm{Deck}(\widetilde{X} \to X) \cong \pi_1(X)$.

Proof. That p is a normal covering follows in a straightforward manner from the universal property of the universal covering.

Fix $x_0 \in X$ and let $y_0 \in \widetilde{X}$ sit above x_0. We define a map

$$\Phi : \mathrm{Deck}(\widetilde{X} \to X) \to \pi_1(X)$$

as follows. Let $\sigma \in \mathrm{Deck}(\widetilde{X} \to X)$. Let γ be a curve in $\widetilde{X}$ from y_0 to $\sigma(y_0)$. Then $p \circ \gamma$ is a closed curve through x_0. Let $\Phi(\sigma)$ be the homotopy class of $p \circ \gamma$. (Note that the homotopy class of γ is uniquely determined because $\widetilde{X}$ is simply connected.)

Let us check that Φ is a group homomorphism. Let $\sigma_1, \sigma_2 \in \mathrm{Deck}(\widetilde{X} \to X)$ and let γ_1, γ_2 be curves from y_0 to $\sigma_1(y_0), \sigma_2(y_0)$, respectively. Then $\sigma_1 \circ \gamma_2$ is a curve from $\sigma_1(y_0)$ to $\sigma_1(\sigma_2(y_0))$ and $\gamma_1 \cdot (\sigma_1 \circ \gamma_2)$ is a curve from y_0 to $\sigma_1(\sigma_2(y_0))$. Thus

$$\begin{aligned}\Phi(\sigma_1 \circ \sigma_2) &= [p \circ (\gamma_1 \cdot (\sigma_1 \circ \gamma_2))] \\ &= [p \circ \gamma_1][p \circ (\sigma \circ \gamma_2)] = [p \circ \gamma_1][p \circ \gamma_2] = \Phi(\sigma_1)\Phi(\sigma_2).\end{aligned}$$

Injectivity of Φ follows from Corollary 3.10: Suppose that $\Phi(\sigma) = [x_0]$. That means that $p \circ \gamma$ is null-homotopic. Since γ is a lifting of $p \circ \gamma$, $\sigma(y_0) = \gamma(1) = \gamma(0) = y_0$. It follows that σ is the identity on $\widetilde{X}$, because it is a homeomorphism.

For surjectivity let $\Gamma \in \pi_1(X, x_0)$ and let γ be a representative of Γ. Then γ has a lifting $\widetilde{\gamma}$ with initial point y_0. Let $y_1 = \widetilde{\gamma}(1)$. Since p is normal, there exists a deck transformation σ with $\sigma(y_0) = y_1$. Then $\Phi(\sigma) = \Gamma$. ☐

Example 4.5. Since $\mathbb{C}$ is simply connected, $\exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering of $\mathbb{C}^*$. Let $\tau_n : \mathbb{C} \to \mathbb{C}$ denote the translation $\tau_n(z) = z + 2\pi i n$. Then τ_n is a deck transformation for every $n \in \mathbb{Z}$. Suppose that σ is any deck transformation. Then $\exp(\sigma(0)) = \exp(0) = 1$ and hence $\sigma(0) = 2\pi i n$ for some $n \in \mathbb{Z}$. It follows that $\sigma = \tau_n$. That shows that $\mathrm{Deck}(\exp : \mathbb{C} \to \mathbb{C}^*) = \{\tau_n : n \in \mathbb{Z}\}$, and we may conclude by Theorem 4.4 that

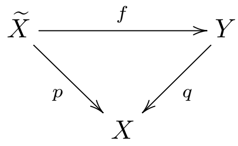
$$\pi_1(\mathbb{C}^*) \cong \mathbb{Z}.$$

Example 4.6 (complex tori, III)**.** Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ be a lattice in $\mathbb{C}$. Then the quotient projection $\mathbb{C} \to \mathbb{C}/\Lambda$ is the universal covering of the torus $\mathbb{C}/\Lambda$. Analogous to the previous example $\mathrm{Deck}(\mathbb{C} \to \mathbb{C}/\Lambda) = \{\tau_w : w \in \Lambda\}$, where τ_w is the translation by w. It follows that

$$\pi_1(\mathbb{C}/\Lambda) \cong \Lambda \cong \mathbb{Z} \times \mathbb{Z}.$$

Theorem 4.7. *Let X, Y be connected manifolds. Let $p : \widetilde{X} \to X$ be the universal covering and $q : Y \to X$ a covering map. Let $f : \widetilde{X} \to Y$ be the fiber-preserving continuous map which exists by the universal property. Then:*

(1) *f is a covering map.*
(2) *There exists a subgroup G of $\mathrm{Deck}(\widetilde{X} \to X)$ such that $f(x) = f(x')$ if and only if there exists $\sigma \in G$ with $\sigma(x) = x'$.*
(3) *$G \cong \pi_1(Y)$.*



Proof. (1) We prove that f is a local homeomorphism and has the curve lifting property. Let $x \in \widetilde{X}$ and set $y = f(x)$ and $z = p(x)$. Since p is a local homeomorphism, there exist open neighborhoods W_1 of x and U_1 of z such that $p|_{W_1} : W_1 \to U_1$ is a homeomorphism. Since q is a covering map, we find an open connected neighborhood $U \subseteq U_1$ of z and an open neighborhood V of y in Y such that $q|_V : V \to U$ is a homeomorphism. If $W := p^{-1}(U) \cap W_1$, then $y \in f(W) \subseteq q^{-1}(U)$. Since $f(W)$ is connected, we may conclude that $f(W) = V$. And $f|_W : W \to V$ is a homeomorphism, since $p|_W : W \to U$ and $q|_V : V \to U$ are.

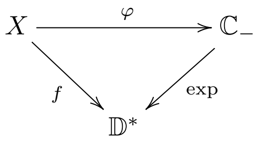
For the curve lifting property, let γ be a curve in Y with initial point y_0. Let $x_0 \in f^{-1}(y_0)$. Then $q \circ \gamma$ is a curve in X which has a lifting $\widetilde{q \circ \gamma}$ to $\widetilde{X}$ with initial point x_0. Then $f \circ \widetilde{q \circ \gamma}$ and γ coincide since they are both liftings of $q \circ \gamma$ with the same initial point. That means that $\widetilde{q \circ \gamma}$ is the desired lifting of γ with initial point x_0.

(2) & (3) Set $G := \mathrm{Deck}(\widetilde{X} \to Y)$ which is a subgroup of $\mathrm{Deck}(\widetilde{X} \to X)$. Since $\widetilde{X}$ is simply connected, $f : \widetilde{X} \to Y$ is the universal covering of Y. By Theorem 4.4, $G \cong \pi_1(Y)$. Now (2) follows from the fact that $f : \widetilde{X} \to Y$ is a normal covering, cf. Theorem 4.4. □

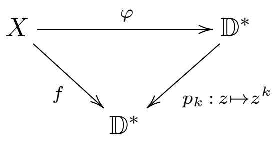
4.3. The covering spaces of the punctured unit disk. Every covering map of $\mathbb{D}^*$ is isomorphic to the covering of the exponential function or to the covering of some power function. More precisely:

Theorem 4.8. *Let X be a Riemann surface and $f : X \to \mathbb{D}^*$ a holomorphic covering map. Then:*

(1) *If the covering has an infinite number of sheets, then there exists a biholomorphism $\varphi : X \to \mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ such that the following diagram commutes.*



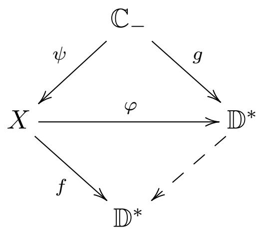
(2) *If the covering has k sheets, then there exists a biholomorphism $\varphi : X \to \mathbb{D}^*$ such that the following diagram commutes.*



Proof. Note that $\exp : \mathbb{C}_- \to \mathbb{D}^*$ is the universal covering of $\mathbb{D}^*$. By Theorem 4.7, there is a holomorphic map $\psi : \mathbb{C}_- \to X$ with $\exp = f \circ \psi$. Let G be the associated subgroup of $\mathrm{Deck}(\mathbb{C}_- \to \mathbb{D}^*)$.

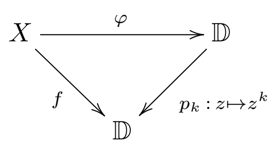
If $G = \{\mathrm{id}\}$ is trivial, then $\psi : \mathbb{C}^- \to X$ is a biholomorphism. This corresponds to the case that the covering f has infinitely many sheets. The inverse of ψ is the desired map φ.

Suppose that G is non-trivial. It is not hard to see that $\mathrm{Deck}(\mathbb{C}_- \to \mathbb{D}^*) = \{\tau_n : n \in \mathbb{Z}\}$, where $\tau_n : \mathbb{C}_- \to \mathbb{C}_-$ is the translation by $2\pi i n$. It follows that there exists a positive integer k such that $G = \{\tau_{kn} : n \in \mathbb{Z}\}$. Let $g : \mathbb{C}_- \to \mathbb{D}^*$ be the covering map defined by $g(z) = \exp(z/k)$. Then $g(z) = g(z')$ if and only if there is $\sigma \in G$ such that $\sigma(z) = z'$. Since G is associated with ψ (i.e., $\psi(z) = \psi(z')$ if and only if there is $\sigma \in G$ such that $\sigma(z) = z'$), there is a bijective map $\varphi : X \to \mathbb{D}^*$ such that $g = \varphi \circ \psi$. Since ψ and g are locally biholomorphic, we may conclude that φ is a biholomorphism. Then the diagram in (2) commutes.



The proof is complete. □

Corollary 4.9. *Let X be a Riemann surface and let $f : X \to \mathbb{D}$ be a branched covering such that $f : f^{-1}(\mathbb{D}^*) \to \mathbb{D}^*$ is a covering map. Then there is an integer $k \geq 1$ and a biholomorphism $\varphi : X \to \mathbb{D}$ such that the following diagram commutes.*



Proof. By Theorem 4.8 and Theorem 3.19, there exists $k \geq 1$ and a biholomorphism $\varphi : f^{-1}(\mathbb{D}^*) \to \mathbb{D}^*$ such that $f = p_k \circ \varphi$. We claim that $f^{-1}(0)$ consists of only one point $a \in X$. Then, by setting $\varphi(a) := 0$, φ extends to a biholomorphism $\varphi : X \to \mathbb{D}$ such that $f = p_k \circ \varphi$, by Riemann's theorem on removable singularities 1.15.

Suppose that $f^{-1}(0)$ consists of $n \geq 2$ points $a_1, \ldots, a_n$. Then there are disjoint open neighborhoods U_i of a_i and $r > 0$ such that $f^{-1}(D_r(0)) \subseteq U_1 \cup \cdots \cup U_n$. Set $D_r^*(0) := D_r(0) \setminus \{0\}$. Then $f^{-1}(D_r^*(0))$ is homeomorphic to $p_k^{-1}(D_r^*(0)) = D^*_{r^{1/k}}(0)$, and thus connected. Every a_i is an accumulation point of $f^{-1}(D_r^*(0))$, and hence also $f^{-1}(D_r(0))$ is connected, a contradiction. □

CHAPTER 2

Analytic continuation

5. Sheaves

The language of sheaves is very useful to organize functions (and other objects) which satisfy local properties. A property of a function defined on an open set which is preserved by restriction to any smaller open set leads to the concept of presheaf. A presheaf is a sheaf if the defining property is local, i.e., it holds if and only if it holds on all open subsets.

5.1. Presheaves and sheaves. Let X be a space with topology $\mathfrak{T}$ (i.e., $\mathfrak{T}$ is the system of open sets in X). A **presheaf** of abelian groups on X is a pair $(\mathscr{F}, \rho)$ consisting of a family $\mathscr{F} = (\mathscr{F}(U))_{U\in\mathfrak{T}}$ of abelian groups and a family $\rho = (\rho_V^U)_{U,V\in\mathfrak{T},V\subseteq U}$ of group homomorphisms $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ such that

- $\rho_U^U = \mathrm{id}_{\mathscr{F}(U)}$ for all $U \in \mathfrak{T}$,
- $\rho_W^V \circ \rho_V^U = \rho_W^U$ for all $W \subseteq V \subseteq U$.

The homomorphisms ρ_V^U are called **restriction homomorphisms**. Often we will write just $f|_V$ for $\rho_V^U(f)$ and $f \in \mathscr{F}(U)$.

Analogously, one defines presheaves of vector spaces, rings, sets, etc.

A presheaf $\mathscr{F}$ on a topological space X is called a **sheaf** if for every open $U \subseteq X$ and every family of open subsets $U_i \subseteq U$, $i \in I$, with $U = \bigcup_{i\in I} U_i$ the following conditions are satisfied:

(1) If $f, g \in \mathscr{F}(U)$ satisfy $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.
(2) Let $f_i \in \mathscr{F}(U_i)$, $i \in I$, be such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$. Then there exists $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

The element f in (2) is unique by (1).

Example 5.1. (1) Let X be a topological space. For each open $U \subseteq X$ let $\mathscr{C}(U)$ denote the vector space of continuous functions $f : U \to \mathbb{C}$. Then $\mathscr{C}$ with the usual restriction mapping is a sheaf on X.

(2) Let X be a Riemann surface. Let $\mathcal{O}(U)$ be the ring of holomorphic functions on the open subset $U \subseteq X$. Taking the usual restriction mapping, we get the sheaf $\mathcal{O}$ of holomorphic functions on X.

(3) Similarly we obtain the sheaf $\mathscr{M}$ of meromorphic functions on a Riemann surface X.

(4) Let X be a Riemann surface. Denote by $\mathcal{O}^*(U)$ the multiplicative group of all holomorphic functions $f : U \to \mathbb{C}^*$. With the usual restriction map we obtain a sheaf $\mathcal{O}^*$ on X.

(5) Similarly we obtain the sheaf $\mathscr{M}^*$. Here $\mathscr{M}^*(U)$ consists of all $f \in \mathscr{M}(U)$ which do not vanish identically on any connected component of U.

(6) Let X be a topological space. For open $U \subseteq X$ let $\mathbb{C}(U)$ denote the vector space of locally constant functions $f : U \to \mathbb{C}$. With the usual restriction map this defines the sheaf $\mathbb{C}$ on X, whereas the constant functions form only a presheaf.

5.2. Stalks. Let $\mathscr{F}$ be a presheaf of sets on a topological space X and let $a \in X$. The **stalk** $\mathscr{F}_a$ of $\mathscr{F}$ at a is defined as the inductive limit

$$\mathscr{F}_a := \varinjlim_{U \ni a} \mathscr{F}(U)$$

over all open neighborhoods U of a. This means the following. Consider the disjoint union

$$\bigsqcup_{U \ni a} \mathscr{F}(U)$$

with the following equivalence relation: $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$ are equivalent, $f \sim g$, if there is an open set W with $a \in W \subseteq U \cap V$ such that $f|_W = g|_W$. Then $\mathscr{F}_a$ is the set of equivalence classes,

$$\mathscr{F}_a = \varinjlim_{U \ni a} \mathscr{F}(U) := \Big(\bigsqcup_{U \ni a} \mathscr{F}(U) \Big) / \sim$$

If $\mathscr{F}$ is a presheaf of abelian groups, then the stalk $\mathscr{F}_a$ is also an abelian group in a natural way (similarly, for presheaves of vector spaces, rings, etc.).

Let U be an open neighborhood of a. Let $\rho_a : \mathscr{F}(U) \to \mathscr{F}_a$ denote the map which assigns to $f \in \mathscr{F}(U)$ its equivalence class modulo $\sim$. Then $\rho_a(f) =: f_a$ is called the **germ** of f at a.

Example 5.2. Consider the sheaf $\mathcal{O}$ of holomorphic functions on a region $X \subseteq \mathbb{C}$ (a region is a connected domain). The stalk $\mathcal{O}_a$ at $a \in X$ is isomorphic to the ring $\mathbb{C}\{z-a\}$ of convergent power series in $z - a$. Analogously, the stalk $\mathscr{M}_a$ is isomorphic to the ring of all convergent Laurent series with finite principal part

$$\sum_{k=m}^{\infty} c_k (z-a)^k, \quad m \in \mathbb{Z}, \quad c_k \in \mathbb{C}.$$

5.3. The topological space associated with a presheaf. Let $\mathscr{F}$ be a presheaf on a topological space X. Let

$$|\mathscr{F}| := \bigsqcup_{x \in X} \mathscr{F}_x$$

be the disjoint union of all stalks. Let $p : |\mathscr{F}| \to X$ be defined by $\mathscr{F}_x \ni \varphi \mapsto x$. For any open $U \subseteq X$ and $f \in \mathscr{F}(U)$, let

$$(U, f) := \{\rho_x(f) : x \in U\}.$$

Theorem 5.3. *The system $\mathfrak{B}$ of all sets (U, f), where U is open in X and $f \in \mathscr{F}(U)$, is a basis for a topology on $|\mathscr{F}|$. With respect to this topology $p : |\mathscr{F}| \to X$ is a local homeomorphism.*

Proof. Let us check that $\mathfrak{B}$ is a basis for a topology on $|\mathscr{F}|$. Clearly, every $\varphi \in |\mathscr{F}|$ is contained in at least one (U, f). We have to verify that if $\varphi \in (U, f) \cap (V, g)$ then there exists $(W, h) \in \mathfrak{B}$ such that $\varphi \in (W, h) \subseteq (U, f) \cap (V, g)$. Let $x = p(\varphi)$. Then $x \in U \cap V$ and $\varphi = \rho_x(f) = \rho_x(g)$. So there is an open neighborhood W of x in $U \cap V$ such that $f|_W = g|_W =: h$. This implies the claim.

To see that $p : |\mathscr{F}| \to X$ is a local homeomorphism, let $\varphi \in |\mathscr{F}|$ and $x = p(\varphi)$. There is $(U, f) \in \mathfrak{B}$ with $\varphi \in (U, f)$. Then (U, f) is an open neighborhood of φ and U is an open neighborhood of x. The restriction $p|_{(U,f)} : (U, f) \to U$ is a homeomorphism. $\square$

We say that a presheaf $\mathscr{F}$ on a topological space X **satisfies the identity theorem** if the following holds: Let $U \subseteq X$ be a connected open set. Let $f, g \in \mathscr{F}(U)$ be such that $\rho_a(f) = \rho_a(g)$ at some $a \in U$. Then $f = g$.

Theorem 5.4. *Let X be a locally connected Hausdorff space. Let $\mathscr{F}$ be a presheaf on X satisfying the identity theorem. Then $|\mathscr{F}|$ is Hausdorff.*

Proof. Let $\varphi_1 \neq \varphi_2 \in |\mathscr{F}|$. Set $x_i = p(\varphi_i)$, $i = 1, 2$. If $x_1 \neq x_2$, then there exist disjoint neighborhoods U_1, U_2 of x_1, x_2, since X is Hausdorff, and $p^{-1}(U_1), p^{-1}(U_2)$ are disjoint neighborhoods of φ_1, φ_2.

Suppose that $x_1 = x_2 =: x$. Let $f_i \in \mathcal{F}(U_i)$ be a representative of the germ φ_i. Let $U \subseteq U_1 \cap U_2$ be a connected open neighborhood of x. Then $(U, f_i|_U)$ is a neighborhood of φ_i. Suppose that $\psi \in (U, f_1|_U) \cap (U, f_2|_U)$ and let $y = p(\psi)$. Then $\psi = \rho_y(f_1) = \rho_y(f_2)$. By assumption, $f_1|_U = f_2|_U$ and thus $\varphi_1 = \varphi_2$, a contradiction. □

Corollary 5.5. *Let X be a Riemann surface and let $\mathcal{O}$ (resp. $\mathscr{M}$) be the sheaf of holomorphic (resp. meromorphic) functions on X. Then $|\mathcal{O}|$ (resp. $|\mathscr{M}|$) is Hausdorff.*

6. Analytic continuation

In this section we study the construction of Riemann surfaces which arise from the analytic continuation of germs of functions.

6.1. Analytic continuation along curves. Let X be a Riemann surface. Let $\gamma : [0, 1] \to X$ be a curve joining $a = \gamma(0)$ and $b = \gamma(1)$. We say that a holomorphic germ $\varphi_1 \in \mathcal{O}_b$ results from the **analytic continuation along** γ of the germ $\varphi_0 \in \mathcal{O}_a$ if the following holds: For every $t \in [0, 1]$ there exists $\varphi_t \in \mathcal{O}_{\gamma(t)}$ such that for every $t_0 \in [0, 1]$ there is a neighborhood $T \subseteq [0, 1]$ of t_0, an open set $U \subseteq X$ with $\gamma(T) \subseteq U$ and a function $f \in \mathcal{O}(U)$ such that $\rho_{\gamma(t)}(f) = \varphi_t$ for all $t \in T$.

Since $[0, 1]$ is compact, this condition is equivalent to the following: There exist a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$, open sets $U_i \subseteq X$ with $\gamma([t_{i-1}, t_i]) \subseteq U_i$, and $f_i \in \mathcal{O}(U_i)$, for $i = 1, \ldots, n$, such that $\rho_a(f_1) = \varphi_0$, $\rho_b(f_n) = \varphi_1$, and $f_i|_{V_i} = f_{i+1}|_{V_i}$, $i = 1, \ldots, n-1$, where V_i is the connected component of $U_i \cap U_{i+1}$ containing $\gamma(t_i)$.

Proposition 6.1. *Let X be a Riemann surface. Let $\gamma : [0, 1] \to X$ be a curve joining $a = \gamma(0)$ and $b = \gamma(1)$. A germ $\varphi_1 \in \mathcal{O}_b$ is the analytic continuation of a germ $\varphi_0 \in \mathcal{O}_a$ along γ if and only if there is a lifting $\tilde{\gamma} : [0, 1] \to |\mathcal{O}|$ of γ such that $\tilde{\gamma}(0) = \varphi_0$ and $\tilde{\gamma}(1) = \varphi_1$.*

Proof. If $\varphi_1 \in \mathcal{O}_b$ is the analytic continuation of a germ $\varphi_0 \in \mathcal{O}_a$ along γ, then $t \mapsto \varphi_t$ is the required lifting.

Conversely, suppose that there is a lifting $\tilde{\gamma} : [0, 1] \to |\mathcal{O}|$ of γ such that $\tilde{\gamma}(0) = \varphi_0$ and $\tilde{\gamma}(1) = \varphi_1$. Define $\varphi_t := \tilde{\gamma}(t) \in \mathcal{O}_{\gamma(t)}$. Let $t_0 \in [0, 1]$ and let (U, f) be an open neighborhood of $\tilde{\gamma}(t_0)$ in $|\mathcal{O}|$. Since $\tilde{\gamma}$ is continuous, there is a neighborhood T of $t_0 \in [0, 1]$ such that $\tilde{\gamma}(T) \subseteq (U, f)$. Hence $\gamma(T) \subseteq U$ and $\varphi_t = \tilde{\gamma}(t) = \rho_{\gamma(t)}(f)$ for $t \in T$. □

We may infer from the uniqueness of liftings 3.6 that the analytic continuation of a function germ is unique (if it exists). The monodromy theorem 3.9 implies the following.

Theorem 6.2. *Let X be a Riemann surface. Let $\gamma_0, \gamma_1 : [0, 1] \to X$ be homotopic curves joining a and b. Let γ_s, $s \in [0, 1]$, be a homotopy of γ_0 and γ_1 and let $\varphi \in \mathcal{O}_a$ be a function germ which has an analytic continuation along all curves γ_s. Then the analytic continuations of φ along γ_0 and γ_1 result in the same germ $\psi \in \mathcal{O}_b$.*

Proof. Apply Corollary 3.10 to the local homeomorphism $|\mathcal{O}| \to X$; $|\mathcal{O}|$ is Hausdorff, by Corollary 5.5. $\square$

Corollary 6.3. *Let X be a simply connected Riemann surface. Let $\varphi \in \mathcal{O}_a$ be a germ at some point $a \in X$ which admits an analytic continuation along every curve starting in a. Then there is a unique holomorphic function $f \in \mathcal{O}(X)$ such that $\rho_a(f) = \varphi$.*

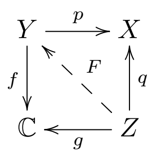
6.2. Riemann surfaces arising from analytic continuation of germs. In general, if X is not simply connected, by considering all germs that arise by analytic continuation from a given germ we obtain a multi-valued function. Let us make this precise.

First we make the following observation. Suppose that X, Y are Riemann surfaces, $\mathcal{O}_X, \mathcal{O}_Y$ the sheaves of holomorphic functions on them, and $p : Y \to X$ is an unbranched holomorphic map. Since p is locally biholomorphic, it induces an isomorphism $p^* : \mathcal{O}_{X,p(y)} \to \mathcal{O}_{Y,y}$ for each $y \in Y$. Let

$$p_* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,p(y)} \tag{6.1}$$

denote the inverse of p^*.

Let X be a Riemann surface, $a \in X$, and $\varphi \in \mathcal{O}_{X,a}$. By an **analytic continuation** (Y, p, f, b) of φ we mean the following data: Y is a Riemann surface and $p : Y \to X$ is an unbranched holomorphic map, $b \in p^{-1}(a)$, and f is a holomorphic function on Y such that $p_*(\rho_b(f)) = \varphi$. An analytic continuation (Y, p, f, b) of φ is called **maximal** if it has the following universal property: if (Z, q, g, c) is another analytic continuation of φ then there is a fiber-preserving holomorphic map $F : Z \to Y$ such that $F(c) = b$ and $F^*(f) = g$.



By the uniqueness of liftings 3.6, a maximal analytic continuation is unique up to isomorphism. Indeed, if also (Z, q, g, c) is a maximal analytic continuation of φ then there is a fiber-preserving holomorphic map $G : Y \to Z$ such that $G(b) = c$ and $G^*(g) = f$. Then $F \circ G$ is a fiber-preserving holomorphic map $Y \to Y$ leaving b fixed. By the uniqueness of liftings 3.6, $F \circ G = \mathrm{id}_Y$. Similarly, $G \circ F = \mathrm{id}_X$ and so $G : Y \to Z$ is biholomorphism.

We will show that there always exists a maximal analytic continuation. We shall need the following lemma.

Lemma 6.4. *Let X be a Riemann surface, $a \in X$, $\varphi \in \mathcal{O}_{X,a}$, and (Y, p, f, b) an analytic continuation of φ. Let $\sigma : [0, 1] \to Y$ be a curve from b to y. Then the germ $\psi := p_*(\rho_y(f)) \in \mathcal{O}_{X,p(y)}$ is an analytic continuation of φ along the curve $\gamma = p \circ \sigma$.*

Proof. For $t \in [0, 1]$ set $\varphi_t := p_*(\rho_{\sigma(t)}(f)) \in \mathcal{O}_{X,p(\sigma(t))} = \mathcal{O}_{X,\gamma(t)}$. Then $\varphi_0 = \varphi$ and $\varphi_1 = \psi$. Let $t_0 \in [0, 1]$. Since $p : Y \to X$ is a local homeomorphism, there exist open neighborhoods $V \subseteq Y$ of $\sigma(t_0)$ and $U \subseteq X$ of $\gamma(t_0)$ such that $p|_V : V \to U$ is a biholomorphism. If $q : U \to V$ is the inverse, then $g := q^*(f|_V) \in \mathcal{O}(U)$. Then $p_*(\rho_z(f)) = \rho_{p(z)}(g)$ for every $z \in V$. There is a neighborhood T of t_0 in $[0, 1]$ such that $\sigma(T) \subseteq V$, and so $\gamma(T) \subseteq U$. For each $t \in T$, we have $\rho_{\gamma(t)}(g) = p_*(\rho_{\sigma(t)}(f)) = \varphi_t$. Thus, ψ is an analytic continuation of φ along γ. $\square$

Theorem 6.5 (maximal analytic continuation). *Let X be a Riemann surface, $a \in X$, and $\varphi \in \mathcal{O}_{X,a}$. There exists a maximal analytic continuation (Y, p, f, b) of φ.*

Proof. Let Y be the connected component of $|\mathcal{O}_X|$ containing φ. Let $p : Y \to X$, be the restriction of the canonical map $|\mathcal{O}_X| \to X$. Then p is a local homeomorphism. By Theorem 3.5, there is a natural complex structure on Y which makes it a Riemann surface and $p : Y \to X$ holomorphic. Let $f : Y \to \mathbb{C}$ be defined by $f(\psi) := \mathrm{ev}_{p(\psi)}(\psi)$, i.e., $\psi \in Y$ is a germ at $p(\psi)$ and $f(\psi)$ is its value. Then f is holomorphic and $p_*(\rho_\psi(f)) = \psi$ for every $\psi \in Y$, in particular, for $b := \varphi$. Thus (Y, p, f, b) is an analytic continuation of φ.

Let us show maximality. Let (Z, q, g, c) be another analytic continuation of φ. Let $z \in Z$ and $q(z) = x$. By Lemma 6.4, the germ $q_*(\rho_z(g)) \in \mathcal{O}_{X,x}$ arises by analytic continuation along a curve from a to x, and hence, by Proposition 6.1, there is precisely one $\psi \in Y$ such that $q_*(\rho_z(g)) = \psi$. Define a mapping $F : Z \to Y$ by setting $F(z) := \psi$. Then F is a fiber-preserving holomorphic map with $F(c) = b$ and $F^*(f) = g$. □

Analytic continuation of meromorphic function germs can be handled in a similar way. Branch points have been disregarded so far. In the next section branch points will also be considered in the special case of algebraic functions.

7. Algebraic functions

An **algebraic function** is a function $w = w(z)$ which satisfies an algebraic equation

$$w^n + a_1(z)w^{n-1} + \cdots a_n(z) = 0, \tag{7.1}$$

where the coefficients a_j are given meromorphic functions in z. A typical example is the square root $w = \sqrt{z}$ which is one of the first examples of a multi-valued functions one encounters in complex analysis.

In this section we will construct the Riemann surface of algebraic functions. It is a branched covering such that the number of sheets equals the degree of the algebraic equation.

7.1. Elementary symmetric functions. Let X and Y be Riemann surfaces and let $p : Y \to X$ be an n-sheeted unbranched holomorphic covering map. Let $f \in \mathscr{M}(Y)$. Fix $x \in X$. Then x has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets $V_1, \ldots, V_n$ and $p|_{V_j} : V_j \to U$ is a biholomorphism for all $j = 1, \ldots, n$. Set $f_j := f \circ p|_{V_j}^{-1}$, $j = 1, \ldots, n$, and consider

$$\prod_{j=1}^{n}(T - f_j) = T^n + c_1T^{n-1} + \cdots + c_n.$$

Then the coefficients c_j, $j = 1, \ldots, n$, are meromorphic functions on U given by

$$c_j = (-1)^j s_j(f_1, \ldots, f_j) = (-1)^j \sum_{1 \le i_1 < \cdots < i_j \le n} f_{i_1} \cdots f_{i_j},$$

where s_j is the jth **elementary symmetric function** in n variables.

If we carry out the same construction on a suitable neighborhood U' of another point $x' \in X$, then we obtain the same functions $c_1, \ldots, c_n$. It follows that they piece together to give global meromorphic functions $c_1, \ldots, c_n \in \mathscr{M}(X)$. Abusing notation we call these functions the **elementary symmetric functions** of f with respect to the covering $p : Y \to X$.

In the next theorem we will see that the elementary symmetric functions of $f \in \mathscr{M}(Y)$ are also defined if $p : Y \to X$ is a branched holomorphic covering.

Theorem 7.1. *Let X and Y be Riemann surfaces and let $p : Y \to X$ be an n-sheeted branched holomorphic covering map. Let $A \subseteq X$ be a closed discrete set containing all critical values of p and set $B := p^{-1}(A)$. Let f be a holomorphic (resp. meromorphic) function on $Y \setminus B$ and let $c_1, \ldots, c_n \in \mathcal{O}(X \setminus A)$ (resp. $\in \mathscr{M}(X \setminus A)$) the elementary symmetric functions of f. Then f can be continued holomorphically (resp. meromorphically) to Y if and only if all c_j can be continued holomorphically (resp. meromorphically) to X.*

Proof. Let $a \in A$ and $p^{-1}(a) = \{b_1, \ldots, b_m\}$. Let (U, z) be a relatively compact coordinate neighborhood of a with $z(a) = 0$ and $U \cap A = \{a\}$. Then $V := p^{-1}(U)$ is a relatively compact neighborhood of $p^{-1}(a)$, since p is proper.

First suppose that f is holomorphic on $Y \setminus B$. If f can be continued holomorphically to all points b_j, then f is bounded on $V \setminus \{b_1, \ldots, b_m\}$. Hence all c_j are bounded on $U \setminus \{a\}$, and so all c_j admit a holomorphic extension to a, by Riemann's theorem on removable singularities 1.15. Conversely, if all c_j extend holomorphically to a, then all c_j are bounded on $U \setminus \{a\}$. Then also f is bounded on $V \setminus \{b_1, \ldots, b_m\}$, since

$$f(y)^n + c_1(p(y))f(y)^{n-1} + \cdots + c_n(p(y)) = 0 \quad \text{for } y \in V \setminus \{b_1, \ldots, b_m\}.$$

By Theorem 1.15, f extends holomorphically to each b_j.

Now let f be meromorphic on $Y \setminus B$. Assume that f can be continued meromorphically to all points b_j. The function $\varphi := p^* z = z \circ p$ is holomorphic on V and vanishes on all points b_j. Then $\varphi^k f$ has a holomorphic extension to all b_j provided that the integer k is chosen large enough. The elementary symmetric functions of $\varphi^k f$ are precisely $z^{kj} c_j$. By the previous paragraph, they admit a holomorphic extension to a, that is, the c_j admit a meromorphic extension to a. Conversely, suppose that all c_j extend meromorphically to a. Then all $z^{kj} c_j$ admit a holomorphic extension to a, where k is a large integer. Thus $\varphi^k f$ extends holomorphically to all b_j, whence f extends meromorphically to all b_j. □

Remark 7.2. For later use we remark that the proof does not use the fact that Y is connected. So in the theorem we may assume that Y is a disjoint union of finitely many Riemann surfaces.

7.2. Associated field extension. Let X and Y be Riemann surfaces and let $p : Y \to X$ be an n-sheeted branched holomorphic covering map. If $f \in \mathscr{M}(Y)$ and $c_1, \ldots, c_n \in \mathscr{M}(X)$ are the elementary symmetric functions of f, then

$$f^n + (p^* c_1) f^{n-1} + \cdots + (p^* c_n) = 0. \tag{7.2}$$

This is clear by the definition of elementary symmetric functions.

Theorem 7.3. *In this situation the monomorphism of fields $p^* : \mathscr{M}(X) \to \mathscr{M}(Y)$ is an algebraic field extension of degree n.*

Proof. Set $K := p^* \mathscr{M}(X) \subseteq \mathscr{M}(Y) =: L$. By the observation above, each $f \in L$ is algebraic over K and the minimal polynomial of f over K has degree $\le n$.

Let $f_0 \in L$ be such that the degree n_0 of its minimal polynomial is maximal. We claim that $L = K(f_0)$. For an arbitrary $f \in L$ consider the field $K(f_0, f)$. By the primitive element theorem (cf. [**8**]), there is $g \in L$ such that $K(f_0, f) = K(g)$. By the definition of n_0, we have $\dim_K K(g) \le n_0$. On the other hand, $\dim_K K(f_0, f) \ge \dim_K K(f_0) = n_0$. It follows that $K(f_0) = K(f_0, f)$ and hence $f \in K(f_0)$.

If the degree of the minimal polynomial of f over K is $m < n$, then f can take at most m different values over every $x \in X$. We will see in Corollary 12.8

and Theorem 26.7, that there exist an $f \in \mathscr{M}(Y)$ and an $x \in X$ with $p^{-1}(x) = \{y_1, \ldots, y_n\}$ such that the values $f(y_j)$, $j = 1, \ldots, n$, are all distinct. $\square$

7.3. Continuation of a covering.

Theorem 7.4. *Let X be a Riemann surface, $A \subseteq X$ a closed discrete set and $X' := X \setminus A$. Suppose that Y' is another Riemann surface and $p' : Y' \to X'$ is a proper unbranched holomorphic covering. Then p' extends to a branched covering of X, i.e., there is a Riemann surface Y, a proper holomorphic map $p : Y \to X$, and a fiber-preserving biholomorphism $\varphi : Y \setminus p^{-1}(A) \to Y'$.*

Proof. For each $a \in A$ choose a coordinate neighborhood (U_a, z_a) on X such that $z_a(a) = 0$, $z_a(U_a) = \mathbb{D}$ and $U_a \cap U_b = \emptyset$ if $a \neq b \in A$. Let $U_a^* = U_a \setminus \{a\}$. Since $p' : Y' \to X'$ is proper, $(p')^{-1}(U_a^*)$ consists of a finite number of connected components V_{aj}^*, $j = 1, \ldots, n_a$. For each j, the map $p'|_{V_{aj}^*} : V_{aj}^* \to U_a^*$ is an unbranched covering with number of sheets k_{aj}. By Theorem 4.8, there are biholomorphisms $\zeta_{aj} : V_{aj}^* \to \mathbb{D}^*$ such that the following diagram commutes:

$$\begin{array}{ccc} V_{aj}^* & \xrightarrow{\ \zeta_{aj}\ } & \mathbb{D}^* \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p_{aj}:\ \zeta \mapsto \zeta^{k_{aj}}} \\ U_a^* & \xrightarrow{\ z_a\ } & \mathbb{D}^* \end{array}$$

Choose pairwise distinct "ideal" points y_{aj}, $a \in A$, $j = 1, \ldots, n_a$, in some set disjoint of Y', and define

$$Y := Y' \cup \{y_{aj} : a \in A,\, j = 1, \ldots, n_a\}.$$

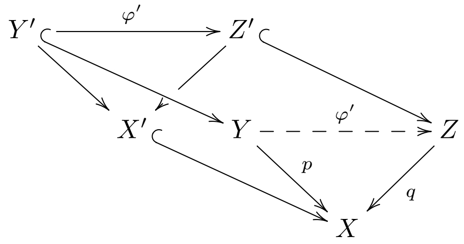
On Y there is precisely one topology with the following property: if W_i, $i \in I$, is a neighborhood basis of a, then $\{y_{aj}\} \cup (p')^{-1}(W_i) \cap V_{aj}^*$, $i \in I$, is a neighborhood basis of y_{aj}, and on Y' it induces the given topology. It makes Y to a Hausdorff space. Define $p : Y \to X$ by $p(y) = p'(y)$ if $y \in Y'$ and $p(y_{aj}) = a$. It is easy to see that p is proper.

In order to make Y into a Riemann surface, we add the following charts to the charts of the complex structure of Y'. Let $V_{aj} := V_{aj}^* \cup \{y_{aj}\}$ and let $\zeta_{aj} : V_{aj} \to \mathbb{D}$ be the extension of $\zeta_{aj} : V_{aj}^* \to \mathbb{D}^*$ by setting $\zeta_{aj}(y_{aj}) := 0$. The new charts $\zeta_{aj} : V_{aj} \to \mathbb{D}$ are compatible with the charts of the complex structure of Y', since $\zeta_{aj} : V_{aj}^* \to \mathbb{D}^*$ is biholomorphic with respect to the complex structure of Y'. It follows that $p : Y \to X$ is holomorphic.

By construction, $Y \setminus p^{-1}(A) = Y'$. So we may take $\varphi : Y \setminus p^{-1}(A) \to Y'$ to be the identity map. $\square$

The continuation of a covering, whose existence was proved in Theorem 7.4, is unique up to isomorphisms, as we shall see next.

Theorem 7.5. *Let X, Y, Z be Riemann surfaces and $p : Y \to X$, $q : Z \to X$ be proper holomorphic maps. Let $A \subseteq X$ be closed discrete and let $X' := X \setminus A$, $Y' := p^{-1}(X')$, and $Z' := q^{-1}(X')$. Then every fiber-preserving biholomorphism $\varphi' : Y' \to Z'$ extends to a fiber-preserving biholomorphism $\varphi : Y \to Z$. In particular,* *every* $\varphi' \in \mathrm{Deck}(Y' \to X')$ *extends to a* $\varphi \in \mathrm{Deck}(Y \to X)$.



Proof. Let $a \in A$ and let (U, z) be a coordinate neighborhood of a such that $z(a) = 0$ and $z(U) = \mathbb{D}$. We may assume that U is so small that p and q are unbranched over $U^* := U \setminus \{a\}$. Let $V_1, \dots, V_n$ be the connected components of $p^{-1}(U)$, and $W_1, \dots, W_m$ those of $q^{-1}(U)$. Then $V_j^* := V_j \setminus p^{-1}(a)$ are the connected components of $p^{-1}(U^*)$, and $W_k^* := W_k \setminus q^{-1}(a)$ are those of $q^{-1}(U^*)$.

Since the map $\varphi'|_{p^{-1}(U^*)} : p^{-1}(U^*) \to q^{-1}(U^*)$ is biholomorphic, we have $m = n$ and after renumbering we may assume that $\varphi'(V_j^*) = W_j^*$. By Corollary 4.9, $V_j \cap p^{-1}(a)$ and $W_j \cap q^{-1}(a)$ both consist of precisely one point, say b_j and c_j, respectively. Thus, $\varphi'|_{p^{-1}(U^*)} : p^{-1}(U^*) \to q^{-1}(U^*)$ can be continued to a bijective map $p^{-1}(U) \to q^{-1}(U)$ which takes b_j to c_j. This map is a homeomorphism, because $p|_{V_j} : V_j \to U$ and $q|_{W_j} : W_j \to U$ are proper and hence closed, by Lemma 3.17. By Riemann's theorem on removable singularities 1.15, it is a biholomorphism; Theorem 1.15 applies since both V_j and W_j are isomorphic to $\mathbb{D}$ by Corollary 4.9. Applying this construction to all $a \in A$, we obtain the desired extension $\varphi : Y \to Z$. □

This theorem allows us to extend the notion of normal covering to branched holomorphic coverings. Let X and Y be Riemann surfaces and let $p : Y \to X$ be a branched holomorphic covering. Let $A \subseteq X$ be the set of critical values of p and set $X' := X \setminus A$ and $Y' := p^{-1}(X')$. Then $p : Y \to X$ is said to be a **normal covering** if the unbranched covering $Y' \to X'$ is normal.

7.4. The roots of holomorphic polynomials.

Lemma 7.6. *Let* $c_1, \dots, c_n$ *be holomorphic functions on* $D_R(0)$. *Suppose that* $w_0 \in \mathbb{C}$ *is a simple root of the polynomial* $T^n + c_1(0)T^{n-1} + \cdots + c_0(0)$. *Then there exists* $r \in (0, R]$ *and a function* φ *holomorphic on* $D_r(0)$ *such that* $\varphi(0) = w_0$ *and*

$$\varphi^n + c_1\varphi^{n-1} + \cdots + c_n = 0 \quad \text{on } D_r(0).$$

Proof. Consider

$$F(z, w) := w^n + c_1(z)w^{n-1} + \cdots + c_n(z), \qquad \text{for } z \in D_R(0),\ w \in \mathbb{C}.$$

There is $\epsilon > 0$ such that the polynomial $F(0, w)$ has no other root than w_0 in the disk $\overline{D}_\epsilon(w_0)$. It follows from the continuity of F that there is $r \in (0, R]$ such that F has no zero in the set $\{(z, w) : |z| < r,\ w \in \partial D_\epsilon(w_0)\}$. By the argument principle, for fixed $z \in D_r(0)$,

$$n(z) := \frac{1}{2\pi i} \int_{\partial D_\epsilon(w_0)} \frac{\partial_w F(z, w)}{F(z, w)}\, dw$$

is the number of roots of the polynomial $F(z, w)$ in $D_\epsilon(w_0)$. Then $n(z) = 1$ for all $z \in D_r(0)$, since $n(0) = 1$ and n depends continuously on z. By the residue theorem, the root of $F(z,w)$ in $D_\epsilon(w_0)$ is given by

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial D_\epsilon(w_0)} w\frac{\partial_w F(z,w)}{F(z,w)}\,dw$$

which is holomorphic in $D_r(0)$. □

Corollary 7.7. *Let X be a Riemann surface and $x \in X$. For a polynomial $P(T) = T^n + c_1T^{n-1} + \cdots + c_n \in \mathcal{O}_x[T]$ which has n distinct roots $w_1,\ldots,w_n$ at x there exist $\varphi_1,\ldots,\varphi_n \in \mathcal{O}_x$ such that $\varphi_j(x) = w_j$, $j = 1,\ldots,n$, and*

$$P(T) = \prod_{j=1}^n (T - \varphi_j).$$

7.5. The Riemann surface of an algebraic function.

Theorem 7.8. *Let X be a Riemann surface and let $P(T) = T^n + c_1T^{n-1} + \cdots + c_n \in \mathscr{M}(X)[T]$ be a irreducible polynomial of degree n. Then there exist a Riemann surface Y, a branched holomorphic n-sheeted covering $p : Y \to X$ and a meromorphic function $F \in \mathscr{M}(Y)$ such that $(p^*P)(F) = 0$. The triple (Y,p,F) is unique in the sense that, if (Z,q,G) is has the same properties, then there is a fiber-preserving biholomorphism $\sigma : Z \to Y$ such that $G = \sigma^*F$.*

The triple (Y,p,F) is called the **algebraic function** defined by the polynomial $P(T)$. Classically, X is the Riemann sphere $\widehat{\mathbb{C}}$. Then the coefficients of $P(T)$ are rational functions. In this case Y is compact, since $\widehat{\mathbb{C}}$ is compact and $p : Y \to \widehat{\mathbb{C}}$ is proper.

Proof. Let $\Delta \in \mathscr{M}(X)$ be the discriminant of $P(T)$. Since $P(T)$ is irreducible, Δ is not identically zero. There is a closed discrete set $A \subseteq X$ such that in the complement $X' := X \setminus A$ all coefficients c_j are holomorphic and Δ is non-vanishing. Then for all $x \in X'$ the polynomial

$$P_x(T) := T^n + c_1(x)T^{n-1} + \cdots + c_n(x) \in \mathbb{C}[T]$$

has n distinct roots. Consider the topological space $|\mathcal{O}| \to X$ associated with the sheaf $\mathcal{O}$ on X. Let

$$Y' := \{\varphi \in \mathcal{O}_x : x \in X',\ P(\varphi) = 0\} \subseteq |\mathcal{O}|$$

and let $p' : Y' \to X'$ be the canonical projection. By Corollary 7.7, for each $x \in X'$, there are a neighborhood $U \subseteq X'$ and holomorphic functions $f_1,\ldots,f_n \in \mathcal{O}(U)$ such that

$$P(T) = \prod_{j=1}^n (T - f_j) \quad \text{on } U.$$

Then $(p')^{-1}(U) = \bigcup_{j=1}^n (U,f_j)$, the (U,f_j) are disjoint, and $p'|_{(U,f_j)} : (U,f_j) \to U$ is a homeomorphism. Hence $p' : Y' \to X'$ is a covering map. The connected components of Y' are Riemann surfaces which are covering spaces over X' (by restricting p').

Define $f : Y' \to \mathbb{C}$ by $f(\varphi) := \varphi(p'(\varphi))$. Then f is holomorphic and

$$f(y)^n + c_1(p'(y))f(y)^{n-1} + \cdots + c_n(p'(y)) = 0 \quad \text{for all } y \in Y'.$$

By Theorem 7.4, the covering $p' : Y' \to X'$ can be continued to a proper holomorphic covering $p : Y \to X$, where we identify Y' with $p^{-1}(X')$. Theorem 7.1 (see Remark 7.2) implies that f can be extended to a meromorphic function $F \in \mathscr{M}(Y)$ such that

$$(p^*P)(F) = F^n + (p^*c_1)F^{n-1} + \cdots + (p^*c_n) = 0.$$

Let us prove that Y is connected and hence a Riemann surface. Let $Y_1, \dots, Y_k$ be the connected components of Y (there are finitely many since p is proper). Then $p|_{Y_j} : Y_j \to X$ is a proper holomorphic n_j-sheeted covering, where $n_1+\cdots+n_k = n$. The elementary symmetric functions of $F|_{Y_i}$ yield polynomials $P_j(T) \in \mathscr{M}(X)[T]$ of degree n_j such that $P(T) = P_1(T)\cdots P_k(T)$, contradicting irreducibility of $P(T)$.

Next we show uniqueness. Let (Z, q, G) be another algebraic function defined by $P(T)$. Let $B \subseteq Z$ be the set of all poles of G and all branch points of q and set $A' := q(B)$. Define $X'' := X' \setminus A'$, $Y'' := p^{-1}(X'')$, and $Z'' := q^{-1}(X'')$.

We define a fiber-preserving map $\sigma'' : Z'' \to Y''$ as follows. Let $z \in Z''$, $q(z) = x$, and $\varphi := q_* G_z$ (cf. (6.1)). Then $P(\varphi) = 0$. Hence φ is a point of Y' over x (see the construction of Y') and consequently $\varphi \in Y''$. Set $\sigma''(z) := \varphi$.

It is clear from the definition that σ'' is continuous and thus holomorphic, since σ'' is fiber-preserving. The map σ'' is proper, since $p|_{Y''} : Y'' \to X''$ is continuous and $q|_{Z''} : Z'' \to X''$ is proper. So σ'' is surjective. Since $p|_{Y''} : Y'' \to X''$ and $q|_{Z''} : Z'' \to X''$ have the same number of sheets, σ'' is biholomorphic. The definition of σ'' implies that $G|_{Z''} = (\sigma'')^*(F|_{Y''})$. By Theorem 7.5, σ'' extends to a fiber-preserving biholomorphic map $\sigma : Z \to Y$ such that $G = \sigma^* F$. Actually, σ is uniquely determined by the property $G = \sigma^* F$. Otherwise, there would exist a non-trivial deck transformation $\alpha : Y \to Y$ such that $\alpha^* F = F$. This is impossible, since F assumes distinct values on the points of the fiber $p^{-1}(x)$ for all $x \in X'$. □

Example 7.9. Let $a_1, \dots, a_n$ be distinct points in $\mathbb{C}$. Consider

$$f(z) := (z - a_1)(z - a_2)\cdots(z - a_n)$$

as a meromorphic function on $\widehat{\mathbb{C}}$. The polynomial $P(T) = T^2 - f$ is irreducible over $\mathscr{M}(\widehat{\mathbb{C}})$. It defines an algebraic function which is denoted by $\sqrt{f(z)}$. Let us describe the Riemann surface $p : Y \to \widehat{\mathbb{C}}$ associated by the above construction.

Let $A := \{a_1, \dots, a_n, \infty\}$, $X' := \widehat{\mathbb{C}} \setminus A$, and $Y' := p^{-1}(X')$. Then $p' : Y' \to X'$ is an unbranched 2-sheeted covering. Every germ $\varphi \in \mathcal{O}_x$, where $x \in X'$, satisfying $\varphi^2 = f$ can be analytically continued along every curve in X'. Let us analyze the covering over neighborhoods of points in A.

Let $r_j > 0$ be such that a_j is the only point of A lying in the disk $D_{r_j}(a_j)$. On $D_{r_j}(a_j)$ we have

$$f(z) = (z - a_j)h^2(z)$$

for a holomorphic function h, because $f/(z - a_j) = \prod_{k \ne j}(z - a_k)$ is non-vanishing in the disk $D_{r_j}(a_j)$ (which is simply-connected). Let $\zeta = a_j + \rho e^{i\theta}$ with $\rho \in (0, r_j)$ and $\theta \in \mathbb{R}$. By Lemma 7.6, there exists $\varphi_\zeta \in \mathcal{O}_\zeta$ such that $\varphi_\zeta^2 = f$ and

$$\varphi_\zeta(\zeta) = \sqrt{\rho}\, e^{\frac{i\theta}{2}} h(\zeta).$$

If we continue this germ along the closed curve $\zeta = a_j + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$, then we get $-\varphi_\zeta$. It follows that $p : p^{-1}(D^*_{r_j}(a_j)) \to D^*_{r_j}(a_j)$ is a connected 2-sheeted covering (as in Theorem 4.8(2)). So the Riemann surface Y has precisely one point over a_j.

Now let us look at the point ∞. Consider $U := \{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$, where $r > |a_j|$ for all j. Then U is a neighborhood of ∞ containing no other point of A. On U we may write $f = z^n g$ for $g \in \mathcal{O}(U)$ having no zeros in U. We must distinguish two cases: If n is odd, then there is a meromorphic h on U such that $f(z) = zh(z)^2$. If n is even, then there is a meromorphic h on U such that $f(z) = h(z)^2$. In the case that n is odd, we find in the same way as above that, above $U \setminus \{\infty\}$, p is a connected 2-sheeted covering and Y has precisely one point over ∞. However, if n is even, p over $U \setminus \{\infty\}$ splits into two 1-sheeted coverings and Y has two points over ∞.

7.6. The field extension associated with an algebraic function. Let X, Y be Riemann surfaces and let $p : Y \to X$ be a branched holomorphic covering map. Then $\operatorname{Deck}(Y \to X)$ (which is defined in analogy to unbranched coverings) has a representation in the automorphism group of $\mathscr{M}(Y)$ defined by $\sigma f := f \circ \sigma^{-1}$, where $\sigma \in \operatorname{Deck}(Y \to X)$ and $f \in \mathscr{M}(Y)$. The map

$$\operatorname{Deck}(Y \to X) \to \operatorname{Aut}(\mathscr{M}(Y)), \quad \sigma \mapsto (f \mapsto \sigma f),$$

is a group homomorphism. Each such automorphism $f \mapsto \sigma f$ leaves invariant the functions in the subfield $p^*\mathscr{M}(X) \subseteq \mathscr{M}(Y)$. Thus it is an element of the Galois group $\operatorname{Aut}(\mathscr{M}(Y)/p^*\mathcal{M}(X))$.

Theorem 7.10. *Let X be a Riemann surface and $P(T) \in \mathscr{M}(X)[T]$ an irreducible monic polynomial of degree n. Let (Y, p, F) be the algebraic function defined by $P(T)$. Consider $\mathscr{M}(X)$ as a subfield of $\mathscr{M}(Y)$ by means of the monomorphism $p^* : \mathscr{M}(X) \to \mathscr{M}(Y)$. Then $\mathscr{M}(Y) : \mathscr{M}(X)$ is a field extension of degree n and $\mathscr{M}(Y) \cong \mathscr{M}(X)[T]/(P(T))$. The map*

$$\operatorname{Deck}(Y \to X) \to \operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(X)) \tag{7.3}$$

defined by the remarks above is a group isomorphism. The covering $Y \to X$ is normal if and only if the field extension $\mathscr{M}(Y) : \mathscr{M}(X)$ is Galois.

Proof. That $\mathscr{M}(Y) : \mathscr{M}(X)$ is a field extension of degree n follows from Theorem 7.3. Since $P(F) = 0$, there is a homomorphism of fields $\mathscr{M}(X)[T]/(P(T)) \to \mathscr{M}(Y)$, which is an isomorphism since both fields are of degree n over $\mathscr{M}(X)$.

The map (7.3) is injective, since $\sigma F \neq F$ for each deck transformation σ which is not the identity. Let us show surjectivity. Let $\alpha \in \operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(X))$. Then $(Y, p, \alpha F)$ is an algebraic function defined by $P(T)$. By the uniqueness statement of Theorem 7.8, there is a deck transformation $\tau \in \operatorname{Deck}(Y \to X)$ such that $\alpha F = \tau^* F$. Then

$$\tau^{-1} F = F \circ \tau = \tau^* F = \alpha F.$$

Since $\mathscr{M}(Y)$ is generated by F over $\mathscr{M}(X)$, the automorphism $f \mapsto \tau^{-1} f$ of $\mathscr{M}(Y)$ coincides with α.

For the last statement observe that $Y \to X$ is normal precisely if $\operatorname{Deck}(Y \to X)$ contains n elements, and $\mathscr{M}(Y) : \mathscr{M}(X)$ is Galois precisely if $\operatorname{Aut}(\mathscr{M}(Y)/\mathscr{M}(X))$ has n elements. □

7.7. Puiseux expansions. As a corollary of Theorem 7.8 we get Puiseux's theorem.

We denote by $\mathbb{C}\{\{z\}\}$ the field of Laurent series with finite principal part $f(z) = \sum_{n=k}^{\infty} c_n z^n$, $k \in \mathbb{Z}$, $c_n \in \mathbb{C}$, which converge is some punctured disk $D_r^*(0)$, $r = r(f) > 0$. The field $\mathbb{C}\{\{z\}\}$ is isomorphic to the stalk $\mathscr{M}_0$ of the sheaf $\mathscr{M}$ of meromorphic function in $\mathbb{C}$ and it is the quotient field of $\mathbb{C}\{z\}$.

Consider an irreducible polynomial

$$P(z, w) = w^n + a_1(z) w^{n-1} + \cdots + a_n(z) \in \mathbb{C}\{\{z\}\}[w].$$

We may consider $P(z, w)$ as an irreducible element of $\mathscr{M}(D_r(0))[w]$ for some $r > 0$. Suppose that $r > 0$ is also chosen such that for each $a \in D_r^*(0)$ the polynomial $P(a, w) \in \mathbb{C}[w]$ has no multiple roots. Let (Y, p, f) be the algebraic function defined by $P(z, w) \in \mathscr{M}(D_r(0))[w]$, see Theorem 7.8. Then $p : Y \to D_r(0)$ is an n-sheeted branched covering which is ramified only over 0. By Corollary 4.9, there is an isomorphism $\alpha : D_{\sqrt[n]{r}}(0) \to Y$ such that $p(\alpha(\zeta)) = \zeta^n$ for all $\zeta \in D_{\sqrt[n]{r}}(0)$. Since $(p^*P)(f) = 0$ we have

$$P(\zeta^n, \varphi(\zeta)) = 0, \qquad \text{where } \varphi = f \circ \alpha.$$

We have proved:

Theorem 7.11 (Puiseux's theorem)**.** *Consider an irreducible polynomial*

$$P(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) \in \mathbb{C}\{\{z\}\}[w].$$

There exists a Laurent series $\varphi(\zeta) = \sum_{m=k}^{\infty} c_m \zeta^m \in \mathbb{C}\{\{z\}\}$ *such that* $P(\zeta^n, \varphi(\zeta)) = 0$ *in* $\mathbb{C}\{\{z\}\}$.

The content of this theorem is often paraphrased by saying that the equation $P(z, w) = 0$ can be solved by a **Puiseux series**

$$w = \varphi(\sqrt[n]{z}) = \sum_{m=k}^{\infty} c_m z^{m/n}.$$

If the coefficients are holomorphic, i.e., $a_j \in \mathbb{C}\{z\}$, then also φ is holomorphic, $\varphi \in \mathbb{C}\{z\}$, since in that case the function f is holomorphic on Y.

CHAPTER 3

Calculus of differential forms

8. Differential forms

8.1. Let $U \subseteq \mathbb{C}$ be a domain. Let $\mathscr{E}(U)$ denote the $\mathbb{C}$-algebra of all C^∞-functions $f : U \to \mathbb{C}$. We identify $\mathbb{C}$ with $\mathbb{R}^2$ by writing $z = x + iy$. Besides the partial derivatives ∂_x, ∂_y we consider the differential operators

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

By the Cauchy–Riemann equations, the space $\mathcal{O}(U)$ of holomorphic functions on U is the kernel of $\partial_{\overline{z}} : \mathscr{E}(U) \to \mathscr{E}(U)$.

8.2. Cotangent space. Let X be a Riemann surface. Let $Y \subseteq X$ be open. Let $\mathscr{E}(Y)$ be the set of all functions $f : Y \to \mathbb{C}$ such that for every chart $z : U \to V \subseteq \mathbb{C}$ on X with $U \subseteq Y$ there exists $\tilde{f} \in \mathscr{E}(V)$ such that $f|_U = \tilde{f} \circ z$ (clearly, $\tilde{f}$ is uniquely determined by f). This defines the sheaf $\mathscr{E}$ of smooth functions on X.

Let (U, z), $z = x+iy$, be a coordinate neighborhood on X. Then the differential operators $\partial_x, \partial_y, \partial_z, \partial_{\overline{z}} : \mathscr{E}(U) \to \mathscr{E}(U)$ are defined in the obvious way.

Let $a \in X$. The stalk $\mathscr{E}_a$ consists of all germs of smooth functions at a. Let $\mathfrak{m}_a := \{\varphi \in \mathscr{E}_a : \varphi(a) = 0\}$. A germ $\varphi \in \mathfrak{m}_a$ is said to vanish of second order if it can be represented by a function f which in some coordinate neighborhood (U, z) of a satisfies $\partial_x f(a) = \partial_y f(a) = 0$. Let $\mathfrak{m}_a^2$ denote the vector subspace of $\mathfrak{m}_a$ of all germs at a that vanish to second order. The quotient vector space

$$T_a^* X := \mathfrak{m}_a / \mathfrak{m}_a^2$$

is the **cotangent space** ot X at a.

Let U be an open neighborhood of a in X, and $f \in \mathscr{E}(U)$. The **differential** $d_a f$ defined by

$$d_a f := (f - f(a)) \bmod \mathfrak{m}_a^2$$

is an element of $T_a^* X$.

Lemma 8.1. *Let X be a Riemann surface, $a \in X$, and (U, z) a coordinate neighborhood of a. Then $(d_a x, d_a y)$ as well as $(d_a z, d_a \overline{z})$ form a basis of $T_a^* X$. If $f \in \mathscr{E}(U)$ then*

$$d_a f = \partial_x f(a)\, d_a x + \partial_y f(a)\, d_a y = \partial_z f(a)\, d_a z + \partial_{\overline{z}} f(a)\, d_a \overline{z}.$$

Proof. Let $\varphi \in \mathfrak{m}_a$. Taylor series expansion about a yields

$$\varphi = c_1(x - x(a)) + c_2(y - y(a)) + \psi, \quad \text{where } c_1, c_2 \in \mathbb{C},\ \psi \in \mathfrak{m}_a^2.$$

This shows that $(d_a x, d_a y)$ spans $T_a^* X$. Now $d_a x, d_a y$ are linearly independent, since $c_1\, d_a x + c_2\, d_a y = 0$ implies $c_1(x - x(a)) + c_2(y - y(a)) \in \mathfrak{m}_a^2$, and applying ∂_x, ∂_y we find $c_1 = c_2 = 0$.

For $f \in \mathscr{E}(U)$, we have

$$f - f(a) = \partial_x f(a)(x - x(a)) + \partial_y f(a)(y - y(a)) + g$$

where g vanishes to second order at a. This gives the first formula.

The proof for $(d_a z, d_a \overline{z})$ is analogous. □

Let $(U, z), (V, w)$ be two coordinate neighborhoods of $a \in X$. Then

$$\partial_z w(a) \in \mathbb{C}^*, \quad \partial_{\overline{z}} \overline{w}(a) = \overline{\partial_z w(a)}, \quad \partial_{\overline{z}} w(a) = \partial_z \overline{w}(a) = 0,$$

and thus $d_a w = \partial_z w(a)\, d_a z$ and $d_a \overline{w} = \overline{\partial_z w(a)}\, d_a \overline{z}$. It follows that the one dimensional subspaces of $T_a^* X$ spanned by $d_a z$ and $d_a \overline{z}$ are independent of the coordinate chart (U, z) at a. We define

$$T_a^{*(1,0)} X := \mathbb{C}\, d_a z, \quad T_a^{*(0,1)} X := \mathbb{C}\, d_a \overline{z}.$$

Then $T_a^* X = T_a^{*(1,0)} X \oplus T_a^{*(0,1)} X$. The elements of $T_a^{*(1,0)} X$ (resp. $T_a^{*(0,1)} X$) are called **cotangent vectors** of type $(1,0)$ (resp. $(0,1)$).

8.3. 1-forms. Let Y be an open subset of a Riemann surface X. A **differential form of degree one** or simply a **1-form** on Y is a map $\omega : Y \to \bigsqcup_{a \in Y} T_a^* X$ such that $\omega(a) \in T_a^* X$ for all $a \in Y$. If $\omega(a) \in T_a^{*(1,0)} X$ (resp. $\omega(a) \in T_a^{*(0,1)} X$) for all $a \in Y$, then ω is said to be of type $(1,0)$ (resp. $(0,1)$).

If (U, z) is a coordinate chart, then every 1-form on U can be written in the form

$$\omega = f\, dx + g\, dy = \varphi\, dz + \psi\, d\overline{z},$$

for functions $f, g, \varphi, \psi : U \to \mathbb{C}$. A 1-form ω on Y is called **smooth** if, for each chart (U, z), we have

$$\omega = \varphi\, dz + \psi\, d\overline{z} \quad \text{on } U \cap Y, \quad \text{where } \varphi, \psi \in \mathscr{E}(U \cap Y).$$

If, for each chart (U, z),

$$\omega = \varphi\, dz \quad \text{on } U \cap Y, \quad \text{where } \varphi \in \mathcal{O}(U \cap Y),$$

then ω is called **holomorphic**.

We denote by $\mathscr{E}^1(Y)$ the vector space of smooth 1-forms on Y, by $\mathscr{E}^{(1,0)}(Y)$ (resp. $\mathscr{E}^{(0,1)}(Y)$) the subspace of smooth 1-forms of type $(1,0)$ (resp. $(0,1)$), and by $\mathcal{O}^1(Y)$ the space of holomorphic 1-forms. Together with the natural restriction maps $\mathscr{E}^1$, $\mathscr{E}^{1,0}$, $\mathscr{E}^{0,1}$, and $\mathcal{O}^1$ are sheaves of vector spaces on X.

8.4. 2-forms. Let V be a complex vector space. Then $\bigwedge^2 V$ is the $\mathbb{C}$-vector space whose elements are finite sums of elements of the form $v_1 \wedge v_2$ for $v_1, v_2 \in V$, where

$$(v_1 + v_2) \wedge v_3 = v_1 \wedge v_3 + v_2 \wedge v_3$$
$$(\lambda v_1) \wedge v_2 = \lambda(v_1 \wedge v_2)$$
$$v_1 \wedge v_2 = -v_2 \wedge v_1,$$

for all $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbb{C}$. If $e_1, \ldots, e_n$ is a basis of V, then $e_i \wedge e_j$, for $i < j$, forms a basis of $\bigwedge^2 V$.

Let (U, z) be a coordinate neighborhood of a. Then $d_a x \wedge d_a y$ is a basis of the vector space $\bigwedge^2 T_a^* X$, another basis is $d_a z \wedge d_a \overline{z} = -2i\, d_a x \wedge d_a y$. So $\bigwedge^2 T_a^* X$ has dimension one.

Let Y be an open subset of a Riemann surface X. A **2-form** on Y is a map $\omega : Y \to \bigsqcup_{a \in Y} \bigwedge^2 T_a^* X$ such that $\omega(a) \in \bigwedge^2 T_a^* X$ for all $a \in Y$. The 2-form ω is called **smooth** if, for each chart (U, z), we have

$$\omega = \varphi\, dz \wedge d\overline{z} \quad \text{on } U \cap Y, \quad \text{where } \varphi \in \mathscr{E}(U \cap Y).$$

The vector space of smooth 2-forms on Y is denoted by $\mathscr{E}^2(Y)$.

Note that, if $\omega_1, \omega_2 \in \mathscr{E}^1(Y)$, then $(\omega_1 \wedge \omega_1)(a) := \omega_1(a) \wedge \omega_1(a)$ defines a 2-form $\omega_1 \wedge \omega_2 \in \mathscr{E}^2(Y)$.

Functions are by definition 0-forms, i.e., $\mathscr{E}^0(Y) := \mathscr{E}(Y)$ and $\mathcal{O}^0(Y) := \mathcal{O}(Y)$. There are no non-trivial k-forms, for $k \geq 3$, on a Riemann surface, since $v \wedge v = 0$.

8.5. Exterior differentiation. Let Y be an open subset of a Riemann surface X. We have a map $d : \mathscr{E}^0(Y) \to \mathscr{E}^1(Y)$ defined by $df(a) = d_a f$. Moreover, $\partial : \mathscr{E}^0(Y) \to \mathscr{E}^{1,0}(Y)$ and $\overline{\partial} : \mathscr{E}^0(Y) \to \mathscr{E}^{0,1}(Y)$ are defined by

$$df = \partial f + \overline{\partial} f.$$

Locally smooth 1-forms can be written as finite sums

$$\omega = \sum f_k \, dg_k, \qquad \text{for smooth functions } f_k, g_k.$$

We define

$$d\omega := \sum df_k \wedge dg_k, \quad \partial\omega := \sum \partial f_k \wedge dg_k, \quad \overline{\partial}\omega := \sum \overline{\partial} f_k \wedge dg_k.$$

One checks easily that this definition is independent of the representation $\omega = \sum f_k \, dg_k$. Thus we obtain operators $d, \partial, \overline{\partial} : \mathscr{E}^1(Y) \to \mathscr{E}^2(Y)$ satisfying

$$d = \partial + \overline{\partial}. \tag{8.1}$$

If $f \in \mathscr{E}(Y)$ then $d^2 f = d(1 \cdot df) = d1 \wedge df = 0$, similarly for $\partial, \overline{\partial}$, i.e.,

$$d^2 = \partial^2 = \overline{\partial}^2 = 0. \tag{8.2}$$

Now (8.1) and (8.2) imply

$$\partial\overline{\partial} = -\overline{\partial}\partial. \tag{8.3}$$

In a local chart,

$$\partial\overline{\partial} f = \partial_z \partial_{\overline{z}} f \, dz \wedge d\overline{z} = \frac{1}{2i}\big(\partial_x^2 f + \partial_y^2 f\big) \, dx \wedge dy.$$

Furthermore we have the product rules

$$\begin{aligned} d(f\omega) &= df \wedge \omega + f d\omega, \\ \partial(f\omega) &= \partial f \wedge \omega + f \partial\omega, \\ \overline{\partial}(f\omega) &= \overline{\partial} f \wedge \omega + f \overline{\partial}\omega. \end{aligned}$$

A 1-form $\omega \in \mathscr{E}^1(Y)$ is called **closed** if $d\omega = 0$ and **exact** if there is $f \in \mathscr{E}(Y)$ such that $df = \omega$. Clearly, every exact form is closed, by (8.2).

Proposition 8.2. *Let Y be an open subset of a Riemann surface. Then:*

1. *Every holomorphic 1-form $\omega \in \mathcal{O}^1(Y)$ is closed.*
2. *Every closed $\omega \in \mathscr{E}^{1,0}(Y)$ is holomorphic.*

Proof. Let $\omega \in \mathscr{E}^{1,0}(Y)$. In a local chart, $\omega = f \, dz$ and so $d\omega = df \wedge dz = -\partial_{\overline{z}} f \, dz \wedge d\overline{z}$. Thus, $d\omega = 0$ if and only if $\partial_{\overline{z}} f = 0$. This implies the statement. □

8.6. Pullbacks. Let $\varphi : X \to Y$ be a holomorphic map between Riemann surfaces. For every open $U \subseteq Y$ the map φ induces a homomorphism

$$\varphi^* : \mathscr{E}(U) \to \mathscr{E}(\varphi^{-1}(U)), \quad \varphi^*(f) := f \circ \varphi.$$

Moreover, we have the **pullback** of differential forms

$$\varphi^* : \mathscr{E}^1(U) \to \mathscr{E}^1(\varphi^{-1}(U)) \quad \text{and} \quad \varphi^* : \mathscr{E}^2(U) \to \mathscr{E}^2(\varphi^{-1}(U))$$

defined by the local formulas

$$\begin{aligned} \varphi^*\Big(\sum f_j \, dg_j\Big) &:= \sum \varphi^*(f_j) \, d(\varphi^*(g_j)), \\ \varphi^*\Big(\sum f_j \, dg_j \wedge dh_j\Big) &:= \sum \varphi^*(f_j) \, d(\varphi^*(g_j)) \wedge d(\varphi^*(h_j)). \end{aligned}$$

One checks easily that these definitions are independent of the local representations. It is clear that the operators $d, \partial, \overline{\partial}$ commute with pullbacks,

$$\varphi^* d = d\varphi^*, \quad \varphi^*\partial = \partial\varphi^*, \quad \varphi^*\overline{\partial} = \overline{\partial}\varphi^*.$$

8.7. Meromorphic differential forms. Let Y be an open subset of a Riemann surface X. Let $a \in Y$ and let ω be a holomorphic 1-form on $Y \setminus \{a\}$. Let (U, z) be a coordinate neighborhood of a with $U \subseteq Y$ and $z(a) = 0$. On $U \setminus \{a\}$ we have $\omega = f\,dz$ for $f \in \mathcal{O}(U \setminus \{a\})$. Let

$$f = \sum_{n=-\infty}^{\infty} c_n z^n$$

be the Laurent series of f at a with respect to the coordinate z. Then:

- a is a **removable singularity** of ω if $c_n = 0$ for all $n < 0$.
- ω has a **pole of order** k if there exists $k < 0$ such that $c_n = 0$ for all $n < k$ and $c_k \neq 0$.
- ω has an **essential singularity** at a if $c_n \neq 0$ for infinitely many $n < 0$.

The **residue** of ω at a is by definition $\operatorname{res}_a(\omega) := c_{-1}$.

Lemma 8.3. *The residue is independent of the chart* (U, z) *and hence well-defined.*

Proof. Let (U, z) be any chart at a with $z(a) = 0$. Let $g \in \mathcal{O}(U \setminus \{a\})$ have the Laurent series $g = \sum_{n=-\infty}^{\infty} c_n z^n$. Then $dg = (\sum_{n=-\infty}^{\infty} n c_n z^{n-1})\,dz$ and hence $\operatorname{res}_a(dg) = 0$. In particular, the residue of dg at a is independent of (U, z).

Let $g \in \mathcal{O}(U)$ have a zero of first order at a. Then $g = zh$ for some holomorphic h which does not vanish at a. Thus $dg = h\,dz + z\,dh$ and

$$\frac{dg}{g} = \frac{dz}{z} + \frac{dh}{h}.$$

It follows that $\operatorname{res}_a(dg/g) = \operatorname{res}_a(dz/z) = 1$. In particular, the residue of dg/g at a is independent of (U, z).

Now let $\omega = f\,dz$ with $f = \sum_{n=-\infty}^{\infty} c_n z^n$. Setting

$$g := \sum_{n=-\infty}^{-2} \frac{c_n}{n+1} z^{n+1} + \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}$$

we have $\omega = dg + c_{-1}dz/z$. The first part of the proof implies $\operatorname{res}_a(\omega) = c_{-1}$ which is independent of the chart. ☐

A 1-form ω on an open subset Y of a Riemann surface X is said to be a **meromorphic differential form** on Y if there is an open subset $Y' \subseteq Y$ such that $Y \setminus Y'$ consists only of isolated points, $\omega \in \mathcal{O}^1(Y')$, and ω has a pole at each point in $Y \setminus Y'$. Let $\mathscr{M}^1(Y)$ be the set of meromorphic 1-forms on Y. Then $\mathscr{M}^1$ forms a sheaf of vector spaces on X. Meromorphic 1-forms are also called **abelian differentials**.

8.8. Integration of 1-forms. Let X be a Riemann surface and $\omega \in \mathscr{E}^1(X)$. Let $\gamma : [0, 1] \to X$ be a piecewise C^1-curve. That means γ is continuous and there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and charts (U_k, z_k), $k = 1, \ldots, n$, such that $\gamma([t_{k-1}, t_k]) \subseteq U_k$, and $z_k \circ \gamma : [t_{k-1}, t_k] \to \mathbb{C}$ are C^1. On U_k we have $\omega = f_k\,dx_k + g_k\,dy_k$ for smooth f_k, g_k. We define

$$\int_\gamma \omega := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \big(f_k(\gamma(t))(x_k \circ \gamma)'(t) + g_k(\gamma(t))(y_k \circ \gamma)'(t)\big)\,dt.$$

This definition is independent of the choice of partition and charts.

Theorem 8.4. *Let X be a Riemann surface, $\gamma : [0,1] \to X$ a piecewise C^1-curve, and $f \in \mathscr{E}(X)$. Then*

$$\int_\gamma df = f(\gamma(1)) - f(\gamma(0)).$$

Proof. Exercise. □

8.9. Primitives. Let X be a Riemann surface and $\omega \in \mathscr{E}^1(X)$. A function $f \in \mathscr{E}(X)$ is called a **primitive** of ω if $df = \omega$. Two primitives differ by an additive constant.

A 1-form with a primitive must be closed. Locally, a closed 1-form has a primitive and hence is exact. In fact, if $\omega = f\,dx + g\,dy$ is a closed 1-form on a disk centered at 0 in $\mathbb{C}$ then

$$F(x,y) := \int_0^1 \big(f(tx,ty)x + g(tx,ty)y\big)\,dt$$

provides a primitive of ω. Globally, a primitive of a closed 1-form exists in general only as a multivalued function:

Theorem 8.5. *Let X be a Riemann surface and let $\omega \in \mathscr{E}^1(X)$ be closed. Then there is a covering map $p : \hat{X} \to X$ with $\hat{X}$ connected and a primitive $F \in \mathscr{E}(\hat{X})$ of $p^*\omega$.*

Proof. For open $U \subseteq X$ define $\mathscr{F}(U) := \{f \in \mathscr{E}(U) : df = \omega\}$. This defines a sheaf $\mathscr{F}$ on X which satisfies the identity theorem (indeed, on a connected open set V any two elements of $\mathscr{F}(V)$ differ by a constant). By Theorem 5.4, $|\mathscr{F}|$ is Hausdorff. We claim that the natural projection $p : |\mathscr{F}| \to X$ is a covering map. Each $a \in X$ has a connected open neighborhood U and a primitive $f \in \mathscr{F}(U)$ of ω. So

$$p^{-1}(U) = \bigcup_{c\in\mathbb{C}} (U, f + c),$$

where the sets $(U, f+c)$ are pairwise disjoint and $p|_{(U,f+c)} : (U, f+c) \to U$ are homeomorphisms.

If $\hat{X}$ is a connected component of $|\mathscr{F}|$, then $p : \hat{X} \to X$ is also a covering map. The function $F : \hat{X} \to \mathbb{C}$ defined by $F(\varphi) := \varphi(p(\varphi))$ is a primitive of $p^*\omega$. □

Corollary 8.6. *Let X be a Riemann surface, $\pi : \widetilde{X} \to X$ its universal covering, and let $\omega \in \mathscr{E}^1(X)$ be closed. Then there is a primitive $F \in \mathscr{E}(\widetilde{X})$ of $\pi^*\omega$.*

Proof. Let $p : \hat{X} \to X$ be the covering provided by Theorem 8.5. There is a holomorphic fiber-preserving map $\tau : \widetilde{X} \to \hat{X}$, by the universal property of π. Then τ^*F is a primitive of $\pi^*\omega = \tau^*(p^*\omega)$. □

Corollary 8.7. *On a simply connected Riemann surface every closed 1-form is exact.*

Theorem 8.8. *Let X be a Riemann surface, $\pi : \tilde{X} \to X$ its universal covering, and let $\omega \in \mathscr{E}^1(X)$ be closed. Let $F \in \mathscr{E}(\tilde{X})$ be a primitive of $\pi^*\omega$. If $\gamma : [0,1] \to X$ is a piecewise C^1-curve and $\tilde{\gamma}$ is a lifting of γ, then*

$$\int_\gamma \omega = F(\tilde{\gamma}(1)) - F(\tilde{\gamma}(0)). \tag{8.4}$$

Proof. This follows from Theorem 8.4 and

$$\int_{\tilde{\gamma}} \pi^*\omega = \int_{\pi\circ\tilde{\gamma}} \omega = \int_{\gamma} \omega. \qquad \square$$

We may take (8.4) as a definition of the integral of a closed 1-form along an *arbitrary continuous* curve $\gamma : [0,1] \to X$. Clearly, this definition is independent of the choice of the primitive F of $\pi^*\omega$. It is also independent of the lifting of the curve γ. Indeed, let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be two liftings. Since the covering π is normal, by Theorem 4.4, there is a deck transformation σ such that $\tilde{\gamma}_1 = \sigma \circ \tilde{\gamma}_2$. Since $\pi \circ \sigma = \pi$, we have $\sigma^*\pi^*\omega = \pi^*\omega$. It follows that $\sigma^* F$ is a primitive of $\pi^*\omega$ and hence $\sigma^* F - F = \text{const}$. Then

$$F(\tilde{\gamma}_1(1)) - F(\tilde{\gamma}_1(0)) = \sigma^* F(\tilde{\gamma}_2(1)) - \sigma^* F(\tilde{\gamma}_2(0)) = F(\tilde{\gamma}_2(1)) - F(\tilde{\gamma}_2(0)).$$

Proposition 8.9. *Let X be a Riemann surface and let $\omega \in \mathscr{E}^1(X)$ be closed. Let γ_1, γ_2 be homotopic curves in X. Then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

Proof. Suppose that γ_1, γ_2 have initial point a and end point b. Let $\pi : \widetilde{X} \to X$ be the universal covering and $\tilde{\gamma}_1, \tilde{\gamma}_2$ liftings of γ_1, γ_2 with the same initial point. Then they also have the same endpoint, by Corollary 3.10. Thus Theorem 8.8 implies the assertion. $\square$

8.10. Integration of 2-forms. Let $U \subseteq \mathbb{C}$ be a domain. A 2-form $\omega \in \mathscr{E}^2(U)$ is of the form

$$\omega = f\, dx \wedge dy = -\frac{1}{2i} f\, dz \wedge d\overline{z} \qquad \text{for } f \in \mathscr{E}(U).$$

If f is integrable on U, we define

$$\int_U \omega := \iint_U f(x,y)\, dx\, dy.$$

Let $\varphi : V \to U$ be a biholomorphism. The Cauchy–Riemann equations imply that the Jacobian determinant of φ equals $|\varphi'|^2$. On the other hand

$$\varphi^*\omega = -\frac{1}{2i}(f\circ\varphi)\, d\varphi \wedge d\overline{\varphi} = -\frac{1}{2i}(f\circ\varphi)|\varphi'|^2\, dz \wedge d\overline{z} = (f\circ\varphi)|\varphi'|^2\, dx \wedge dy.$$

The transformation formula implies

$$\int_{\varphi(V)} \omega = \int_V \varphi^*\omega. \tag{8.5}$$

Let $\omega \in \mathscr{E}^2(X)$ be a 2-form on a Riemann surface with compact support. There are finitely many charts $\varphi_k : U_k \to V_k$, $k = 1, \ldots, n$, such that the support of ω is contained in the union of the U_k. Let (f_k) be a smooth partition of unity subordinate to (U_k). We define

$$\int_X \omega := \sum_{k=1}^n \int_X f_k\, \omega := \sum_{k=1}^n \int_{V_k} (\varphi_k^{-1})^*(f_k\, \omega).$$

Using the transformation formula, one checks that this definition is independent of the choice of charts and of the partition of unity.

Let us recall Stokes' theorem in the plane.

Theorem 8.10 (Stokes' theorem)**.** *Let A be a compact subset of the plane $\mathbb{C}$ with smooth boundary ∂A which is oriented so that the outward pointing normal of A and the tangent vector to ∂A form a positively oriented basis. For every smooth 1-form defined in a neighborhood of A,*

$$\int_A d\omega = \int_{\partial A} \omega.$$

Theorem 8.11. *Let X be a Riemann surface and let $\omega \in \mathscr{E}^1(X)$ have compact support. Then $\int_X dw = 0$.*

Proof. In the plane this follows immediately from Theorem 8.10. In general one may use a partition of unity to decompose ω into 1-forms each of which with has compact support in one chart. □

8.11. The residue theorem.

Theorem 8.12 (residue theorem)**.** *Let X be a compact Riemann surface, $a_1, \ldots, a_n$ distinct points in X, and $X' := X \setminus \{a_1, \ldots, a_n\}$. For every holomorphic 1-form $\omega \in \mathcal{O}^1(X')$,*

$$\sum_{k=1}^n \operatorname{res}_{a_k}(\omega) = 0.$$

Proof. We may choose coordinate neighborhoods (U_k, z_k) of the a_k which are pairwise disjoint and such that $z_k(a_k) = 0$ and each $z_k(U_k)$ is a disk in $\mathbb{C}$. Choose functions f_k with $\operatorname{supp} f_k \subseteq U_k$ such that $f_k|_{U_k'} = 1$ for neighborhoods $U_k' \subseteq U_k$ of a_k. Then $g := 1 - (f_1 + \cdots + f_n)$ vanishes on each U_k'. Thus $g\omega$ defines a smooth 1-form on X. By Theorem 8.11,

$$\int_X d(g\omega) = 0$$

We have $d\omega = 0$ on X', since ω is holomorphic, by Proposition 8.2. On $U_k' \cap X'$ we have $d(f_k\omega) = d\omega = 0$. Thus, we may consider $d(f_k\omega)$ as a smooth 2-form on X with compact support in $U_k \setminus \{a_k\}$. Hence $d(g\omega) = -\sum_k d(f_k\omega)$ and so

$$\sum_{k=1}^n \int_X d(f_k\omega) = 0.$$

To finish the proof we show

$$\int_X d(f_k\omega) = \int_{U_k} d(f_k\omega) = -2\pi i \operatorname{res}_{a_k}(\omega).$$

Using the coordinate z_k we may assume that $U_k = \mathbb{D}$. There are $0 < r < R < 1$ such that $\operatorname{supp} f_k \subseteq D_R(0)$ and $f|_{\overline{D_r(0)}} = 1$. Then, by Theorem 8.10,

$$\begin{aligned} \int_{U_k} d(f_k\omega) &= \int_{r \le |z_k| \le R} d(f_k\omega) = \int_{|z_k|=R} f_k\omega - \int_{|z_k|=r} f_k\omega \\ &= -\int_{|z_k|=r} \omega = -2\pi i \operatorname{res}_{a_k}(\omega), \end{aligned} \tag{8.6}$$

by the residue theorem in the plane. □

Corollary 8.13. *Any non-constant meromorphic function f on a compact Riemann surface X has as many zeros as poles (counting multiplicities).*

Proof. Apply the residue theorem to $\omega = df/f$. □

Compare this with Corollary 3.20.

9. Periods and summands of automorphy

9.1. Periods. Let X be a Riemann surface and let $\omega \in \mathscr{E}^1(X)$ be closed. By Proposition 8.9, we may define the **periods** of ω,

$$a_\sigma := \int_\sigma \omega, \quad \sigma \in \pi_1(X).$$

For all $\sigma, \tau \in \pi_1(X)$,

$$a_{\sigma\cdot\tau} = a_\sigma + a_\tau,$$

whence we obtain a homomorphism $a : \pi_1(X) \to \mathbb{C}$ of the fundamental group of X to the additive group $\mathbb{C}$, the so-called **period homomorphism** associated with ω.

Example 9.1. For $X = \mathbb{C}^*$ we have $\pi_1(X) \cong \mathbb{Z}$, see Example 4.5. The curve $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = e^{2\pi i t}$ represents a generator of $\pi_1(X)$. For $\omega = dz/z$, the period homomorphism is $\mathbb{Z} \to \mathbb{C}$, $n \mapsto 2\pi i n$, since $\int_\gamma \omega = 2\pi i$.

9.2. Summands of automorphy. Let X be a Riemann surface and $\widetilde{X}$ its universal covering. We saw in Theorem 4.4, that $\mathrm{Deck}(\widetilde{X} \to X) \cong \pi_1(X)$. There is a natural action of $\mathrm{Deck}(\widetilde{X} \to X)$ on the set of functions $f : \widetilde{X} \to \mathbb{C}$, namely, for $\sigma \in \mathrm{Deck}(\widetilde{X} \to X)$, we set $\sigma f := f \circ \sigma^{-1}$.

A function $f : \widetilde{X} \to \mathbb{C}$ is called **additively automorphic** with constant summands of automorphy if there exists constants $a_\sigma \in \mathbb{C}$ such that

$$f - \sigma f = a_\sigma \quad \text{for all } \sigma \in \mathrm{Deck}(\widetilde{X} \to X).$$

The constants a_σ are called the **summands of automorphy** of f. We have

$$a_{\sigma\tau} = f - \sigma\tau f = (f - \sigma f) + (\sigma f - \sigma\tau f) = a_\sigma + a_\tau,$$

whence $a : \mathrm{Deck}(\widetilde{X} \to X) \to \mathbb{C}$ is a group homomorphism.

9.3. Connection between periods and summands of automorphy.

Theorem 9.2. *Let X be a Riemann surface and $p : \widetilde{X} \to X$ its universal covering.*

(1) *Let $\omega \in \mathscr{E}^1(X)$ be a closed 1-form and let $F \in \mathscr{E}(\widetilde{X})$ be a primitive of $p^*\omega$. Then F is additively automorphic with constant summands of automorphy. Its summands of automorphy are precisely the periods of ω (relative to the isomorphism $\mathrm{Deck}(\widetilde{X} \to X) \cong \pi_1(X)$).*

(2) *Let $F \in \mathscr{E}(\widetilde{X})$ be an additively automorphic function with constant summands of automorphy. Then there exists a unique closed 1-form $\omega \in \mathscr{E}^1(X)$ such that $dF = p^*\omega$.*

Proof. (1) Let $\sigma \in \mathrm{Deck}(\widetilde{X} \to X)$. Then σF is a primitive of $p^*\omega$, indeed, $d(\sigma F) = d((\sigma^{-1})^* F) = (\sigma^{-1})^*(dF) = (\sigma^{-1})^* p^*\omega = p^*\omega$ because $p \circ \sigma^{-1} = p$. Since two primitives differ by a constant, $F - \sigma F =: a_\sigma$ is a constant. Let $x_0 \in X$ and $z_0 \in p^{-1}(x_0)$. By the proof of Theorem 4.4, the element $\hat\sigma \in \pi_1(X, x_0)$ associated with σ can be represented as follows. Choose a curve $\gamma : [0,1] \to \widetilde{X}$ with $\gamma(0) = \sigma^{-1}(z_0)$ and $\gamma(1) = z_0$. Then $p \circ \gamma$ is a closed curve in X and $\hat\sigma$ is its homotopy class. By Theorem 8.8,

$$\int_{\hat\sigma} \omega = F(\gamma(1)) - F(\gamma(0)) = F(z_0) - F(\sigma^{-1}(z_0)) = a_\sigma.$$

(2) Suppose that F has summands of automorphy a_σ. Then, for each $\sigma \in \mathrm{Deck}(\widetilde{X} \to X)$,

$$\sigma^*(dF) = d\sigma^* F = d(F + a_\sigma) = dF,$$

that is, dF is invariant under deck transformations. Since $p : \widetilde{X} \to X$ is locally biholomorphic, there exists $\omega \in \mathscr{E}^1(X)$ with $dF = p^*\omega$. Evidently ω is closed and uniquely determined. □

Example 9.3 (complex tori, IV). Let $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ be a lattice and $X = \mathbb{C}/\Lambda$. The quotient map $\pi : \mathbb{C} \to X$ is the universal covering and $\mathrm{Deck}(\mathbb{C} \to X)$ is the group of translations by vectors $\lambda \in \Lambda$. Let $z : \mathbb{C} \to \mathbb{C}$ denote the identity map. It is additively automorphic with constant summands of automorphy $\lambda \in \Lambda$. The 1-form dz is invariant under deck transformations. Thus there is a holomorphic 1-form $\omega \in \mathcal{O}^1(X)$ such that $\pi^*\omega = dz$ and whose periods are the elements of Λ.

Corollary 9.4. *Let X be a Riemann surface. A closed 1-form $\omega \in \mathscr{E}^1(X)$ has a primitive $f \in \mathscr{E}(X)$ if and only if all periods of ω vanish.*

Proof. Suppose that all periods of ω vanish. There is a primitive $F \in \mathscr{E}(\widetilde{X})$ of $p^*\omega$ on the universal covering $p : \widetilde{X} \to X$, by Corollary 8.6. By Theorem 9.2, F has summands of automorphy 0, which means that F is invariant under deck transformations. It follows that there exists $f \in \mathscr{E}(X)$ with $F = p^*f$. Then f is a primitive of ω, in fact, $p^*\omega = dF = d(p^*f) = p^*df$ implies $\omega = df$.

The other direction is evident, by Theorem 8.4. □

Corollary 9.5. *Let X be a compact Riemann surface. Two holomorphic 1-forms which define the same period homomorphism coincide.*

Proof. Let ω_1 and ω_2 be holomorphic 1-forms which define the same period homomorphism. Then all periods of $\omega := \omega_1 - \omega_2$ vanish. So there exists $f \in \mathcal{O}(X)$ such that $\omega = df$. Since X is compact, f must be constant, and so $\omega = df = 0$. □

CHAPTER 4

Čech cohomology

In the language of sheaves local statements are easy to formulate. Often a problem can be solved locally easily, by finding sections of some sheaf. But one really is interested in a global solution, i.e., a global section. By the sheaf axioms, global sections exist if the local sections coincide on the overlap domains. The cohomology construction turns this condition into an algebraic condition.

The cohomology theory we are using is the so-called **Čech cohomology**. It assigns groups to sheaves $\mathscr{F}$ on topological spaces X which are usually denoted by $\check{H}^q(X, \mathscr{F})$; we will simply write $H^q(X, \mathscr{F})$. For us it will be enough to have the cohomology groups of zeroth and first order, so we will not bother to define them for $q \geq 2$.

10. Cohomology groups

10.1. The first cohomology group $H^1(\mathfrak{U}, \mathscr{F})$. Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. For $k = 0, 1, 2, \dots$ the **kth cochain group** of $\mathscr{F}$ with respect to $\mathfrak{U}$ is defined by

$$C^k(\mathfrak{U}, \mathscr{F}) := \prod_{(i_0, \dots, i_k) \in I^{k+1}} \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_k}).$$

Its elements are called **k-cochains**. Component-wise addition gives the group structure. We shall use the **coboundary operators**

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}) \quad \text{and} \quad \delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$$

defined as follows:

- For $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathscr{F})$ let $\delta((f_i)_{i \in I}) = (g_{ij})_{i,j \in I} \in C^1(\mathfrak{U}, \mathscr{F})$, where $g_{ij} := f_j - f_i \in \mathscr{F}(U_i \cap U_j)$.
- For $(f_{ij})_{i,j \in I} \in C^1(\mathfrak{U}, \mathscr{F})$ let $\delta((f_{ij})_{i,j \in I}) = (g_{ijk})_{i,j,k \in I} \in C^2(\mathfrak{U}, \mathscr{F})$, where $g_{ijk} := f_{jk} - f_{ik} + f_{ij} \in \mathscr{F}(U_i \cap U_j \cap U_k)$.

The coboundary operators are group homomorphisms. Set

$$Z^1(\mathfrak{U}, \mathscr{F}) := \ker \big(\delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})\big),$$
$$B^1(\mathfrak{U}, \mathscr{F}) := \operatorname{im} \big(\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})\big).$$

The elements of $Z^1(\mathfrak{U}, \mathscr{F})$ are called **1-cocycles**. A 1-cochain $(f_{ij})_{i,j \in I}$ is a 1-cocycle if and only if it satisfies the cocycle relation

$$f_{ik} = f_{ij} + f_{jk} \quad \text{on } U_i \cap U_j \cap U_k \text{ for all } i, j, k \in I.$$

The cocycle relation implies

$$f_{ii} = 0 \quad \text{and} \quad f_{ij} = -f_{ji}.$$

The elements of $B^1(\mathfrak{U}, \mathscr{F})$ are called **1-coboundaries**. Every 1-coboundary is a 1-cocycle. The quotient group

$$H^1(\mathfrak{U}, \mathscr{F}) := Z^1(\mathfrak{U}, \mathscr{F}) / B^1(\mathfrak{U}, \mathscr{F})$$

is the **first cohomology group** of $\mathfrak{U}$ with coefficients in $\mathscr{F}$. Its elements are called **cohomology classes**. Two cocycles which belong to the same cohomology class are called **cohomologous**.

The cohomology group $H^1(\mathfrak{U}, \mathscr{F})$ depends on the cover $\mathfrak{U}$. Our next goal is to define cohomology groups that depend only on X and $\mathscr{F}$.

10.2. The first cohomology group $H^1(X, \mathscr{F})$. An open cover $\mathfrak{V} = (V_j)_{j\in J}$ of X is called **finer** than $\mathfrak{U} = (U_i)_{i\in I}$ if every V_j is contained is some U_i. We write $\mathfrak{V} < \mathfrak{U}$ in this case. So there is a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for each $j \in J$. Consider the map $t^{\mathfrak{U}}_{\mathfrak{V}} : Z^1(\mathfrak{U}, \mathscr{F}) \to Z^1(\mathfrak{V}, \mathscr{F})$ defined by $t^{\mathfrak{U}}_{\mathfrak{V}}((f_{ij})) = (g_{k\ell})$, where $g_{k\ell} := f_{\tau(k)\tau(\ell)}|_{V_k\cap V_\ell}$ for all $k, \ell \in J$. Since $t^{\mathfrak{U}}_{\mathfrak{V}}$ takes coboundaries to coboundaries, it descends to a homomorphism $t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$.

Lemma 10.1. *$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is independent of τ.*

Proof. Suppose that $\tau' : J \to I$ is another map such that $V_j \subseteq U_{\tau'(j)}$ for all $j \in J$. Let $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and set $g_{k\ell} := f_{\tau(k)\tau(\ell)}|_{V_k\cap V_\ell}$ and $g'_{k\ell} := f_{\tau'(k)\tau'(\ell)}|_{V_k\cap V_\ell}$. We have to show that (g_{kl}) and (g'_{kl}) are cohomologous. Since $V_k \subseteq U_{\tau(k)} \cap U_{\tau'(k)}$ we may put $h_k := f_{\tau(k)\tau'(k)}|_{V_k} \in \mathscr{F}(V_k)$. Then on $V_k \cap V_\ell$,

$$\begin{aligned} g_{k\ell} - g'_{k\ell} = f_{\tau(k)\tau(\ell)} - f_{\tau'(k)\tau'(\ell)} &= f_{\tau(k)\tau(\ell)} + f_{\tau(\ell)\tau'(k)} - f_{\tau(\ell)\tau'(k)} - f_{\tau'(k)\tau'(\ell)} \\ &= f_{\tau(k)\tau'(k)} - f_{\tau(\ell)\tau'(\ell)} = h_k - h_\ell. \end{aligned}$$

The proof is complete. □

Lemma 10.2. *$t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is injective.*

Proof. Let $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ be a cocycle such that its image in $Z^1(\mathfrak{V}, \mathscr{F})$ is a coboundary. We must show that (f_{ij}) itself is a coboundary. Suppose $f_{\tau(k)\tau(\ell)} = g_k - g_\ell$ on $V_k \cap V_\ell$ for $g_k \in \mathscr{F}(V_k)$. Then, on $U_i \cap V_k \cap V_\ell$,

$$g_k - g_\ell = f_{\tau(k)\tau(\ell)} = f_{\tau(k)i} + f_{i\tau(\ell)} = -f_{i\tau(k)} + f_{i\tau(\ell)}$$

and thus $f_{i\tau(k)} + g_k = f_{i\tau(\ell)} + g_\ell$. Since $\mathscr{F}$ is a sheaf, there exists $h_i \in \mathscr{F}(U_i)$ such that $h_i = f_{i\tau(k)} + g_k$ on $U_i \cap V_k$. Then, on $U_i \cap U_j \cap V_k$,

$$f_{ij} = f_{i\tau(k)} + f_{\tau(k)j} = f_{i\tau(k)} + g_k - f_{j\tau(k)} - g_k = h_i - h_j.$$

Since k is arbitrary and since $\mathscr{F}$ is a sheaf, this holds on $U_i \cap U_j$, and hence $(f_{ij}) \in B^1(\mathfrak{U}, \mathscr{F})$. □

If we have three covers $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$ of X, then $t^{\mathfrak{V}}_{\mathfrak{W}} \circ t^{\mathfrak{U}}_{\mathfrak{V}} = t^{\mathfrak{U}}_{\mathfrak{W}}$. So we define the **first cohomology group** of X with coefficients in $\mathscr{F}$ by

$$H^1(X, \mathscr{F}) := \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}),$$

where $\mathfrak{U}$ runs through all open covers of X. That means $H^1(X, \mathscr{F})$ is the quotient with respect to the following equivalence relation on $\bigsqcup_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F})$: two cohomology classes $\xi_1 \in H^1(\mathfrak{U}_1, \mathscr{F})$ and $\xi_2 \in H^1(\mathfrak{U}_2, \mathscr{F})$ are equivalent if there is an open cover $\mathfrak{V} < \mathfrak{U}_i$, $i = 1, 2$, such that $t^{\mathfrak{U}_1}_{\mathfrak{V}}(\xi_1) = t^{\mathfrak{U}_2}_{\mathfrak{V}}(\xi_2)$.

Addition in $H^1(X, \mathscr{F})$ is defined as follows. Let $x_1, x_2 \in H^1(X, \mathscr{F})$ be represented by $\xi_i \in H^1(\mathfrak{U}_i, \mathscr{F})$, respectively. Let $\mathfrak{V} < \mathfrak{U}_i$, $i = 1, 2$. Then $x_1 + x_2 \in H^1(X, \mathscr{F})$ is defined to be the equivalence class of $t^{\mathfrak{U}_1}_{\mathfrak{V}}(\xi_1) + t^{\mathfrak{U}_2}_{\mathfrak{V}}(\xi_2) \in H^1(\mathfrak{V}, \mathscr{F})$. This definition is independent of the various choices made and makes $H^1(X, \mathscr{F})$ into an abelian group. If $\mathscr{F}$ is a sheaf of vector spaces, then $H^1(\mathfrak{U}, \mathscr{F})$ and $H^1(X, \mathscr{F})$ are vector spaces in a natural way.

Lemma 10.3. *We have $H^1(X,\mathscr{F}) = 0$ if and only if $H^1(\mathfrak{U},\mathscr{F}) = 0$ for every open cover $\mathfrak{U}$ of X.*

Proof. Lemma 10.2 implies that for each open cover $\mathfrak{U}$ of X the canonical map $H^1(\mathfrak{U},\mathscr{F}) \to H^1(X,\mathscr{F})$ is injective. □

10.3. Some cohomology groups on Riemann surfaces.

Theorem 10.4. *Let X be a Riemann surface and let $\mathscr{E}$ be the sheaf of smooth functions on X. Then $H^1(X,\mathscr{E}) = 0$.*

Proof. Let $\mathfrak{U} = (U_i)_{i\in I}$ be any open cover of X. We prove that $H^1(\mathfrak{U},\mathscr{F}) = 0$. There is a partition of unity $(\varphi_i)_{i\in I}$ subordinate to $\mathfrak{U}$, since X is second countable. Let $(f_{ij}) \in Z^1(\mathfrak{U},\mathscr{E})$. The function

$$\begin{cases} \varphi_j(x) f_{ij}(x) & \text{if } x \in U_i \cap U_j, \\ 0 & \text{if } x \in U_i \setminus (U_i \cap U_j), \end{cases}$$

is in $\mathscr{E}(U_i)$; we denote this function simply by $\varphi_j f_{ij}$. Define

$$f_i := \sum_{j\in I} \varphi_j f_{ij} \quad \text{on } U_i.$$

This sum contains only finitely many nonzero terms near any point of U_i, since the family $(\operatorname{supp}\varphi_i)$ is locally finite. Thus $f_i \in \mathscr{E}(U_i)$. Then,

$$f_k - f_\ell = \sum_{j\in I} \varphi_j (f_{kj} - f_{\ell j}) = \Big(\sum_{j\in I} \varphi_j\Big) f_{k\ell} = f_{k\ell}$$

on $U_k \cap U_\ell$. Thus (f_{ij}) is a coboundary. □

Remark 10.5. The same proof shows that on a Riemann surface the first cohomology groups with coefficients in the sheaves $\mathscr{E}^1$, $\mathscr{E}^{1,0}$, $\mathscr{E}^{0,1}$, and $\mathscr{E}^2$ also vanish.

Theorem 10.6. *Let X be a simply connected Riemann surface. Then $H^1(X,\mathbb{C}) = 0$ and $H^1(X,\mathbb{Z}) = 0$. Here $\mathbb{C}$ (resp. $\mathbb{Z}$) denotes the sheaf of locally constant function with values in $\mathbb{C}$ (resp. $\mathbb{Z}$).*

Proof. Let $\mathfrak{U} = (U_i)_{i\in I}$ be any open cover of X and let $c_{ij} \in Z^1(\mathfrak{U},\mathbb{C})$. By Theorem 10.4, there is a cochain $(f_i) \in C^0(\mathfrak{U},\mathscr{E})$ such that $c_{ij} = f_i - f_j$ on $U_i \cap U_j$. It follows that $df_i = df_j$ on $U_i \cap U_j$. So there is a 1-form $\omega \in \mathscr{E}^1(X)$ such that $\omega|_{U_i} = df_i$. In particular, ω is closed. Since X is simply connected, there exists $f \in \mathscr{E}(X)$ such that $\omega = df$, by Corollary 8.7. Set $c_i := f_i - f$ on U_i. Then $dc_i = 0$ on U_i, thus c_i is locally constant on U_i, i.e., $(c_i) \in C^0(\mathfrak{U},\mathbb{C})$. On $U_i \cap U_j$ we have $c_{ij} = f_i - f_j = c_i - c_j$, that is $(c_{ij}) \in B^1(\mathfrak{U},\mathbb{C})$.

Let $(a_{jk}) \in Z^1(\mathfrak{U},\mathbb{Z})$. By the first part, there is a cochain $(c_j) \in C^0(\mathfrak{U},\mathbb{C})$ such that $a_{jk} = c_j - c_k$ on $U_j \cap U_k$. This implies $\exp(2\pi i c_j) = \exp(2\pi i c_k)$ on $U_j \cap U_k$, since $\exp(2\pi i a_{jk}) = 1$. Since X is connected, there is $b \in \mathbb{C}^*$ such that $b = \exp(2\pi i c_j)$ for all $j \in I$. Choose $c \in \mathbb{C}$ such that $\exp(2\pi i c) = b$. Set $a_j := c_j - c$. Then $\exp(2\pi i a_j) = \exp(2\pi i c_j)\exp(-2\pi i c) = 1$ and hence a_j is an integer. On $U_j \cap U_k$ we have $a_{jk} = c_j - c_k = a_j - a_k$, that is $(a_{jk}) \in B^1(\mathfrak{U},\mathbb{Z})$. □

10.4. Leray cover. Sometimes it is possible to compute $H^1(X,\mathscr{F})$ using only a single cover of X.

Theorem 10.7 (Leray's theorem)**.** *Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X. Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of X such that $H^1(U_i,\mathscr{F}) = 0$ for all $i \in I$. Then $H^1(X,\mathscr{F}) \cong H^1(\mathfrak{U},\mathscr{F})$.*

The cover in the theorem is called a **Leray cover** (of first order) for $\mathscr{F}$.

Proof. It suffices to show that for each open cover $\mathfrak{V} = (V_\alpha)_{\alpha\in A}$ with $\mathfrak{V} < \mathfrak{U}$ the map $t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{V}, \mathscr{F})$ is an isomorphism. By Lemma 10.2, the map is injective. Let $\tau : A \to I$ be such that $V_\alpha \subseteq U_{\tau(\alpha)}$ for all $\alpha \in A$. Let $(f_{\alpha\beta}) \in Z^1(\mathfrak{V}, \mathscr{F})$. We claim that there exists $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ such that $(F_{\tau(\alpha)\tau(\beta)}) - (f_{\alpha\beta})$ is cohomologous to zero with respect to $\mathfrak{V}$. This will show surjectivity. By assumption, $H^1(U_i \cap \mathfrak{V}, \mathscr{F}) = 0$, i.e., there exist $g_{i\alpha} \in \mathscr{F}(U_i \cap V_\alpha)$ such that

$$f_{\alpha\beta} = g_{i\alpha} - g_{i\beta} \quad \text{on } U_i \cap V_\alpha \cap V_\beta.$$

So on $U_i \cap U_j \cap V_\alpha \cap V_\beta$,

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}.$$

Since $\mathscr{F}$ is a sheaf, there exist $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$F_{ij} = g_{j\alpha} - g_{i\alpha} \quad \text{on } U_i \cap U_j \cap V_\alpha.$$

Clearly, $(F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. On $V_\alpha \cap V_\beta$,

$$F_{\tau(\alpha)\tau(\beta)} - f_{\alpha\beta} = (g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}) - (g_{\tau(\beta)\alpha} - g_{\tau(\beta)\beta}) = g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha}$$

and the claim follows, by setting $h_\alpha = g_{\tau(\alpha)\alpha}|_{V_\alpha} \in \mathscr{F}(V_\alpha)$. □

Example 10.8. We claim that $H^1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}$. Let $\mathfrak{U} = (U_1, U_2)$ be the open cover of $\mathbb{C}^*$ by $U_1 := \mathbb{C}^* \setminus (-\infty, 0)$ and $U_2 := \mathbb{C}^* \setminus (0, \infty)$. Theorem 10.6 and Theorem 10.7 imply $H^1(\mathbb{C}^*, \mathbb{Z}) = H^1(\mathfrak{U}, \mathbb{Z})$. Note that $Z^1(\mathfrak{U}, \mathbb{Z}) \cong \mathbb{Z}(U_1 \cap U_2)$, since every cocycle (a_{ij}) is completely determined by a_{12} (in fact $a_{11} = a_{22} = 0$ and $a_{21} = -a_{12}$). The intersection $U_1 \cap U_2$ has two connected components, whence $\mathbb{Z}(U_1 \cap U_2) \cong \mathbb{Z} \times \mathbb{Z}$. Moreover, $\mathbb{Z}(U_i) \cong \mathbb{Z}$, since U_i is connected, and so $C^0(\mathfrak{U}, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. The coboundary operator $\delta : C^0(\mathfrak{U}, \mathbb{Z}) \to Z^1(\mathfrak{U}, \mathbb{Z})$ takes the form

$$\mathbb{Z} \times \mathbb{Z} \ni (b_1, b_2) \mapsto (b_2 - b_1, b_2 - b_1) \in \mathbb{Z} \times \mathbb{Z}.$$

Thus $B^1(\mathfrak{U}, \mathbb{Z})$ corresponds to the diagonal in $\mathbb{Z} \times \mathbb{Z}$ and hence $H^1(\mathfrak{U}, \mathbb{Z}) \cong \mathbb{Z}$.

10.5. The zeroth cohomology group. Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X and let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of X. Define

$$\begin{aligned} Z^0(\mathfrak{U}, \mathscr{F}) &:= \ker\big(\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})\big), \\ B^0(\mathfrak{U}, \mathscr{F}) &:= 0, \\ H^0(\mathfrak{U}, \mathscr{F}) &:= Z^0(\mathfrak{U}, \mathscr{F}). \end{aligned}$$

Thus $(f_i) \in C^0(\mathfrak{U}, \mathscr{F})$ belongs to $Z^0(\mathfrak{U}, \mathscr{F})$ if and only if $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$. Since $\mathscr{F}$ is a sheaf, there is an element $f \in \mathscr{F}(X)$ such that $f|_{U_i} = f_i$ for all $i \in I$. So there is a natural isomorphism

$$H^0(\mathfrak{U}, \mathscr{F}) \cong \mathscr{F}(X),$$

and thus the groups $H^0(\mathfrak{U}, \mathscr{F})$ are independent of the cover $\mathfrak{U}$. Hence one defines

$$H^0(X, \mathscr{F}) := \mathscr{F}(X).$$

10.6. Solution of the inhomogeneous Cauchy–Riemann equation in the plane. We recall the following consequence of Runge's approximation theorem which is a special case of the **Dolbeault lemma** in several complex variables. For a proof see e.g. [**14**, Theorem 12.2]. A generalization on non-compact Riemann surfaces will be obtained in Corollary 25.11.

Theorem 10.9. *Let $X \subseteq \mathbb{C}$ be a domain and let $f \in \mathscr{E}(X)$. Then there exists $u \in \mathscr{E}(X)$ such that*

$$\partial_{\overline{z}} u = f. \tag{10.1}$$

In the case that f has compact support in $\mathbb{C}$, a solution $u \in \mathscr{E}(\mathbb{C})$ is given by

$$u(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

Theorem 10.10. *Let $X \subseteq \mathbb{C}$ be a domain. Then $H^1(X, \mathcal{O}) = 0$.*

Proof. Let $\mathfrak{U} = (U_i)_{i \in I}$ be any open cover of X. Let $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$. By Theorem 10.4, there is $(g_i) \in C^0(\mathfrak{U}, \mathscr{E})$ such that $f_{ij} = g_i - g_j$ on $U_i \cap U_j$. This implies $\partial_{\bar{z}} g_i = \partial_{\bar{z}} g_j$ on $U_i \cap U_j$, and so there is $h \in \mathscr{E}(X)$ with $h|_{U_i} = \partial_{\bar{z}} g_i$ for all $i \in I$. By Theorem 10.9, there exists $g \in \mathscr{E}(X)$ such that $\partial_{\bar{z}} g = h$. Then $f_i := g_i - g$ is holomorphic on U_i, hence $(f_i) \in C^0(\mathfrak{U}, \mathcal{O})$. On $U_i \cap U_j$, $f_i - f_j = g_i - g_j = f_{ij}$. That is $(f_{ij}) \in B^1(\mathfrak{U}, \mathcal{O})$. □

We shall see in Theorem 26.1 that actually $H^1(X, \mathcal{O}) = 0$ for every non-compact Riemann surface X.

Theorem 10.11. *We have $H^1(\widehat{\mathbb{C}}, \mathcal{O}) = 0$.*

Proof. Let $U_1 := \widehat{\mathbb{C}} \setminus \{\infty\}$ and $U_2 := \widehat{\mathbb{C}} \setminus \{0\}$. Then $U_1 = \mathbb{C}$ and U_2 is biholomorphic to $\mathbb{C}$. By Theorem 10.10, $H^1(U_i, \mathcal{O}) = 0$. By Theorem 10.7, $H^1(\widehat{\mathbb{C}}, \mathcal{O}) \cong H^1(\mathfrak{U}, \mathcal{O})$ for $\mathfrak{U} = (U_1, U_2)$. Let $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$. It suffices to find $f_i \in \mathcal{O}(U_i)$ such that $f_{12} = f_1 - f_2$ on $U_1 \cap U_2 = \mathbb{C}^*$. Consider the Laurent expansion $f_{12}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ on $\mathbb{C}^*$. Then $f_1(z) := \sum_{n=0}^{\infty} c_n z^n$ and $f_2(z) = -\sum_{n=-\infty}^{-1} c_n z^n$ are as required. □

11. The exact cohomology sequence

In this section we develop some tools for the computation of cohomology groups.

11.1. Sheaf homomorphisms. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of abelian groups on a topological space X. A **sheaf homomorphism** $\alpha : \mathscr{F} \to \mathscr{G}$ is a family of group homomorphisms $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$, $U \subseteq X$ open, which is compatible with the restriction homomorphisms: for all open $V, U \subseteq X$ with $V \subseteq U$ the diagam

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\alpha_U} & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}(V) & \xrightarrow{\alpha_V} & \mathscr{G}(V) \end{array}$$

commutes. If all α_U are isomorphisms, then α is called a **sheaf isomorphism**. Similarly, for homomorphisms of vector spaces, etc.

Example 11.1. (1) The exterior derivative induces sheaf homomorphisms $d : \mathscr{E}^0 \to \mathscr{E}^1$ and $d : \mathscr{E}^1 \to \mathscr{E}^2$.

(2) Natural inclusions such as $\mathcal{O} \to \mathscr{E}$, $\mathbb{C} \to \mathscr{E}$, etc., are sheaf homomorphisms.

(3) Let X be a Riemann surface. The exponential function defines a sheaf homomorphism $e : \mathcal{O} \to \mathcal{O}^*$ from the sheaf of holomorphic functions to the multiplicative sheaf of holomorphic functions with values in $\mathbb{C}^*$ by $e_U(f) = \exp(2\pi i f)$ for open $U \subseteq X$ and $f \in \mathcal{O}(U)$.

Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a sheaf homomorphism. For open $U \subseteq X$ let

$$\ker(\alpha)(U) := \ker\big(\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)\big).$$

Then $\ker(\alpha)$ (together with the restriction homomorphisms induced from $\mathscr{F}$) is again a sheaf. It is called the **kernel** of α.

Example 11.2. On a Riemann surface

(1) $\mathcal{O} = \ker\big(\overline{\partial} : \mathscr{E}^0 \to \mathscr{E}^{0,1}\big)$, by the Cauchy–Riemann equations.
(2) $\mathcal{O}^1 = \ker\big(d : \mathscr{E}^{1,0} \to \mathscr{E}^2\big)$, by Proposition 8.2.
(3) $\mathbb{Z} = \ker\big(e : \mathcal{O} \to \mathcal{O}^*\big)$, by Example 11.1(3).

In analogy to the kernel of α one may define the **image** $\operatorname{im}(\alpha)$ of α by setting

$$\operatorname{im}(\alpha)(U) := \operatorname{im}\big(\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)\big), \quad \text{for open } U \subseteq X.$$

Note that $\operatorname{im}\alpha$ is a presheaf but in general not necessarily a sheaf.

Example 11.3. Consider the sheaf homomorphism $e : \mathcal{O} \to \mathcal{O}^*$ on $X = \mathbb{C}^*$, cf. Example 11.1(3). Let $U_1 = \mathbb{C}^* \setminus (-\infty, 0)$ and $U_2 = \mathbb{C}^* \setminus (0, \infty)$ and $f_i \in \mathcal{O}^*(U_i)$ with $f_i(z) = z$, for $i = 1, 2$. Since U_i is simply connected, $f_i \in \operatorname{im}\big(e : \mathcal{O}(U_i) \to \mathcal{O}^*(U_i)\big)$. Clearly, $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$. But there is no $f \in \operatorname{im}\big(e : \mathcal{O}(X) \to \mathcal{O}^*(X)\big)$ such that $f|_{U_i} = f_i$, since $z \mapsto z$ has no single valued logarithm on all of $X = \mathbb{C}^*$.

11.2. Exact sequences of sheaf homomorphisms. Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a sheaf homomorphism on a topological space X. For each $x \in X$ we obtain an induced homomorphism of stalks

$$\alpha_x : \mathscr{F}_x \to \mathscr{G}_x.$$

A sequence of sheaf homomorphisms

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is called **exact** if for all $x \in X$ the sequence

$$\mathscr{F}_x \xrightarrow{\alpha_x} \mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$$

is exact, i.e., $\ker \beta_x = \operatorname{im} \alpha_x$. A sequence

$$\mathscr{F}_1 \to \mathscr{F}_2 \to \cdots \to \mathscr{F}_{n-1} \to \mathscr{F}_n$$

is exact if $\mathscr{F}_k \to \mathscr{F}_{k+1} \to \mathscr{F}_{k+2}$ is exact for all $k = 1, \ldots, n-2$. An exact sequence of the form

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is called a **short exact sequence**. A sheaf homomorphism $\alpha : \mathscr{F} \to \mathscr{G}$ is called a **monomorphism** if $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G}$ is exact, and an **epimorphism** if $\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \to 0$ is exact.

Lemma 11.4. *Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a sheaf monomorphism on a topological space X. Then, for each open $U \subseteq X$ the map $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

Proof. Let $f \in \mathscr{F}(U)$ with $\alpha_U(f) = 0$. Thus $\alpha_x(f) = 0$ for all $x \in U$. Since $\alpha_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for all x, every $x \in U$ has a neighborhood $V_x \subseteq U$ such that $f|_{V_x} = 0$. Since $\mathscr{F}$ is a sheaf, we have $f = 0$. □

The analogue for sheaf epimorphisms is not necessarily true, by Example 11.3: for each $x \in \mathbb{C}^*$ the map $e : \mathcal{O}_x \to \mathcal{O}_x^*$ is surjective, since every non-vanishing holomorphic function locally has a logarithm, but $e : \mathcal{O}(\mathbb{C}^*) \to \mathcal{O}^*(\mathbb{C}^*)$ is not surjective.

Lemma 11.5. *Let $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ be an exact sequence of sheaves on a topological space X. Then for each open $U \subseteq X$ the sequence*

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha_U} \mathscr{G}(U) \xrightarrow{\beta_U} \mathscr{H}(U)$$

is exact.

Proof. By Lemma 11.4, it remains to prove exactness at $\mathscr{G}(U)$.

To show $\operatorname{im}(\alpha_U) \subseteq \ker(\beta_U)$ let $f \in \mathscr{F}(U)$ and $g := \alpha_U(f)$. Since $\mathscr{F}_x \overset{\alpha_x}{\to} \mathscr{G}_x \overset{\beta_x}{\to} \mathscr{H}_x$ is exact for all $x \in X$, each $x \in U$ has a neighborhood $V_x \subseteq U$ such that $\beta_U(g)|_{V_x} = 0$. Since $\mathscr{H}$ is a sheaf, $\beta_U(g) = 0$ and the claim is proved.

For $\ker(\beta_U) \subseteq \operatorname{im}(\alpha_U)$ let $g \in \mathscr{G}(U)$ with $\beta(g) = 0$. By assumption, $\ker(\beta_x) = \operatorname{im}(\alpha_x)$ for all $x \in X$. So there is an open cover $(V_i)_{i \in I}$ of U and $f_i \in \mathscr{F}(V_i)$ such that $\alpha_U(f_i) = g|_{V_i}$ for all $i \in I$. On any intersection $V_i \cap V_j$ we have $\alpha(f_i - f_j) = 0$. By the exactness at $\mathscr{F}(U)$, $f_i = f_j$ on $V_i \cap V_j$ for all $i, j \in I$. Since $\mathscr{F}$ is a sheaf, there exists $f \in \mathscr{F}(U)$ with $f_i = f|_{V_i}$ for all $i \in I$. Then $\alpha_U(f)|_{V_i} = \alpha_U(f|_{V_i}) = \alpha_U(f_i) = g|_{V_i}$ for all $i \in I$. Since $\mathscr{G}$ is a sheaf, $\alpha_U(f) = g$. □

Example 11.6. Let X be a Riemann surface. We have the following short exact sequences on X.

(1) $0 \to \mathcal{O} \to \mathscr{E} \overset{\bar\partial}{\to} \mathscr{E}^{0,1} \to 0$, by Theorem 10.9.
(2) $0 \to \mathbb{C} \to \mathscr{E} \overset{d}{\to} \mathscr{Z} \to 0$, where $\mathscr{Z} = \ker\big(d : \mathscr{E}^1 \to \mathscr{E}^2\big)$ is the sheaf of closed 1-forms. Here $d : \mathscr{E} \to \mathscr{Z}$ is an epimorphism, since locally every closed form is exact.
(3) $0 \to \mathbb{C} \to \mathcal{O} \overset{d}{\to} \mathcal{O}^1 \to 0$, by Proposition 8.2.
(4) $0 \to \mathcal{O}^1 \to \mathscr{E}^{1,0} \overset{d}{\to} \mathscr{E}^2 \to 0$. Proposition 8.2 implies exactness at $\mathscr{E}^{1,0}$. Let us prove that $d : \mathscr{E}^{1,0} \to \mathscr{E}^2$ is an epimorphism. In a local chart (U, z), we have $d(f\,dz) = \partial_{\bar z} f\, d\bar z \wedge dz$. So the assertion follows from Theorem 10.9.
(5) $0 \to \mathbb{Z} \to \mathcal{O} \overset{e}{\to} \mathcal{O}^* \to 0$.

11.3. Induced homomorphisms of cohomology groups. Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves on a topological space X. It induces homomorphisms

$$\alpha^0 : H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}), \quad \alpha^1 : H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G})$$

as follows. The homomorphism α^0 is just the map $\alpha_X : \mathscr{F}(X) \to \mathscr{G}(X)$.

Let us construct the homomorphism α^1. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. The map

$$\alpha_{\mathfrak{U}} : C^1(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{G}), \quad (f_{ij}) \mapsto (\alpha(f_{ij}))$$

takes cocycles to cocycles and coboundaries to coboundaries. It thus induces a homomorphism

$$\tilde\alpha_{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{U}, \mathscr{G}).$$

The collection of all $\tilde\alpha_{\mathfrak{U}}$, where $\mathfrak{U}$ runs over all open covers of X, induces the homomorphism α^1.

11.4. The connecting homomorphism. Let $0 \to \mathscr{F} \overset{\alpha}{\to} \mathscr{G} \overset{\beta}{\to} \mathscr{H} \to 0$ be a short exact sequence of sheaves on a topological space X. Let us define a **connecting homomorphism**

$$\delta^* : H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F})$$

in the following way. Let $h \in H^0(X, \mathscr{H}) = \mathscr{H}(X)$. Since all the homomorphisms $\beta_x : \mathscr{G}_x \to \mathscr{H}_x$ are surjective, we find an open cover $\mathfrak{U} = (U_i)_{i \in I}$ of X and a cochain $(g_i) \in C^0(\mathfrak{U}, \mathscr{G})$ such that $\beta(g_i) = h|_{U_i}$ for all $i \in I$. Then $\beta(g_i - g_j) = 0$ on $U_i \cap U_j$ and so, by Lemma 11.5, there exists $f_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $\alpha(f_{ij}) = g_j - g_i$. On $U_i \cap U_j \cap U_k$ we have $\alpha(f_{ij} + f_{jk} + f_{ki}) = 0$ and hence $f_{ij} + f_{jk} + f_{ki} = 0$, by Lemma 11.4. That is $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. Now let $\delta^* h \in H^1(X, \mathscr{F})$ be the cohomology class represented by (f_{ij}). This definition is independent of the various choices made.

11.5. The exact cohomology sequence.

Theorem 11.7. *Let $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ be a short exact sequence of sheaves on a topological space X. Then the induced sequence of cohomology groups*

$$0 \longrightarrow H^0(X,\mathscr{F}) \xrightarrow{\alpha^0} H^0(X,\mathscr{G}) \xrightarrow{\beta^0} H^0(X,\mathscr{H}) \xrightarrow{\delta^*}$$

$$\xrightarrow{\delta^*} H^1(X,\mathscr{F}) \xrightarrow{\alpha^1} H^1(X,\mathscr{G}) \xrightarrow{\beta^1} H^1(X,\mathscr{H})$$

is exact.

Proof. Exactness at $H^0(X,\mathscr{F})$ and $H^0(X,\mathscr{G})$ follows from Lemma 11.5.

($\operatorname{im}\beta^0 \subseteq \ker\delta^*$) Let $g \in H^0(X,\mathscr{G}) = \mathscr{G}(X)$ and $h = \beta^0(g)$. In the construction of $\delta^* h$ one may choose $g_i := g|_{U_i}$ which results in $f_{ij} = 0$ and hence $\delta^* h = 0$.

($\ker\delta^* \subseteq \operatorname{im}\beta^0$) Let $h \in \ker\delta^*$. Let $(f_{ij}) \in Z^1(\mathfrak{U},\mathscr{F})$ be the representative of $\delta^* h$, as in subsection 11.4. Since $\delta^* h = 0$, there is a cochain $(f_i) \in C^0(\mathfrak{U},\mathscr{F})$ such that $f_{ij} = f_j - f_i$ on $U_i \cap U_j$. Set $\tilde g_i := g_i - \alpha(f_i)$, where g_i is as in subsection 11.4. Then, on $U_i \cap U_j$,

$$\tilde g_i - \tilde g_j = g_i - g_j - \alpha(f_i - f_j) = g_i - g_j + \alpha(f_{ij}) = 0.$$

It follows that there is $g \in H^0(X,\mathscr{G})$ with $\tilde g_i = g|_{U_i}$ for all $i \in I$. On U_i, we have $\beta(g) = \beta(\tilde g_i) = \beta(g_i) = h$ (by the exactness of $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$), that is $h \in \operatorname{im}\beta^0$.

($\operatorname{im}\delta^* \subseteq \ker\alpha^1$) This is obvious by the condition $\alpha(f_{ij}) = g_j - g_i$ in the definition of δ^* in subsection 11.4.

($\ker\alpha^1 \subseteq \operatorname{im}\delta^*$) Let $\xi \in \ker\alpha^1$ be represented by $(f_{ij}) \in Z^1(\mathfrak{U},\mathscr{F})$. Since $\alpha^1(\xi) = 0$, there exists $(g_i) \in C^0(\mathfrak{U},\mathscr{G})$ such that $\alpha(f_{ij}) = g_j - g_i$ on $U_i \cap U_j$. Then $0 = \beta(\alpha(f_{ij})) = \beta(g_j) - \beta(g_i)$ on $U_i \cap U_j$. Therefore, there exists $h \in \mathscr{H}(X)$ such that $h|_{U_i} = \beta(g_i)$ for all $i \in I$. By subsection 11.4, $\delta^* h = \xi$.

($\operatorname{im}\alpha^1 \subseteq \ker\beta^1$) By Lemma 11.5, the sequence $\mathscr{F}(U_i \cap U_j) \xrightarrow{\alpha} \mathscr{G}(U_i \cap U_j) \xrightarrow{\beta} \mathscr{H}(U_i \cap U_j)$ is exact, which implies the assertion.

($\ker\beta^1 \subseteq \operatorname{im}\alpha^1$) Let $\eta \in \ker\beta^1$ be represented by $(g_{ij}) \in Z^1(\mathfrak{U},\mathscr{G})$. Since $\beta^1(\xi) = 0$, there exists $(h_i) \in C^0(\mathfrak{U},\mathscr{H})$ such that $\beta(g_{ij}) = h_j - h_i$ on $U_i \cap U_j$. For each $x \in X$ choose $\tau(x) \in I$ such that $x \in U_{\tau(x)}$. Since $\beta_x : \mathscr{G}_x \to \mathscr{H}_x$ is surjective, we find an open neighborhood $V_x \subseteq U_{\tau(x)}$ of x and $g_x \in \mathscr{G}(V_x)$ such that $\beta(g_x) = h_{\tau(x)}|_{V_x}$. Let $\mathfrak{V} = (V_x)_{x\in X}$ be the family of all such neighborhoods V_x and set $\tilde g_{xy} := g_{\tau(x)\tau(y)}|_{V_x \cap V_y}$. Then $(\tilde g_{xy}) \in Z^1(\mathfrak{V},\mathscr{G})$ also represents the cohomology class η. Let $\psi_{xy} := \tilde g_{xy} - g_y + g_x$. Then the cocycle (ψ_{xy}) is cohomologous to $(\tilde g_{xy})$ and $\beta(\psi_{xy}) = 0$. So there is $f_{xy} \in \mathscr{F}(V_x \cap V_y)$ such that $\alpha(f_{xy}) = \psi_{xy}$, by Lemma 11.5. This defines a cocycle $(f_{xy}) \in Z^1(\mathfrak{V},\mathscr{F})$, since $\alpha : \mathscr{F}(V_x \cap V_y \cap V_z) \to \mathscr{G}(V_x \cap V_y \cap V_z)$ is injective by Lemma 11.4. Then the cohomology class $\xi \in H^1(X,\mathscr{F})$ of (f_{xy}) satisfies $\alpha^1(\xi) = \eta$. □

Remark 11.8. On a paracompact space X a short exact sequence of sheaves gives a long exact sequence in cohomology which extends indefinitely past the H^1 level.

Corollary 11.9. *Let $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ be a short exact sequence of sheaves on a topological space X. If $H^1(X,\mathscr{G}) = 0$, then $H^1(X,\mathscr{F}) \cong \mathscr{H}(X)/\beta\mathscr{G}(X)$.*

Proof. By Theorem 11.7, we have the exact sequence

$$\mathscr{G}(X) \xrightarrow{\beta} \mathscr{H}(X) \xrightarrow{\delta^*} H^1(X,\mathscr{F}) \to 0.$$

□

Remark 11.10. It is sometimes important to have an explicit description of the isomorphism $\Phi : H^1(X, \mathscr{F}) \cong \mathscr{H}(X)/\beta\mathscr{G}(X)$. By Lemma 11.5, we can assume that $\mathscr{F} = \ker\beta$ and $\alpha : \mathscr{F} \to \mathscr{G}$ is the inclusion. Let $\xi \in H^1(X, \mathscr{F})$ be represented by $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F}) \subseteq Z^1(\mathfrak{U}, \mathscr{G})$. Since $H^1(\mathfrak{U}, \mathscr{G}) = 0$, there exists $(g_i) \in C^0(\mathfrak{U}, \mathscr{G})$ with $f_{ij} = g_j - g_i$ on $U_i \cap U_j$. Then $\beta(g_j) = \beta(g_i)$ on $U_i \cap U_j$, since $\beta(f_{ij}) = 0$. Thus, there exists $h \in \mathscr{H}(X)$ such that $h|_{U_i} = \beta(g_i)$ for all $i \in I$. Then $\Phi(\xi)$ is the coset of h modulo $\beta\mathscr{G}(X)$. That this map Φ is the isomorphism $H^1(X, \mathscr{F}) \cong \mathscr{H}(X)/\beta\mathscr{G}(X)$ induced by the long exact sequence in Theorem 11.7 follows from the part ($\ker\alpha^1 \subseteq \operatorname{im}\delta^*$) in the proof of Theorem 11.7.

11.6. Dolbeault's theorem.

Theorem 11.11 (Dolbeault's theorem)**.** *Let X be a Riemann surface. We have the isomorphisms*

$$H^1(X, \mathcal{O}) \cong \mathscr{E}^{0,1}(X)/\overline{\partial}\mathscr{E}(X), \quad H^1(X, \mathcal{O}^1) \cong \mathscr{E}^2(X)/d\mathscr{E}^{1,0}(X)$$

Proof. By Theorem 10.4 and Remark 10.5, $H^1(X, \mathscr{E}) = H^1(X, \mathscr{E}^{1,0}) = 0$. So the statement follows from Corollary 11.9 applied to Example 11.6(1) and Example 11.6(4). □

Note that Theorem 10.10 is a special case of this result.

11.7. The deRham cohomology groups. Let X be a Riemann surface. Consider the **first deRham cohomology group**

$$\mathrm{Rh}^1(X) := \frac{\ker\big(d : \mathscr{E}^1(X) \to \mathscr{E}^2(X)\big)}{\operatorname{im}\big(d : \mathscr{E}^0(X) \to \mathscr{E}^1(X)\big)}$$

of closed smooth 1-form modulo exact 1-forms. Note that $\mathrm{Rh}^1(X) = 0$ if and only if every closed 1-form $\omega \in \mathscr{E}^1(X)$ has a primitive. In particular, $\mathrm{Rh}^1(X) = 0$ if X is simply connected; cf. Corollary 8.7.

Theorem 11.12 (deRham's theorem)**.** *Let X be a Riemann surface. Then $H^1(X, \mathbb{C}) \cong \mathrm{Rh}^1(X)$.*

Proof. Apply Corollary 11.9 to Example 11.6(2); cf. Theorem 10.4. □

Remark 11.13. More general versions of Dolbeault's and deRham's theorems are valid on manifolds of arbitrary dimension.

11.8. Cohomology of locally constant sheaves. Let G be a group and consider the locally constant sheaf G of locally constant functions $X \to G$. It is evident that all cohomological constructions for this sheaf depend only on the topology of X. A fundamental result in algebraic topology states that the Čech cohomology groups for locally constant sheaves coincide with the simplicial cohomology groups for any triangulable space.

We state the following results without proofs.

Theorem 11.14. *Let X be a contractible Riemann surface and G an abelian group. Then $H^0(X, G) \cong G$ and $H^n(X, G) = 0$ for all $n \geq 1$.*

For compact Riemann surfaces X there is a connection to the genus of X which will be introduced in the next chapter.

Theorem 11.15. *Let X be a compact Riemann surface of genus g and G an abelian group. Then $H^0(X,G) \cong G$, $H^1(X,G) \cong G^{2g}$, $H^2(X,G) \cong G$, and $H^n(X,G) = 0$ for all $n \geq 3$.*

For $H^1(X,G) \cong G^{2g}$ with $G = \mathbb{C}$ see also Corollary 18.9.

CHAPTER 5

Compact Riemann surfaces

12. A finiteness result

In this section we will see that, for any compact Riemann surface X, the cohomology group $H^1(X, \mathcal{O})$ is a finite dimensional complex vector space. Its dimension is called the genus of X.

12.1. A finiteness result. We will use a functional-analytic result due to L. Schwartz. The presentation follows [**6**, Appendix B].

Recall that a linear map $\varphi : E \to F$ between Fréchet spaces is called **compact** if there is a neighborhood U of zero in E such that $\varphi(U)$ is relatively compact in F. Any compact linear map φ is continuous.

Lemma 12.1. *Let E, F be Banach spaces and let $\varphi, \psi : E \to F$ be linear continuous maps, where φ is surjective and ψ is compact. Then $(\varphi + \psi)(E)$ is closed.*

Proof. We will work with the adjoints $\varphi^*, \psi^* : F^* \to E^*$. By Schauder's theorem [**12**, 15.3], a continuous linear map $\psi : E \to F$ is compact if and only if $\psi^* : F^* \to E^*$ is compact. By the closed range theorem [**12**, 9.4], a continuous linear map $\varphi : E \to F$ has closed range if and only if $\varphi^* : F^* \to E^*$ has closed range. Thus φ^* is injective and has closed range and ψ^* is compact. And it suffices to show that $(\varphi^* + \psi^*)(F^*)$ is closed.

The kernel of $\varphi^* + \psi^*$ is finite dimensional. Indeed, let x_n be a bounded sequence with $(\varphi^* + \psi^*)(x_n) = 0$. Since ψ^* is compact, there is a subsequence x_{n_k} such that $\psi^*(x_{n_k}) = \varphi^*(-x_{n_k})$ converges. It follows that x_{n_k} must converge, because φ^* is injective and has closed range. So the kernel of $\varphi^* + \psi^*$ is locally compact and hence finite dimensional.

We may assume that $\varphi^* + \psi^*$ is injective, since the kernel of $\varphi^* + \psi^*$ is finite dimensional and hence complemented.

Let $x_n \in F^*$ be such that $(\varphi^* + \psi^*)(x_n) \to z$. We can assume that x_n is bounded. For, if x_n is not bounded, then setting $y_n := \|x_n\|^{-1} x_n$ implies $(\varphi^* + \psi^*)(y_n) \to 0$. There is a subsequence y_{n_k} such that $\psi^*(y_{n_k})$ converges. Then $\varphi^*(y_{n_k}) = (\varphi^* + \psi^*)(y_{n_k}) - \psi^*(y_{n_k})$ converges. By the open mapping theorem, y_{n_k} converges, say, to y. But then $\|y\| = \lim \|y_{n_k}\| = 1$ and $(\varphi^* + \psi^*)(y) = \lim(\varphi^* + \psi^*)(y_{n_k}) = 0$, contradicting injectivity.

Since x_n is bounded, there exists a subsequence x_{n_k} such that $\psi^*(x_{n_k})$ converges. Then, as before, $\varphi^*(x_{n_k})$ converges and there exists $x \in E^*$ such that $x_{n_k} \to x$. Clearly, $(\varphi^* + \psi^*)(x) = z$. □

Lemma 12.2. *Let $E = \operatorname{proj}_{n \in \mathbb{N}} E_n$ and $F = \operatorname{proj}_{n \in \mathbb{N}} F_n$ be Fréchet spaces and $\varphi : E \to F$ a continuous linear map which induces continuous maps $\varphi_n : E_n \to F_n$ such that $\varphi_n(E_n)$ is closed. Then $\varphi(E)$ is closed in F.*

Proof. Let $\| \cdot \|_n$ denote the norm of E_n as well as F_n; we may assume that $\| \cdot \|_n \leq \| \cdot \|_{n+1}$ by replacing $\| \cdot \|_n$ by $\sup\{\| \cdot \|_1, \dots, \| \cdot \|_n\}$. Since $\varphi_n : E_n \to F_n$ has closed range, there is, by the open mapping theorem, a constant C_n such that for all $y \in \varphi_n(E_n)$ there exists $x \in E_n$ with $y = \varphi_n(x)$ and $\|x\|_n \le C_n \|y\|_n$. We may assume that $C_n \le C_{n+1}$. Let y be in the closure of $\varphi(E)$. Without loss of generality we assume that $\|y\|_1 > 0$. We claim that there is a sequence (x_k) in E such that

$$\|x_n\|_{n-1} < 2^{-n} + 2^{-n+1}, \quad \|\varphi(x_1 + \cdots + x_n) - y\|_n < C_n^{-1} 2^{-n}.$$

Then $x = \sum_{k=1}^{\infty} x_k \in E$ and $y = \varphi(x)$.

Let us construct x_n. Choose x_1 such that $\|\varphi(x_1) - y\|_1 < C_1^{-1} 2^{-1}$. Suppose that $x_1, \ldots, x_{n-1}$ have already be found. Then $\varphi(x_1 + \cdots + x_{n-1}) - y$ lies in the closure of $\varphi(E)$. So there exists $x_n' \in E$ with

$$\|\varphi(x_n') + \varphi(x_1 + \cdots + x_{n-1}) - y\|_n < C_n^{-1} 2^{-n},$$

whence

$$\|\varphi(x_n')\|_{n-1} \le C_n^{-1} 2^{-n} + C_{n-1}^{-1} 2^{-n+1}.$$

Thus there exists $x_n'' \in E_{n-1}$ such that $\varphi_{n-1}(x_n'') = \varphi_{n-1}(x_n')$ and

$$\|x_n''\|_{n-1} \le C_{n-1} \|\varphi(x_n')\|_{n-1} < 2^{-n} + 2^{-n+1}.$$

It suffices to choose x_n sufficiently close to x_n''. □

Theorem 12.3 (L. Schwartz). *Let E, F be Fréchet spaces and let $\varphi, \psi : E \to F$ be linear continuous maps, where φ is surjective and ψ is compact. Then $(\varphi + \psi)(E)$ is closed and $F/(\varphi + \psi)(E)$ is finite dimensional.*

Proof. We have $E = \operatorname{proj}_{n \in \mathbb{N}} E_n$ and $F = \operatorname{proj}_{n \in \mathbb{N}} F_n$, where E_n, F_n are Banach spaces defined by the norms $\|\cdot\|_n$. We may assume that $\|\cdot\|_1$ is chosen so that $\psi(\{x \in E : \|x\|_1 < 1\})$ is relatively compact in F. Since $\varphi : E \to F$ is continuous and open, by the open mapping theorem, for each $\|\cdot\|_n$ on F there is $\|\cdot\|_{m_n}$ on E and a constant K_n such that

$$\|\varphi(x)\|_n \le K_n \|x\|_{m_n}, \quad x \in E,$$

and for all $x \in E$ there exists $y \in E$ with $\varphi(x) = \varphi(y)$ such that

$$\|y\|_{m_n} \le K_n \|\varphi(x)\|_n.$$

Thus φ induces a continuous open, hence surjective map $\varphi_n : E_{m_n} \to F_n$. We may assume that m_n is chosen such that ψ induces a continuous compact map $\psi_n : E_{m_n} \to F_n$. The sequence of seminorms $\|\cdot\|_{m_n}$ defines the topology of E, i.e., $E = \operatorname{proj}_{n \in \mathbb{N}} E_{m_n}$.

By Lemma 12.1 and Lemma 12.2, $(\varphi + \psi)(E)$ is closed in F. Thus $G := F/(\varphi + \psi)(E)$ is a Fréchet space. We claim that G is locally compact and thus finite dimensional. Let $\pi : F \to G$ be the canonical projection. Let $V = \pi(\{y \in F : \|y\|_1 < \epsilon\})$. If $\pi(y_k)$ is a sequence in V, then we can write $y_k = \varphi(x_k)$, where $x_k \in E$ and (x_k) is bounded. Then $(\psi(-x_k))$ has a convergent subsequence, since ψ is compact. We have $y_k = \varphi(x_k) = (\varphi + \psi)(x_k) + \psi(-x_k)$, and hence $\pi(y_k) = \pi(\psi(-x_k))$ has a convergent subsequence. The proof is complete. □

12.2. The genus.

Theorem 12.4. *Let X be a Riemann surface. Let $Y \Subset X$ be an open relatively compact subset. Then $H^1(Y, \mathcal{O})$ is finite dimensional.*

Proof. There is an open set Y' with $Y \Subset Y' \Subset X$ and there exist finitely many open sets $V_i \Subset U_i$, $i = 1, \ldots, r$, in X such that $\bigcup_{i=1}^r V_i = Y$, $\bigcup_{i=1}^r U_i = Y'$, and each U_i is biholomorphic to an open subset of $\mathbb{C}$. By Theorem 10.10, both $\mathfrak{U} = (U_i)$

and $\mathfrak{V} = (V_i)$ are Leray covers of Y' and Y, respectively. By Theorem 10.7, the restriction map $H^1(\mathfrak{U}, \mathcal{O}) \to H^1(\mathfrak{V}, \mathcal{O})$ is an isomorphism. It follows that the map

$$\varphi : C^0(\mathfrak{V}, \mathcal{O}) \oplus Z^1(\mathfrak{U}, \mathcal{O}) \to Z^1(\mathfrak{V}, \mathcal{O})$$
$$((g_i), (f_{ij})) \mapsto \delta((g_i)) + (f_{ij}|_{V_i \cap V_j})$$

is surjective; let us denote the map $Z^1(\mathfrak{U}, \mathcal{O}) \to Z^1(\mathfrak{V}, \mathcal{O})$, $(f_{ij}) \mapsto (f_{ij}|_{V_i \cap V_j})$ by β.

The spaces $Z^1(\mathfrak{U}, \mathcal{O})$, $Z^1(\mathfrak{V}, \mathcal{O})$, and $C^0(\mathfrak{V}, \mathcal{O})$ can be made into Fréchet spaces in the following way. The space $\mathcal{O}(U_i \cap U_j)$ with the topology of uniform convergence on compact sets is a Fréchet space. Hence so is $C^1(\mathfrak{U}, \mathcal{O}) = \prod_{i,j} \mathcal{O}(U_i \cap U_j)$ with the product topology. Then $Z^1(\mathfrak{U}, \mathcal{O})$ is a closed subspace of $C^1(\mathfrak{U}, \mathcal{O})$, thus also a Fréchet space. Similarly, for $Z^1(\mathfrak{V}, \mathcal{O})$, and $C^0(\mathfrak{V}, \mathcal{O})$. With respect to these topologies, the maps $\delta : C^0(\mathfrak{V}, \mathcal{O}) \to Z^1(\mathfrak{V}, \mathcal{O})$ and $\beta : Z^1(\mathfrak{U}, \mathcal{O}) \to Z^1(\mathfrak{V}, \mathcal{O})$ are continuous. By Montel's theorem, β is even compact. Thus, also the map

$$\psi : C^0(\mathfrak{V}, \mathcal{O}) \oplus Z^1(\mathfrak{U}, \mathcal{O}) \to Z^1(\mathfrak{V}, \mathcal{O})$$
$$((g_i), (f_{ij})) \mapsto \beta((f_{ij}))$$

is compact. By Theorem 12.3, the map

$$\varphi - \psi : C^0(\mathfrak{V}, \mathcal{O}) \oplus Z^1(\mathfrak{U}, \mathcal{O}) \to Z^1(\mathfrak{V}, \mathcal{O})$$
$$((g_i), (f_{ij})) \mapsto \delta((g_i))$$

as a difference of a surjective and a compact map between Fréchet spaces has an image with finite codimension. But $\operatorname{im}(\varphi - \psi) = B^1(\mathfrak{V}, \mathcal{O})$. And hence $H^1(Y, \mathcal{O}) \cong H^1(\mathfrak{V}, \mathcal{O})$ (by Theorem 10.7) is finite dimensional. □

The proof of this theorem actually shows the following corollary which we state for later reference.

Corollary 12.5. *Let X be a Riemann surface. Let $Y_1 \Subset Y_2 \subseteq X$ be open subsets. Then the restriction homomorphism $H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$ has a finite dimensional image.*

Proof. We may assume that in the notation of the above proof $\bigcup_{i=1}^r V_i =: Y$, $\bigcup_{i=1}^r U_i =: Y'$ are such that $Y_1 \subseteq Y \Subset Y' \subseteq Y_2$. The proof of the theorem implies that the restriction map $H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})$ has finite dimensional image. This entails the assertion since the restriction map $H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$ factors as $H^1(Y_2, \mathcal{O}) \to H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O}) \to H^1(Y_1, \mathcal{O})$. □

Corollary 12.6. *Let X be a compact Riemann surface. Then $\dim H^1(X, \mathcal{O}) < \infty$.*

Proof. Choose $X = Y$ in the previous theorem. □

Let X be a compact Riemann surface. The number

$$g := \dim H^1(X, \mathcal{O})$$

is called the **genus** of X. The genus of the Riemann sphere is zero, by Theorem 10.11. We shall see below, in Corollary 18.9, that the genus is a purely topological invariant.

12.3. Existence of meromorphic functions.

Theorem 12.7. *Let X be a Riemann surface. Let $Y \Subset X$ be a relatively compact open subset and let $a \in Y$. Then there exists a meromorphic function $f \in \mathscr{M}(Y)$ which has a pole at a and is holomorphic on $Y \setminus \{a\}$.*

Proof. By Corollary 12.5, the image of $H^1(X,\mathcal{O}) \to H^1(Y,\mathcal{O})$ is finite dimensional, say, its dimension is k. Let (U_1, z) be a coordinate chart at a with $z(a) = 0$. Let $U_2 := X \setminus \{a\}$. Then $\mathfrak{U} = (U_1, U_2)$ is an open cover of X. On $U_1 \cap U_2 = U_1 \setminus \{a\}$ the functions z^{-j}, $j = 1, \ldots, k+1$, are holomorphic and (trivially) represent cocycles in $Z^1(\mathfrak{U},\mathcal{O})$. Their restrictions to Y belong to $Z^1(\mathfrak{U}\cap Y,\mathcal{O})$ and are linearly dependent modulo coboundaries. So there exist $c_1, \ldots, c_{k+1} \in \mathbb{C}$, not all zero, and a cochain $(f_1, f_2) \in C^0(\mathfrak{U} \cap Y,\mathcal{O})$ such that

$$\sum_{j=1}^{k+1} c_j z^{-j} = f_2 - f_1 \quad \text{on } U_1 \cap U_2 \cap Y.$$

Hence, there is a meromorphic function $f \in \mathscr{M}(Y)$, which coincides with $f_1 + \sum_{j=1}^{k+1} c_j z^{-j}$ on $U_1 \cap Y$ and equals f_2 on $U_2 \cap Y = Y \setminus \{a\}$. □

Corollary 12.8. *Let X be a compact Riemann surface. Let $a_1, \ldots, a_n$ be distinct points in X and let $c_1, \ldots, c_n$ be complex numbers. Then there exists $f \in \mathscr{M}(X)$ such that $f(a_i) = c_i$ for $i = 1, \ldots, n$.*

Proof. For every pair $i \neq j$ there exists $f_{ij} \in \mathscr{M}(X)$ with a pole at a_i but holomorphic at a_j, by Theorem 12.7. Choose a constant $\lambda_{ij} \in \mathbb{C}^*$ such that $f_{ij}(a_k) \neq f_{ij}(a_j) - \lambda_{ij}$ for every $k = 1, \ldots, n$. Then the meromorphic function

$$g_{ij} := \frac{f_{ij} - f_{ij}(a_j)}{f_{ij} - f_{ij}(a_j) + \lambda_{ij}}$$

is holomorphic at the points a_k, $k = 1, \ldots, n$, and satisfies $g_{ij}(a_i) = 1$ and $g_{ij}(a_j) = 0$. Then

$$f = \sum_{i=1}^{n} c_i h_i \quad \text{with} \quad h_i := \prod_{j \neq i} g_{ij}$$

is as required. □

12.4. Consequences for non-compact Riemann surfaces. We deduce some consequences for non-compact Riemann surfaces which shall be needed in the proof of the Runge approximation theorem 25.10.

Corollary 12.9. *Let Y be a relatively compact open subset of a non-compact Riemann surface X. There exists a holomorphic function $f : Y \to \mathbb{C}$ which is not constant an any connected component of Y.*

Proof. Let Y_1 be an open subset of X with $Y \Subset Y_1 \Subset X$. Fix $a \in Y_1 \setminus Y$ (note that $Y_1 \setminus Y$ is non-empty, since X is non-compact and connected). The statement follows from Theorem 12.7 applied to Y_1 and a. □

Theorem 12.10. *Let X be a non-compact Riemann surface. Let $Y \Subset Y' \subseteq X$ be open subsets. Then* $\operatorname{im}\big(H^1(Y',\mathcal{O}) \to H^1(Y,\mathcal{O})\big) = 0$.

Proof. By Corollary 12.5, the vector space $L := \operatorname{im}\big(H^1(Y',\mathcal{O}) \to H^1(Y,\mathcal{O})\big)$ is finite dimensional. Let $\xi_1, \ldots, \xi_n \in H^1(Y',\mathcal{O})$ be cohomology classes whose restrictions to Y span L. By Corollary 12.9, there is a holomorphic function $f \in \mathcal{O}(Y')$ which is not constant an any connected component of Y'. There exist constants $c_{ij} \in \mathbb{C}$ such that

$$f\xi_i = \sum_{j=1}^{n} c_{ij}\xi_j \quad \text{on } Y \text{ for } i = 1, \ldots, n. \tag{12.1}$$

Now $F := \det(f\delta_{ij} - c_{ij})_{i,j}$ is a holomorphic function on Y' which is not identically zero on any connected component of Y'. Moreover, by (12.1),

$$F\xi_i|_Y = 0 \quad \text{for } i = 1, \ldots, n. \tag{12.2}$$

An arbitrary cohomology class $\zeta \in H^1(Y', \mathcal{O})$ can be represented by a cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$, where $\mathfrak{U} = (U_i)$ is an open cover of Y' such that each zero of F is contained in at most one U_i. Then $F|_{U_i \cap U_j}$ is holomorphic and non-vanishing in $U_i \cap U_j$ if $i \neq j$. So there exists $(g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ such that $f_{ij} = Fg_{ij}$. Let $\xi \in H^1(Y', \mathcal{O})$ be the cohomology class of (g_{ij}). Then $\zeta = F\xi$. By (12.2), $\zeta|_Y = 0$. □

Corollary 12.11. *Let X be a non-compact Riemann surface. Let $Y \Subset Y' \subseteq X$ be open subsets. Then for every $\omega \in \mathscr{E}^{0,1}(Y')$ there exists a function $f \in \mathscr{E}(Y)$ such that $\overline{\partial} f = \omega|_Y$.*

Proof. The problem has a solution locally, by Theorem 10.9. So there exist an open cover $\mathfrak{U} = (U_i)$ of Y' and functions $f_i \in \mathscr{E}(U_i)$ such that $\overline{\partial} f_i = \omega|_{U_i}$. The differences $f_i - f_j$ define a cocycle in $Z^1(\mathfrak{U}, \mathcal{O})$. By Theorem 12.10, this cocyle is cohomologous to zero on Y. Thus there exist $g_i \in \mathcal{O}(U_i \cap Y)$ such that

$$f_i - f_j = g_i - g_j \quad \text{on } U_i \cap U_j \cap Y.$$

It follows that there is a function $f \in \mathscr{E}(Y)$ such that $f|_{U_i \cap Y} = f_i - g_i$ for all i, and, consequently, $\overline{\partial} f = \omega|_Y$. □

13. The Riemann–Roch theorem

The Riemann–Roch theorem is central in the theory of compact Riemann surfaces. It tells us how many linearly independent meromorphic functions with prescribed zeros and poles there are on a compact Riemann surface, relating the complex analysis with the genus of the Riemann surface.

13.1. Divisors. Let X be a Riemann surface. A **divisor** on X is a map $D : X \to \mathbb{Z}$ such that for each compact subset $K \subseteq X$ the set of $x \in K$ with $D(x) \neq 0$ is finite. The set $\mathrm{Div}(X)$ of all divisors on X forms an abelian group with respect to addition. There is a natural partial ordering on $\mathrm{Div}(X)$: for $D_1, D_2 \in \mathrm{Div}(X)$ we set $D_1 \leq D_2$ if $D_1(x) \leq D_2(x)$ for all $x \in X$.

For any meromorphic function $f \in \mathscr{M}(X) \setminus \{0\}$ the map $x \mapsto \mathrm{ord}_x(f)$ is a divisor on X. It is called the **divisor of** f and will be denoted by (f).

We say that f is a **multiple** of the divisor D if $(f) \geq D$. Note that f is holomorphic if and only if $(f) \geq 0$.

The divisor (ω) of a meromorphic 1-form $\omega \in \mathscr{M}^1(X) \setminus \{0\}$ is the map $x \mapsto \mathrm{ord}_x(\omega)$; here $\mathrm{ord}_x(\omega) = \mathrm{ord}_x(f)$ where $\omega = f\,dz$ in a local chart (U, z) at x.

For $f, g \in \mathscr{M}(X) \setminus \{0\}$ and $\omega \in \mathscr{M}^1(X) \setminus \{0\}$ we have

$$(fg) = (f) + (g), \quad (1/f) = -(f), \quad (f\omega) = (f) + (\omega).$$

A divisor $D \in \mathrm{Div}(X)$ is a **principal divisor** if there exists $f \in \mathscr{M}(X) \setminus \{0\}$ such that $D = (f)$. Two divisors D_1, D_2 are said to be **equivalent** if their difference $D_1 - D_2$ is principal.

A divisor $D \in \mathrm{Div}(X)$ is a **canonical divisor** if there exists $\omega \in \mathscr{M}^1(X) \setminus \{0\}$ such that $D = (\omega)$. Note that any two canonical divisors are equivalent. Indeed, for any two $\omega_1, \omega_2 \in \mathscr{M}^1(X) \setminus \{0\}$ there exists $f \in \mathscr{M}(X) \setminus \{0\}$ with $\omega_1 = f\omega_2$, whence $(\omega_1) = (f) + (\omega_2)$.

13.2. The degree of a divisor. Let X be a *compact* Riemann surface. Then the **degree**, i.e., the map

$$\deg : \mathrm{Div}(X) \to \mathbb{Z}, \quad \deg D = \sum_{x \in X} D(x),$$

is well-defined. It is a group homomorphism. For every principal divisor (f) we have $\deg (f) = 0$, since there are as many zeros as poles on a compact Riemann surface. It follows that equivalent divisors have the same degree.

13.3. The sheaf $\mathscr{L}_D$. Let D be a divisor on a Riemann surface X. For any open $U \subseteq X$ let

$$\mathscr{L}_D(U) := \{f \in \mathscr{M}(U) : \mathrm{ord}_x(f) \geq -D(x) \text{ for all } x \in U\}$$

be the set of multiples of the divisor $-D$. Then $\mathscr{L}_D$ (together with the natural restriction maps) forms a sheaf on X. Note that $\mathscr{L}_0 = \mathcal{O}$. For equivalent divisors $D_1, D_2 \in \mathrm{Div}(X)$, the sheaves $\mathscr{L}_{D_1}$ and $\mathscr{L}_{D_2}$ are isomorphic. In fact, if $D_1 - D_2 = (g)$, then $\mathscr{L}_{D_1} \ni f \mapsto gf \in \mathscr{L}_{D_2}$ is a sheaf isomorphism.

Lemma 13.1. *Let X be a compact Riemann surface and $D \in \mathrm{Div}(X)$ a divisor with $\deg D < 0$. Then $H^0(X, \mathscr{L}_D) = 0$.*

Proof. If there exists $f \in H^0(X, \mathscr{L}_D) = \mathscr{L}_D(X)$ such that $f \neq 0$, then $(f) \geq -D$ and thus $\deg (f) \geq -\deg D > 0$, a contradiction, by subsection 13.2. □

13.4. The skyscraper sheaf. Fix a point P of a Riemann surface X. The **skyscraper sheaf** $\mathbb{C}_P$ on X is defined by

$$\mathbb{C}_P(U) = \begin{cases} \mathbb{C} & \text{if } P \in U, \\ 0 & \text{if } P \notin U, \end{cases}$$

with the obvious restriction maps.

Lemma 13.2. *We have $H^0(X, \mathbb{C}_P) \cong \mathbb{C}$ and $H^1(X, \mathbb{C}_P) = 0$.*

Proof. Clearly, $H^0(X, \mathbb{C}_P) = \mathbb{C}_P(X) = \mathbb{C}$. Let $\xi \in H^1(X, \mathbb{C}_P)$ be represented by a cocycle in $Z^1(\mathfrak{U}, \mathbb{C}_P)$. The cover $\mathfrak{U}$ has a refinement $\mathfrak{V}$ such that P is contained in just one $V \in \mathfrak{V}$. It follows that $Z^1(\mathfrak{V}, \mathbb{C}_P) = 0$ and so $\xi = 0$. □

Lemma 13.3. *Let $D_1 \leq D_2$ be divisors on a compact Riemann surface X. Then the inclusion map $\mathscr{L}_{D_1} \to \mathscr{L}_{D_2}$ induces an epimorphism*

$$H^1(X, \mathscr{L}_{D_1}) \to H^1(X, \mathscr{L}_{D_2}) \to 0.$$

Proof. Let $D \in \mathrm{Div}(X)$ and $P \in X$. We denote by P the divisor which equals 1 at the point P and zero otherwise. Then $D \leq D' := D + P$ and we have a natural inclusion map $\mathscr{L}_D \to \mathscr{L}_D$.

Let us define a sheaf homomorphism $\beta : \mathscr{L}_{D'} \to \mathbb{C}_P$ as follows. Let (V, z) be a local coordinate neighborhood of P such that $z(P) = 0$. Let $U \subseteq X$ be open. If $P \notin U$, set $\beta_U = 0$. If $P \in U$ and $f \in \mathscr{L}_{D'}(U)$, then f has a Laurent series expansion about P with respect to z,

$$f = \sum_{n=-k-1}^{\infty} c_n z^n, \quad \text{where } k = D(P).$$

Define $\beta_U(f) := c_{-k-1} \in \mathbb{C} = \mathbb{C}_P(U)$. Clearly, β is a sheaf epimorphism and $0 \to \mathscr{L}_D \to \mathscr{L}_{D'} \stackrel{\beta}{\to} \mathbb{C}_P \to 0$ is a short exact sequence. By Theorem 11.7 and Lemma 13.2, we have the exact sequence

$$0 \to H^0(X, \mathscr{L}_D) \to H^0(X, \mathscr{L}_{D'}) \to \mathbb{C} \to H^1(X, \mathscr{L}_D) \to H^1(X, \mathscr{L}_{D'}) \to 0. \quad (13.1)$$

This implies the lemma for $D_1 = D$ and $D_2 = D' = D + P$.

In general, $D_2 = D + P_1 + P_2 + \cdots + P_m$, where $P_j \in X$. The lemma follows by induction. □

13.5. The Riemann–Roch theorem.

Theorem 13.4 (Riemann–Roch theorem)**.** *Let D be a divisor on a compact Riemann surface X of genus g. Then $H^0(X, \mathscr{L}_D)$ and $H^1(X, \mathscr{L}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(X, \mathscr{L}_D) - \dim H^1(X, \mathscr{L}_D) = 1 - g + \deg D.$$

Proof. The theorem holds for $D = 0$. In fact, $H^0(X, \mathcal{O}) = \mathcal{O}(X)$ only consists of constant functions, whence $\dim H^0(X, \mathscr{L}_D) = 1$. By the definition of g, the result follows.

We will use the notation of the proof of Lemma 13.3. Let $D \in \mathrm{Div}(X)$, $P \in X$, and $D' = D + P$. Suppose that the result holds for one of the divisors D, D'. We will prove that it also holds for the other. Since any divisor on X is of the form $P_1 + \cdots + P_m - P_{m+1} - \cdots - P_n$, the theorem will follow by induction.

Consider the exact sequence (13.1). Let $V := \mathrm{im}\big(H^0(X, \mathscr{L}_{D'}) \to \mathbb{C}\big)$ and $W := \mathbb{C}/V$. Then $\dim V + \dim W = 1 = \deg D' - \deg D$ and the sequences

$$0 \to H^0(X, \mathscr{L}_D) \to H^0(X, \mathscr{L}_{D'}) \to V \to 0$$

and

$$0 \to W \to H^1(X, \mathscr{L}_D) \to H^1(X, \mathscr{L}_{D'}) \to 0$$

are exact. It follows that all vector spaces are finite dimensional (since this holds for either D or D' by assumption),

$$\dim H^0(X, \mathscr{L}_{D'}) = \dim H^0(X, \mathscr{L}_D) + \dim V,$$
$$\dim H^1(X, \mathscr{L}_D) = \dim H^1(X, \mathscr{L}_{D'}) + \dim W,$$

and hence

$$\dim H^0(X, \mathscr{L}_{D'}) - \dim H^1(X, \mathscr{L}_{D'}) - \deg D' = \dim H^0(X, \mathscr{L}_D) - H^1(X, \mathscr{L}_D) - \deg D.$$

The claim and thus the theorem follows. □

Typically, one is interested in the quantity $\dim H^0(X, \mathscr{L}_D)$, i.e., *the maximal number of linearly independent meromorphic functions on X which are multiples of $-D$*. The correction term

$$i(D) := \dim H^1(X, \mathscr{L}_D)$$

is called the **index of speciality** of the divisor D. Thus

$$\dim H^0(X, \mathscr{L}_D) = 1 - g + \deg D + i(D).$$

Clearly, $i(D) \geq 0$ and hence we always have **Riemann's inequality**

$$\dim H^0(X, \mathscr{L}_D) \geq 1 - g + \deg D.$$

By Lemma 13.1,

$$i(D) = g - 1 - \deg D \quad \text{if } \deg D < 0.$$

13.6. Consequences for the existence of meromorphic functions.

Theorem 13.5. *Let X be a compact Riemann surface of genus g and let $a \in X$. There exists a non-constant meromorphic function f on X which has a pole of order $\le g+1$ at a and is holomorphic on $X \setminus \{a\}$.*

Proof. Let $D \in \operatorname{Div}(X)$ be defined by $D(a) = g+1$ and $D(x) = 0$ if $x \neq a$. By Theorem 13.4, $\dim H^0(X, \mathscr{L}_D) \ge 1 - g + \deg D = 2$. So there exists a non-constant function $f \in H^0(X, \mathscr{L}_D)$. It is clear that f has the required properties. □

Corollary 13.6. *Let X be a Riemann surface of genus g. There exists a branched holomorphic covering map $f : X \to \widehat{\mathbb{C}}$ with at most $g+1$ sheets.*

Proof. The function from the previous theorem is the required covering map in view of Theorem 3.19 (in fact, ∞ is assumed with multiplicity $\le g+1$). □

A strengthened statement will be proved in Corollary 19.7.

Corollary 13.7. *Every Riemann surface with genus zero is isomorphic to the Riemann sphere.*

Proof. A one-sheeted holomorphic covering map is a biholomorphism. □

14. Serre duality

In this section we will prove the Serre duality theorem which states that there is an isomorphism $H^1(X, \mathscr{L}_D)^* \cong H^0(X, \mathscr{L}^1_{-D})$. Thus,

$$\dim H^1(X, \mathscr{L}_D) = \dim H^0(X, \mathscr{L}^1_{-D})$$

is the maximal number of linearly independent 1-forms on X which are multiples of the divisor D. As a special case (for $D = 0$) we get that the genus

$$g = \dim H^1(X, \mathcal{O}) = \dim H^0(X, \mathcal{O}^1)$$

is the maximal number of linearly independent holomorphic 1-forms on X.

14.1. Mittag–Leffler distributions and their residues. Let X be a Riemann surface. Let $\mathscr{M}^1$ be the sheaf of meromorphic 1-forms on X. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. A cochain $\mu = (\omega_i) \in C^0(\mathfrak{U}, \mathscr{M}^1)$ is called a **Mittag–Leffler distribution** if $\delta\mu \in Z^1(\mathfrak{U}, \mathcal{O}^1)$, i.e., the differences $\omega_j - \omega_i$ are holomorphic on $U_i \cap U_j$. We denote by $[\delta\mu] \in H^1(\mathfrak{U}, \mathcal{O}^1)$ the cohomology class of $\delta\mu$.

Let $\mu = (\omega_i)$ be a Mittag–Leffler distribution. Its **residue** at a point $a \in X$ is defined as follows. Choose $i \in I$ such that $a \in U_i$ and set

$$\operatorname{res}_a(\mu) := \operatorname{res}_a(\omega_i).$$

This is well-defined, since if $a \in U_i \cap U_j$ then $\operatorname{res}_a(\omega_i) = \operatorname{res}_a(\omega_j)$ because $\omega_j - \omega_i$ is holomorphic.

Let X be a compact Riemann surface. Then we define

$$\operatorname{res}(\mu) := \sum_{a \in X} \operatorname{res}_a(\mu).$$

Note that $\operatorname{res}_a(\mu)$ is non-zero only for finitely many $a \in X$.

14.2. A formula for the residue. Let X be a compact Riemann surface. By Dolbeault's theorem 11.11,

$$H^1(X, \mathcal{O}^1) \cong \mathscr{E}^2(X)/d\mathscr{E}^{1,0}(X). \tag{14.1}$$

Let $\xi \in H^1(X, \mathcal{O}^1)$ and let $\omega \in \mathscr{E}^2(X)$ be a representative of ξ with respect to the above isomorphism. We define the linear form $\operatorname{res} : H^1(X, \mathcal{O}^1) \to \mathbb{C}$ by

$$\operatorname{res}(\xi) := \frac{1}{2\pi i} \int_X \omega. \tag{14.2}$$

Since $\int_X d\sigma = 0$ for each $\sigma \in \mathscr{E}^1(X)$, by Theorem 8.11, the definition is independent of the choice of the representative ω.

Theorem 14.1. *Let X be a compact Riemann surface. Let μ be a Mittag–Leffler distribution. Then*

$$\operatorname{res}(\mu) = \operatorname{res}([\delta\mu]).$$

Proof. For the computation of $\operatorname{res}([\delta\mu])$ we have to know the isomorphism (14.1) explicitly (cf. Remark 11.10). We have $\delta\mu = (\omega_j - \omega_i) \in Z^1(\mathfrak{U}, \mathcal{O}^1) \subseteq Z^1(\mathfrak{U}, \mathscr{E}^{1,0})$. Since $H^1(X, \mathscr{E}^{1,0}) = 0$ (see Remark 10.5), there is a cochain $(\sigma_i) \in C^0(\mathfrak{U}, \mathscr{E}^{1,0})$ such that

$$\omega_j - \omega_i = \sigma_j - \sigma_i \quad \text{on } U_i \cap U_j.$$

By Proposition 8.2, $d(\omega_j - \omega_i) = 0$ and hence $d\sigma_j = d\sigma_i$ on $U_i \cap U_j$. So there exists $\tau \in \mathscr{E}^2(X)$ such that $\tau|_{U_i} = d\sigma_i$ for all $i \in I$. This 2-form represents the cohomology class $[\delta\mu]$ (cf. Remark 11.10), and thus

$$\operatorname{res}([\delta\mu]) = \frac{1}{2\pi i} \int_X \tau.$$

Let $a_1, \ldots, a_n \in X$ be the poles of μ and set $X' := X \setminus \{a_1, \ldots, a_n\}$. Then $\sigma_i - \omega_i = \sigma_j - \omega_j$ on $X' \cap U_i \cap U_j$, and hence there exists $\sigma \in \mathscr{E}^{1,0}(X')$ with $\sigma|_{X' \cap U_i} = \sigma_i - \omega_i$ for all $i \in I$. It follows that $\tau = d\sigma$ on X'.

For each a_k there exists $i(k)$ such that $a_k \in U_{i(k)}$. Choose a coordinate chart (V_k, z_k) such that $V_k \subseteq U_{i(k)}$ and $z_k(a_k) = 0$. We may assume that the V_k are pairwise disjoint and that each $z_k(V_k)$ is a disk in $\mathbb{C}$. Choose a function $f_k \in \mathscr{E}(X)$ with support contained in V_k and equal to 1 on an open neighborhood $V_k' \subseteq V_k$ of a_k. Set $g = 1 - (f_1 + \cdots + f_n)$. Then $g\sigma$ can be considered as an element in $\mathscr{E}^{1,0}(X)$ (by setting it 0 on the points a_k). By Theorem 8.11,

$$\int_X d(g\sigma) = 0.$$

On $V_k' \setminus \{a_k\}$ we have $d(f_k\sigma) = d\sigma = d\sigma_{i(k)} - d\omega_{i(k)} = d\sigma_{i(k)}$. Thus $d(f_k\sigma)$ extends smoothly to a_k and can be considered as an element of $\mathscr{E}^2(X)$, since it vanishes outside the support of f_k. Since $\tau = d(g\sigma) + \sum_k d(f_k\sigma)$, we obtain

$$\int_X \tau = \sum_{k=1}^n \int_X d(f_k\sigma) = \sum_{k=1}^n \int_{V_k} d(f_k\sigma_{i(k)} - f_k\omega_{i(k)}).$$

By Theorem 8.11, $\int_{V_k} d(f_k\sigma_{i(k)}) = 0$. As in the computation (8.6),

$$\int_{V_k} d(f_k\omega_{i(k)}) = -2\pi i \operatorname{res}_{a_k}(\omega_{i(k)}).$$

It follows that

$$\operatorname{res}([\delta\mu]) = \frac{1}{2\pi i} \int_X \tau = \sum_{k=1}^n \operatorname{res}_{a_k}(\omega_{i(k)}) = \operatorname{res}(\mu).$$

□

14.3. The sheaf of meromorphic 1-forms which are multiples of $-D$. Let X be a compact Riemann surface. Let $D \in \mathrm{Div}(X)$ be a divisor. We denote by $\mathscr{L}_D^1$ the sheaf of meromorphic 1-forms which are multiples of $-D$, i.e., for any open $U \subseteq X$,

$$\mathscr{L}_D^1(U) = \{\omega \in \mathscr{M}^1(U) : \mathrm{ord}_x(\omega) \geq -D(x) \text{ for all } x \in U\}$$

In particular, $\mathscr{L}_0^1 = \mathcal{O}^1$.

Fix a non-trivial meromorphic 1-form $\omega \in \mathscr{M}^1(X)$ and let K be its divisor. For any $D \in \mathrm{Div}(X)$ we obtain a sheaf isomorphism

$$\mathscr{L}_{D+K} \to \mathscr{L}_D^1, \quad f \mapsto f\omega. \tag{14.3}$$

Then, by the Riemann–Roch theorem 13.4,

$$\begin{aligned}\dim H^0(X, \mathscr{L}_D^1) &= \dim H^0(X, \mathscr{L}_{D+K}) \\ &= \dim H^1(X, \mathscr{L}_{D+K}) + 1 - g + \deg(D+K) \\ &\geq \deg D + 1 - g + \deg K,\end{aligned}$$

where g is the genus of X. So there is an integer k_0 such that

$$\dim H^0(X, \mathscr{L}_D^1) \geq \deg D + k_0 \tag{14.4}$$

for all $D \in \mathrm{Div}(X)$.

14.4. A dual pairing. Let X be a compact Riemann surface. Let $D \in \mathrm{Div}(X)$ be a divisor. The product

$$\mathscr{L}_{-D}^1 \times \mathscr{L}_D \to \mathcal{O}^1, \quad (\omega, f) \mapsto f\omega,$$

induces a map

$$H^0(X, \mathscr{L}_{-D}^1) \times H^1(X, \mathscr{L}_D) \to H^1(X, \mathcal{O}^1).$$

Composition with the map $\mathrm{res} : H^1(X, \mathcal{O}^1) \to \mathbb{C}$ (from (14.2)) yields a bilinear map

$$H^0(X, \mathscr{L}_{-D}^1) \times H^1(X, \mathscr{L}_D) \to \mathbb{C}, \quad \langle \omega, \xi \rangle := \mathrm{res}(\omega\xi).$$

This induces a linear map

$$\iota_D : H^0(X, \mathscr{L}_{-D}^1) \to H^1(X, \mathscr{L}_D)^*, \quad \iota_D(\omega)(\xi) = \mathrm{res}(\omega\xi),$$

where $H^1(X, \mathscr{L}_D)^*$ is the dual space of $H^1(X, \mathscr{L}_D)$. The Serre duality theorem holds that ι_D is an isomorphism, i.e., $\langle \cdot, \cdot \rangle$ is a dual pairing.

Lemma 14.2. *The map ι_D is injective.*

Proof. We have to show that for every non-zero $\omega \in H^0(X, \mathscr{L}_{-D}^1)$ there exists $\xi \in H^1(X, \mathscr{L}_D)$ such that $\langle \omega, \xi \rangle \neq 0$. We may choose a point $a \in X$ such that $D(a) = 0$ and a coordinate neighborhood (U_0, z) of a with $z(a) = 0$ and $D|_{U_0} = 0$. On U_0 we have $\omega = f\,dz$ for $f \in \mathcal{O}(U_0)$. By shrinking U_0 if necessary we may assume that f does not vanish in $U_0 \setminus \{a\}$. Let $U_1 := X \setminus \{a\}$ and $\mathfrak{U} = (U_0, U_1)$. Let $\eta = ((zf)^{-1}, 0) \in C^0(\mathfrak{U}, \mathscr{M})$. Then $\omega\eta = (z^{-1}\,dz, 0) \in C^0(\mathfrak{U}, \mathscr{M}^1)$ is a Mittag–Leffler distribution with $\mathrm{res}(\omega\eta) = 1$. We have $\delta\eta \in Z^1(\mathfrak{U}, \mathscr{L}_D)$. For the cohomology class $\xi := [\delta\eta] \in H^1(\mathfrak{U}, \mathscr{L}_D)$ of $\delta\eta$, we have $\omega\xi = [\delta(\omega\eta)]$ and hence, by Theorem 14.1,

$$\langle \omega, \xi \rangle = \mathrm{res}(\omega\xi) = \mathrm{res}([\delta(\omega\eta)]) = \mathrm{res}(\omega\eta) = 1.$$

□

14.5. Working up to the Serre duality theorem. Before we can prove surjectivity of ι_D we need some preparation.

Let X be a compact Riemann surface. Let $D' \le D$ be two divisors on X. The inclusion $0 \to \mathscr{L}_{D'} \to \mathscr{L}_D$ induces an epimorphism $H^1(X, \mathscr{L}_{D'}) \to H^1(X, \mathscr{L}_D) \to 0$, by Lemma 13.3. This in turn induces a monomorphism $0 \to H^1(X, \mathscr{L}_D)^* \to H^1(X, \mathscr{L}_{D'})^*$ which we denote by $i^D_{D'}$. We obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(X,\mathscr{L}_D)^* & \xrightarrow{i^D_{D'}} & H^1(X,\mathscr{L}_{D'})^* \\ & & \uparrow{\scriptstyle \iota_D} & & \uparrow{\scriptstyle \iota_{D'}} \\ 0 & \longrightarrow & H^0(X,\mathscr{L}^1_{-D}) & \longrightarrow & H^0(X,\mathscr{L}^1_{-D'}) \end{array}$$

Lemma 14.3. *Let $\lambda \in H^1(X, \mathscr{L}_D)^*$ and $\omega \in H^0(X, \mathscr{L}^1_{-D'})$ satisfy $i^D_{D'}(\lambda) = \iota_{D'}(\omega)$. Then ω lies in $H^0(X, \mathscr{L}^1_{-D})$ and $\lambda = \iota_D(\omega)$.*

Proof. Suppose that $\omega \notin H^0(X, \mathscr{L}^1_{-D})$. So there is $a \in X$ with $\mathrm{ord}_a(\omega) < D(a)$. Let (U_0, z) be a coordinate chart at a with $z(a) = 0$. On U_0, $\omega = f\,dz$ for $f \in \mathscr{M}(U_0)$. Shrinking U_0 if necessary we can assume that $D|_{U_0\setminus\{a\}} = D'|_{U_0\setminus\{a\}} = 0$ and f has no zeros and poles in $U_0 \setminus \{a\}$. Let $U_1 := X \setminus \{a\}$ and $\mathfrak{U} := (U_0, U_1)$. Let $\eta = ((zf)^{-1}, 0) \in C^0(\mathfrak{U}, \mathscr{M})$. Since $\mathrm{ord}_a(\omega) < D(a)$, we have $\eta \in C^0(\mathfrak{U}, \mathscr{L}_D)$. Thus $\delta\eta \in Z^1(\mathfrak{U}, \mathcal{O}) = Z^1(\mathfrak{U}, \mathscr{L}_D) = Z^1(\mathfrak{U}, \mathscr{L}_{D'})$, because $U_0 \cap U_1 = U_0 \setminus \{a\}$. The cohomology class of $\delta\eta$ in $H^1(\mathfrak{U}, \mathscr{L}_D)$ is $\xi = 0$. Let ξ' denote the cohomology class of $\delta\eta$ in $H^1(\mathfrak{U}, \mathscr{L}_{D'})$. By assumption,

$$\langle \omega, \xi' \rangle = \iota_{D'}(\omega)(\xi') = i^D_{D'}(\lambda)(\xi') = \lambda(\xi) = 0.$$

Since $\omega\eta = (z^{-1}\,dz, 0)$, we also have

$$\langle \omega, \xi' \rangle = \mathrm{res}(\omega\eta) = 1,$$

a contradiction. Thus $\omega \in H^0(X, \mathscr{L}^1_{-D})$. We have $\lambda = \iota_D(\omega)$, since $i^D_{D'}(\lambda) = \iota_{D'}(\omega) = i^D_{D'}(\iota_D(\omega))$ and $i^D_{D'}$ is a monomorphism. $\square$

Let X be a compact Riemann surface. Let $D, B \in \mathrm{Div}(X)$ be two divisors. Let $\psi \in H^0(X, \mathscr{L}_B) = \mathscr{L}_B(X)$. We have the sheaf morphism

$$\mathscr{L}_{D-B} \to \mathscr{L}_D, \quad f \mapsto \psi f \tag{14.5}$$

which induces a linear map $H^1(X, \mathscr{L}_{D-B}) \to H^1(X, \mathscr{L}_D)$ and hence a linear map (also denoted by ψ)

$$\psi : H^1(X, \mathscr{L}_D)^* \to H^1(X, \mathscr{L}_{D-B})^*.$$

Then, by definition, $(\psi\lambda)(\xi) = \lambda(\psi\xi)$ for $\lambda \in H^1(X, \mathscr{L}_D)^*$ and $\xi \in H^1(X, \mathscr{L}_{D-B})$. The following diagram, where the arrow in the bottom row is defined by multiplication by ψ, commutes.

$$\begin{array}{ccc} H^1(X,\mathscr{L}_D)^* & \xrightarrow{\psi} & H^1(X,\mathscr{L}_{D-B})^* \\ \uparrow{\scriptstyle \iota_D} & & \uparrow{\scriptstyle \iota_{D-B}} \\ H^0(X,\mathscr{L}^1_{-D}) & \xrightarrow{\psi} & H^0(X,\mathscr{L}^1_{-D+B}) \end{array} \tag{14.6}$$

Indeed, if $\omega \in H^0(X, \mathscr{L}^1_{-D})$ and $\xi \in H^1(X, \mathscr{L}_{D-B})$, then

$$\psi\iota_D(\omega)(\xi) = \iota_D(\omega)(\psi\xi) = \langle \omega, \psi\xi \rangle = \langle \psi\omega, \xi \rangle = \iota_{D-B}(\psi\omega)(\xi).$$

Lemma 14.4. *If $\psi \in H^0(X, \mathscr{L}_B) = \mathscr{L}_B(X)$ is not the zero function, then the top row in the diagram is injective.*

Proof. Let A be the divisor of ψ. Then $A \geq -B$. The map (14.5) factors through $\mathscr{L}_{D+A}$, i.e., $\mathscr{L}_{D-B} \to \mathscr{L}_{D+A} \to \mathscr{L}_D$, where $\mathscr{L}_{D+A} \to \mathscr{L}_D$ (induced by multiplication with ψ) is an isomorphism. By Lemma 13.3, $H^1(X, \mathscr{L}_{D-B}) \to H^1(X, \mathscr{L}_{D+A})$ is an epimorphism. Thus also $H^1(X, \mathscr{L}_{D-B}) \to H^1(X, \mathscr{L}_D)$ is an epimorphism. This implies the statement. □

14.6. The Serre duality theorem.

Theorem 14.5 (Serre duality theorem)**.** *Let X be a compact Riemann surface, and $D \in \mathrm{Div}(X)$. Then the map $\iota_D : H^0(X, \mathscr{L}^1_{-D}) \to H^1(X, \mathscr{L}_D)^*$ is an isomorphism.*

Proof. It remains to prove surjectivity of ι_D; cf. Lemma 14.2. Let $\lambda \in H^1(X, \mathscr{L}_D)^*$ and $\lambda \neq 0$. Fix $P \in \mathrm{Div}(X)$ with $\deg P = 1$. Set $D_n := D - nP$, for $n \in \mathbb{N}$. Consider the linear subspace $\Lambda := \{\psi\lambda \in H^1(X, \mathscr{L}_{D_n})^* : \psi \in H^0(X, \mathscr{L}_{nP})\}$. Then Λ is isomorphic to $H^0(X, \mathscr{L}_{nP})$; indeed, if $\psi\lambda = 0$ and $\psi \neq 0$ then $\lambda = 0$, by Lemma 14.4, a contradiction. So, by the Riemann–Roch theorem 13.4,

$$\dim \Lambda \geq 1 - g + n.$$

By (14.4), the linear subspace $\operatorname{im} \iota_{D_n} \subseteq H^1(X, \mathscr{L}_{D_n})^*$ satisfies (we already know that ι_{D_n} is injective, by Lemma 14.2)

$$\dim \operatorname{im} \iota_{D_n} = \dim H^0(X, \mathscr{L}^1_{-D_n}) \geq n - \deg D + k_0$$

for some integer k_0. If $n > \deg D$ then $\deg D_n < 0$ and $H^0(X, \mathscr{L}_{D_n}) = 0$, by Lemma 13.1. In that case the Riemann–Roch theorem 13.4 implies

$$\dim H^1(X, \mathscr{L}_{D_n})^* = g - 1 - \deg D_n = n + g - 1 - \deg D.$$

Thus, by choosing n sufficiently large we achieve

$$\dim \Lambda + \dim \operatorname{im} \iota_{D_n} > \dim H^1(X, \mathscr{L}_{D_n})^*.$$

It follows that the linear subspaces Λ and $\operatorname{im} \iota_{D_n}$ of $H^1(X, \mathscr{L}_{D_n})^*$ have non-trivial intersection. So there exists a non-trivial $\psi \in H^0(X, \mathscr{L}_{nP})$ and $\omega \in H^0(X, \mathscr{L}^1_{-D_n})$ such that $\psi\lambda = \iota_{D_n}(\omega)$. Let A be the divisor of ψ and set $D' = D_n - A$. Then $1/\psi \in H^0(X, \mathscr{L}_A)$ and (since the diagram (14.6) commutes)

$$i^D_{D'}(\lambda) = \frac{1}{\psi}(\psi\lambda) = \frac{1}{\psi}\iota_{D_n}(\omega) = \iota_{D'}\Big(\frac{1}{\psi}\omega\Big).$$

By Lemma 14.3, $(1/\psi)\omega$ lies in $H^0(X, \mathscr{L}^1_{-D})$ and $\lambda = \iota_D((1/\psi)\omega)$. □

14.7. Consequences. A first consequence of Theorem 14.5 is

$$\dim H^1(X, \mathscr{L}_D) = \dim H^0(X, \mathscr{L}^1_{-D}),$$

in particular, for $D = 0$,

$$g = \dim H^1(X, \mathcal{O}) = \dim H^0(X, \mathcal{O}^1) = \dim \mathcal{O}^1(X). \tag{14.7}$$

So *the genus of a compact Riemann surface X is equal to the maximal number of linearly independent holomorphic 1-forms on X.*

The Riemann–Roch theorem 13.4 takes the form

$$\dim H^0(X, \mathscr{L}_{-D}) - \dim H^0(X, \mathscr{L}^1_D) = 1 - g - \deg D,$$

which means: *the maximal number of linearly independent meromorphic functions which are multiples of a divisor D minus the maximal number of linearly independent meromorphic 1-forms which are multiples of a divisor $-D$ is equal to $1 - g - \deg D$.*

Corollary 14.6. *Let X be a compact Riemann surface, and let $D \in \mathrm{Div}(X)$. Then $H^0(X, \mathscr{L}_{-D}) \cong H^1(X, \mathscr{L}^1_D)^*$.*

Proof. Let ω be a non-trivial meromorphic 1-form on X and $K = (\omega)$ its divisor. As in (14.3) we have sheaf isomorphisms $\mathscr{L}_{D+K} \cong \mathscr{L}^1_D$ and $\mathscr{L}_{-D} \cong \mathscr{L}^1_{-D-K}$. So the statement follows from Theorem 14.5. □

Corollary 14.7. *Let X be a compact Riemann surface. The map* res : $H^1(X, \mathcal{O}^1) \to \mathbb{C}$ *is an isomorphism.*

Proof. For $D = 0$, Corollary 14.6 gives $\dim H^1(X, \mathcal{O}^1) = \dim H^0(X, \mathcal{O}) = 1$. Clearly, res is not identically zero. □

Corollary 14.8. *Let X be a compact Riemann surface of genus g. Let $\omega \in \mathscr{M}^1(X)$ be non-trivial. Then* $\deg(\omega) = 2g - 2$*.*

Proof. Let $K = (\omega)$ be the divisor of ω. We have a sheaf isomorphism $\mathscr{L}_K \cong \mathcal{O}^1$, by (14.3). By the Riemann–Roch theorem 13.4,

$$\begin{aligned} 1 - g + \deg K &= \dim H^0(X, \mathscr{L}_K) - \dim H^1(X, \mathscr{L}_K) \\ &= \dim H^0(X, \mathcal{O}^1) - \dim H^1(X, \mathcal{O}^1) \\ &= g - 1, \end{aligned}$$

where the last identity follows from (14.7) and Corollary 14.7. □

Corollary 14.9 (complex tori, V)**.** *For any lattice $\Lambda \subseteq \mathbb{C}$ the complex torus $\mathbb{C}/\Lambda$ has genus* 1*.*

Proof. The 1-form dz on $\mathbb{C}$ induces a 1-form ω on $\mathbb{C}/\Lambda$ having no zeros or poles; cf. Example 9.3. Thus $0 = \deg(\omega) = 2g - 2$, by Corollary 14.8. □

15. The Riemann–Hurwitz formula

The Riemann–Hurwitz formula allows one the calculate the genus of a holomorphic covering from the genus of the base space, the number of sheets, and the branching order.

15.1. The branching order. Let $f : X \to Y$ be a non-constant holomorphic map between two compact Riemann surfaces. Let $m_x(f)$ be the multiplicity with which f takes the value $f(x)$ at x; cf. Theorem 1.7. Then

$$b_x(f) := m_x(f) - 1$$

is called the **branching order** of f at x, and

$$b(f) := \sum_{x \in X} b_x(f)$$

is called the **total branching order** of f. Since X is compact, $b_x(f)$ is zero but for a finite number of points $x \in X$.

15.2. The Riemann–Hurwitz formula.

Theorem 15.1. *Let $f : X \to Y$ be an n-sheeted holomorphic covering map between compact Riemann surfaces with total branching order b. If g is the genus of X and g' is the genus of Y, then*

$$g = \frac{b}{2} + n(g' - 1) + 1.$$

Proof. Let ω be a non-trivial meromorphic 1-form on Y. Then, by Corollary 14.8, $\deg(\omega) = 2g' - 2$ and $\deg(f^*\omega) = 2g - 2$.

Let $x \in X$ and $y = f(x)$. There is a coordinate neighborhood (U, z) of x with $z(x) = 0$ and a coordinate neighborhood (V, w) of y with $w(y) = 0$ such that in these coordinates f takes the form $w = z^k$, where $k = m_x(f)$; see Theorem 1.7. Take $\omega = \psi(w)\,dw$ on V. Then on U,

$$f^*\omega = \psi(z^k)\,d(z^k) = kz^{k-1}\psi(z^k)\,dz.$$

This implies that $\operatorname{ord}_x(f^*\omega) = k - 1 + k\operatorname{ord}_y(\omega) = b_x(f) + m_x(f)\operatorname{ord}_y(\omega)$, hence

$$\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega) = \sum_{x\in f^{-1}(y)} b_x(f) + n\operatorname{ord}_y(\omega),$$

and therefore

$$\begin{aligned}\deg(f^*\omega) &= \sum_{y\in Y}\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega)\\ &= \sum_{y\in Y}\sum_{x\in f^{-1}(y)} b_x(f) + n\sum_{y\in X}\operatorname{ord}_y(\omega) = b + n\deg(\omega).\end{aligned}$$

Hence $2g - 2 = b + n(2g' - 2)$. □

15.3. Coverings of the Riemann sphere. For an n-sheeted holomorphic covering $\pi : X \to \widehat{\mathbb{C}}$ of the Riemann sphere we obtain

$$g = \frac{b}{2} - n + 1.$$

In particular, for a double covering of $\widehat{\mathbb{C}}$, b is the number of branch points and

$$g = \frac{b}{2} - 1. \tag{15.1}$$

A compact Riemann surface of genus > 1 which admits a double covering of $\widehat{\mathbb{C}}$ is called **hyperelliptic**.

Example 15.2. Let $P(z) = (z - a_1)\cdots(z - a_k)$ be a polynomial of degree k with distinct roots a_j. Let $p : X \to \widehat{\mathbb{C}}$ be the Riemann surface of $\sqrt{P(z)}$. Then X is branched over ∞ precisely if k is odd; cf. Example 7.9. The total branching order b is k or $k + 1$, depending on whether k is even or odd. By (15.1), we have

$$g = \left\lfloor \frac{k-1}{2} \right\rfloor.$$

An explicit basis $\omega_1, \dots, \omega_g$ for $\mathcal{O}^1(X)$ is given by

$$\omega_j := \frac{z^{j-1}\,dz}{\sqrt{P(z)}}, \quad j = 1, \dots, g,$$

where z denotes the function $p : X \to \widehat{\mathbb{C}}$. Using local coordinates (cf. Theorem 4.8) one shows that the ω_j are holomorphic on all of X.

16. A vanishing theorem

A further consequence of the Serre duality theorem 14.5 is the following vanishing theorem for $H^1(X, \mathscr{L}_D)$. It will lead to an embedding theorem for compact Riemann surfaces into projective space in the next section.

Theorem 16.1. *Let X be a compact Riemann surface of genus g. Let $D \in \operatorname{Div}(X)$ be such that $\deg D > 2g - 2$. Then $H^1(X, \mathscr{L}_D) = 0$.*

Proof. Let ω be a non-trivial meromorphic 1-form on X and $K = (\omega)$ its divisor. Then $\deg K = 2g-2$, by Corollary 14.8. By (14.3), we have a sheaf isomorphism $\mathscr{L}_{-D+K} = \mathscr{L}^1_{-D}$ and thus, using the Serre duality theorem 14.5, $H^1(X, \mathscr{L}_D)^* \cong H^0(X, \mathscr{L}^1_{-D}) \cong H^0(X, \mathscr{L}_{-D+K})$. We have $\deg(-D+K) < 0$, since $\deg D > 2g-2$, and thus $H^0(X, \mathscr{L}_{-D+K}) = 0$, by Lemma 13.1. □

Corollary 16.2. *Let X be a compact Riemann surface. Then $H^1(X, \mathscr{M}) = 0$.*

Proof. Let $\xi \in H^1(X, \mathscr{M})$ be represented by $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{M})$. Passing to a refinement of $\mathfrak{U}$ if necessary, we may assume that the total number of poles of all the f_{ij} is finite. Thus we may find a divisor D with $\deg D > 2g-2$ such that $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{L}_D)$. By Theorem 16.1, $\xi \in B^1(\mathfrak{U}, \mathscr{L}_D) \subseteq B^1(\mathfrak{U}, \mathscr{M})$. □

Let X be a Riemann surface and $D \in \operatorname{Div}(X)$. The sheaf $\mathscr{L}_D$ is called **globally generated** if for each $x \in X$ there exists $f \in H^0(X, \mathscr{L}_D) = \mathscr{L}_D(X)$ such that $\mathscr{L}_{D,x} = \mathcal{O}_x f$, i.e., every $\varphi \in \mathscr{L}_{D,x}$ can be written $\varphi = \psi f$ for some $\psi \in \mathcal{O}_x$. Note that the condition $\mathscr{L}_{D,x} = \mathcal{O}_x f$ is equivalent to $\operatorname{ord}_x(f) = -D(x)$.

Corollary 16.3. *Let X be a compact Riemann surface of genus g. Let $D \in \operatorname{Div}(X)$ be such that $\deg D \ge 2g$. Then $\mathscr{L}_D$ is globally generated.*

Proof. Let $x \in X$ and let D' be the divisor defined by

$$D'(y) := \begin{cases} D(y) & \text{if } y \neq x, \\ D(y) - 1 & \text{if } y = x. \end{cases}$$

Then $\deg D > \deg D' > 2g-2$. So $H^1(X, \mathscr{L}_D) = H^1(X, \mathscr{L}_{D'}) = 0$, by Theorem 16.1. The Riemann–Roch theorem 13.4 implies

$$\dim H^0(X, \mathscr{L}_D) > \dim H^0(X, \mathscr{L}_{D'}).$$

Thus there is an element $f \in H^0(X, \mathscr{L}_D) \setminus H^0(X, \mathscr{L}_{D'})$; in particular, it satisfies $\operatorname{ord}_x(f) = -D(x)$. □

17. Embedding of compact Riemann surfaces into projective space

We will now see that every compact Riemann surface can be embedded into some projective space $\mathbb{P}^N$.

17.1. Projective space. The N-dimensional **projective space** is the quotient space $\mathbb{P}^N = (\mathbb{C}^{N+1} \setminus \{0\})/\sim$ with respect to the equivalence relation

$$z \sim w \quad :\Leftrightarrow \quad \exists \lambda \in \mathbb{C}^* : z = \lambda w.$$

The equivalence class of $(z_0, \dots, z_N) \in \mathbb{C}^{N+1} \setminus \{0\}$ is denoted by $(z_0 : \cdots : z_N) \in \mathbb{P}^N$. With the quotient topology $\mathbb{P}^N$ is a compact Hausdorff space (indeed, $\mathbb{P}^N$ is the image of the unit sphere in $\mathbb{C}^{N+1}$). The sets

$$U_i := \{(z_0 : \cdots : z_N) \in \mathbb{P}^N : z_i \neq 0\}, \quad \text{for } i = 0, \dots, N$$

form an open cover of $\mathbb{P}^N$. The homeomorphisms $\varphi_i : U_i \to \mathbb{C}^N$ defined by

$$\varphi_i(z_0 : \cdots : z_N) := \Big(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_N}{z_i}\Big)$$

induce a complex structure on $\mathbb{P}^N$ which makes it a N-dimensional complex manifold.

Let X be a compact Riemann surface and let $F : X \to \mathbb{P}^N$ be continuous. It is then clear what it means that F is holomorphic, an immersion, an embedding: Indeed, $W_i := F^{-1}(U_i)$ is open in X and we may consider

$$F_i = (F_{i1}, \dots, F_{iN}) := \varphi_i \circ F : W_i \to \mathbb{C}^N.$$

The map F is holomorphic if and only if all F_{ij}, $0 \le i \le N$, $1 \le j \le N$, are holomorphic. And F is an immersion if and only if F is holomorphic and for each $x \in X$ there exists some F_{ij} such that $x \in W_i$ and $dF_{ij}(x) \neq 0$. An embedding is an injective immersion.

Example 17.1. Let X be a compact Riemann surface. Let $f_0, \dots, f_N \in \mathscr{M}(X)$ be non-trivial meromorphic functions on X. Fix $x \in X$ and let (U, z) be a coordinate neighborhood of x with $z(x) = 0$. If $k := \min_i \operatorname{ord}_x(f_i)$, then we have $f_i = z^k g_i$ for all i on U, where the g_i are holomorphic and for some i we have $g_i(x) \neq 0$. Define

$$F(x) := (g_0(x) : \cdots : g_N(x)).$$

This defines a map $F : X \to \mathbb{P}^N$ which we shall denote by $F = (f_0 : \cdots : f_N)$. Clearly, the definition is independent of the local coordinate chosen. The map F is holomorphic, since if $g_i(x) \neq 0$, then near x we have

$$\varphi_i \circ F = \Big(\frac{g_0}{g_i}, \dots, \frac{g_{i-1}}{g_i}, \frac{g_{i+1}}{g_i}, \dots, \frac{g_N}{g_i}\Big).$$

17.2. Embedding theorem.

Theorem 17.2. *Let X be a compact Riemann surface of genus g. Let D be a divisor with $\deg D \ge 2g + 1$. Let $f_0, \dots, f_N$ be a basis of $H^0(X, \mathscr{L}_D)$. Then the map $F = (f_0 : \cdots : f_N) : X \to \mathbb{P}^N$ is an embedding.*

Proof. First we show that F is injective. Let x_1, x_2 be distinct points in X. Let D' be the divisor defined by

$$D'(x) := \begin{cases} D(x) & \text{if } x \neq x_2, \\ D(x) - 1 & \text{if } x = x_2. \end{cases}$$

Then $\deg D > \deg D' \ge 2g$, whence the sheaves $\mathscr{L}_{D'}$ and $\mathscr{L}_D$ are globally generated, by Corollary 16.3. Thus there exists $f \in H^0(X, \mathscr{L}_{D'})$ such that

$$\operatorname{ord}_{x_1}(f) = -D'(x_1) = -D(x_1). \tag{17.1}$$

On the other hand,

$$\operatorname{ord}_{x_2}(f) \ge -D(x_2) + 1. \tag{17.2}$$

Clearly, also $f \in H^0(X, \mathscr{L}_D)$ and hence $f = \sum \lambda_i f_i$ for some $\lambda_i \in \mathbb{C}$. For $j = 1, 2$, let (U_j, z_j) be a coordinate neighborhood of x_j such that $z_j(x_j) = 0$. Since $\mathscr{L}_D$ is globally generated,

$$k_j := \min_i \operatorname{ord}_{x_j}(f_i) = -D(x_j). \tag{17.3}$$

Factor $f_i = z_j^{k_j} g_{ji}$ and $f = z_j^{k_j} g_j$ near x_j. Then $F(x_j) = (g_{j0}(x_j) : \cdots : g_{jN}(x_j))$ and

$$g_j(x_j) = \sum_i \lambda_i g_{ji}(x_j).$$

We have $g_1(x_1) \neq 0$ and $g_2(x_2) = 0$, by (17.1), (17.2), and (17.3), and consequently $F(x_1) \neq F(x_2)$.

Next we prove that F is an immersion. Let $x_0 \in X$. Let D' be the divisor defined by

$$D'(x) := \begin{cases} D(x) & \text{if } x \neq x_0, \\ D(x) - 1 & \text{if } x = x_0. \end{cases}$$

As above we may conclude that $\mathscr{L}_{D'}$ is globally generated and thus there exists $f \in H^0(X, \mathscr{L}_{D'})$ such that

$$\operatorname{ord}_{x_0}(f) = -D(x_0) + 1. \tag{17.4}$$

As before $f = \sum \lambda_i f_i$ for some $\lambda_i \in \mathbb{C}$. Let (U, z) be a coordinate neighborhood of x_0 such that $z(x_0) = 0$. Then for $k := \min_i \operatorname{ord}_{x_0}(f_i) = -D(x_0)$ we have $f_i = z^k g_i$ and $f = z^k g$, and some $g_i(x_0) \neq 0$. Without loss of generality assume that $g_0(x_0) \neq 0$. In a neighborhood of x_0,

$$F_0 := \varphi_0 \circ F = \Big(\frac{g_1}{g_0}, \dots, \frac{g_N}{g_0}\Big).$$

Then, as $f = \sum_{i=0}^N \lambda_i f_i$ implies $g = \sum_{i=0}^N \lambda_i g_i$,

$$\sum_{i=1}^N \lambda_i F_{0i} = \sum_{i=1}^N \lambda_i \frac{g_i}{g_0} = \frac{g}{g_0} - \lambda_0$$

and therefore

$$\sum_{i=1}^N \lambda_i \, dF_{0i} = d\Big(\frac{g}{g_0}\Big).$$

We have $d(g/g_0)(x_0) \neq 0$, since $g_0(x_0) \neq 0$ and since g vanishes of first order at x_0 thanks to (17.4). It follows that $dF_{0i}(x_0) \neq 0$ for some i, and so F is an immersion. □

Remark 17.3. Actually, every compact Riemann surface admits an embedding into $\mathbb{P}^3$.

18. Harmonic differential forms

In this section we introduce and study harmonic 1-forms on a compact Riemann surface X. We will see that every smooth closed 1-form on X can be uniquely written as the sum of a harmonic and an exact 1-form. So the first deRham cohomology group of X is isomorphic to the vector space of harmonic 1-forms on X. This will imply that the genus is a topological invariant.

18.1. The $*$-operator and harmonic 1-forms. Let X be a Riemann surface. Let $\omega \in \mathscr{E}^1(X)$. Locally, $\omega = \sum f_j \, dg_j$ for smooth function f_j, g_j, and we may consider the complex conjugate $\overline{\omega} = \sum \overline{f}_j \, d\overline{g}_j$. This defines the **complex conjugate** $\overline{\omega} \in \mathscr{E}^1(X)$ of ω. The **real part** of ω is defined by $\operatorname{Re}(\omega) = (\omega + \overline{\omega})/2$. We say that ω is real if $\omega = \operatorname{Re}(\omega)$. We have $\operatorname{Re}(\int_\gamma \omega) = \int_\gamma \operatorname{Re}(\omega)$, because $\overline{\int_\gamma \omega} = \int_\gamma \overline{\omega}$.

If $\omega \in \mathcal{O}^1(X)$ is holomorphic, then $\overline{\omega}$ is called **antiholomorphic**. The vector space of all antiholomorphic 1-forms on X is denoted by $\overline{\mathcal{O}}^1(X)$.

Let $\omega \in \mathscr{E}^1(X)$. There is a unique decomposition

$$\omega = \omega_1 + \omega_2, \qquad \text{where } \omega_1 \in \mathscr{E}^{1,0}(X),\ \omega_2 \in \mathscr{E}^{0,1}(X).$$

Defining

$$*\omega := i(\overline{\omega}_1 - \overline{\omega}_2)$$

we obtain an $\mathbb{R}$-linear automorphism $* : \mathscr{E}^1(X) \to \mathscr{E}^1(X)$ with $*\mathscr{E}^{1,0}(X) = \mathscr{E}^{0,1}(X)$ and $*\mathscr{E}^{0,1}(X) = \mathscr{E}^{1,0}(X)$.

Lemma 18.1. *We have the following properties.*

(1) $**\omega = -\omega$ *and* $\overline{*\omega} = *\overline{\omega}$ *for all* $\omega \in \mathscr{E}^1(X)$.
(2) $d*(\omega_1 + \omega_2) = i\partial\overline{\omega}_1 - i\overline{\partial}\overline{\omega}_2$ *for all* $\omega_1 \in \mathscr{E}^{1,0}(X)$, $\omega_2 \in \mathscr{E}^{0,1}(X)$.
(3) $*\partial f = i\overline{\partial}\,\overline{f}$, $*\overline{\partial} f = -i\partial\overline{f}$, *and* $d*df = 2i\partial\overline{\partial}\,\overline{f}$ *for all* $f \in \mathscr{E}(X)$.

Proof. (1) is obvious. For (2), $d*(\omega_1 + \omega_2) = (\partial + \overline{\partial})(i(\overline{\omega}_1 - \overline{\omega}_2)) = i\partial\overline{\omega}_1 - i\overline{\partial}\overline{\omega}_2$. By (2), $d*df = d*(\partial f + \overline{\partial} f) = i\partial\overline{\partial}\,\overline{f} - i\overline{\partial}\partial\overline{f} = 2i\partial\overline{\partial}\,\overline{f}$. The rest follows easily from the definition. □

Proposition 18.2. *Let $\omega \in \mathscr{E}^1(X)$. The following conditions are equivalent.*

(1) $d\omega = d{*}\omega = 0$.
(2) $\partial\omega = \overline{\partial}\omega = 0$.
(3) $\omega = \omega_1 + \omega_2$, *where* $\omega_1 \in \mathcal{O}^1(X)$ *and* $\omega_2 \in \overline{\mathcal{O}}^1(X)$.
(4) ω *is locally the exterior derivative of a harmonic function.*

Proof. The equivalence of the first three items follows from Lemma 18.1.

(1) $\Rightarrow$ (4) Since $d\omega = 0$, locally $\omega = df$ for some smooth function f. Since $0 = d{*}\omega = d{*}df = 2i\partial\overline{\partial}\,\overline{f}$, f is harmonic.

(4) $\Rightarrow$ (1) Assume that $\omega = df$ and f is harmonic, then $d\omega = ddf = 0$ and $d{*}\omega = d{*}df = 2i\partial\overline{\partial}\,\overline{f} = 0$. □

A 1-form $\omega \in \mathscr{E}^1(X)$ on a Riemann surface X which satisfies the equivalent conditions in the proposition is called **harmonic**. We denote the vector space of all harmonic 1-forms on X by $\mathscr{E}^1_{\text{har}}(X)$. By Proposition 18.2,

$$\mathscr{E}^1_{\text{har}}(X) = \mathcal{O}^1(X) \oplus \overline{\mathcal{O}}^1(X). \tag{18.1}$$

If X is a compact Riemann surface of genus g, then

$$\dim \mathscr{E}^1_{\text{har}}(X) = 2g, \tag{18.2}$$

by (14.7).

Theorem 18.3. *Let $\sigma \in \mathscr{E}^1_{\text{har}}(X)$ be a real harmonic 1-form. There exists a unique $\omega \in \mathcal{O}^1(X)$ such that $\sigma = \operatorname{Re}(\omega)$.*

Proof. We may write $\sigma = \omega_1 + \overline{\omega}_2$ for $\omega_1, \omega_2 \in \mathcal{O}^1(X)$. Since σ is real, $\omega_1 + \overline{\omega}_2 = \sigma = \overline{\sigma} = \overline{\omega}_1 + \omega_2$, and so $\omega_1 = \omega_2$. Then $\sigma = \operatorname{Re}(2\omega_1)$.

If $\sigma = \operatorname{Re}(\omega) = \operatorname{Re}(\omega')$ for $\omega, \omega' \in \mathcal{O}^1(X)$, then $\tau := \omega - \omega' \in \mathcal{O}^1(X)$ and $\operatorname{Re}(\tau) = 0$. Locally, there is a holomorphic function f such that $\tau = df$. Thus, f has constant real part and hence is constant itself. Therefore, $\tau = 0$. □

18.2. A scalar product on $\mathscr{E}^1(X)$.

Let X be a compact Riemann surface. For $\omega_1, \omega_2 \in \mathscr{E}^1(X)$ let

$$\langle \omega_1, \omega_2 \rangle := \int_X \omega_1 \wedge *\omega_2.$$

This defines a (sesquilinear) scalar product on $\mathscr{E}^1(X)$. Let us check that $\langle \cdot, \cdot \rangle$ is positive definite. In a local chart $\omega \in \mathscr{E}^1(X)$ has the form $\omega = f\,dz + g\,d\overline{z}$, thus $*\omega = i(\overline{f}\,d\overline{z} - \overline{g}\,dz)$ and

$$\omega \wedge *\omega = i(|f|^2 + |g|^2)\,dz \wedge d\overline{z} = 2(|f|^2 + |g|^2)\,dx \wedge dy$$

Consequently, $\langle \omega, \omega \rangle \ge 0$ and $\langle \omega, \omega \rangle = 0$ if and only if $\omega = 0$.

Thus $(\mathscr{E}^1(X), \langle \cdot, \cdot \rangle)$ is a unitary space, which however is not complete.

Lemma 18.4. *Let X be a compact Riemann surface.*

(1) $\partial\mathscr{E}(X)$, $\overline{\partial}\mathscr{E}(X)$, $\mathcal{O}^1(X)$, *and* $\overline{\mathcal{O}}^1(X)$ *are pairwise orthogonal subspaces of* $\mathscr{E}^1(X)$.
(2) $d\mathscr{E}(X)$ *and* $*d\mathscr{E}(X)$ *are orthogonal subspaces of* $\mathscr{E}^1(X)$ *and*

$$d\mathscr{E}(X) \oplus *d\mathscr{E}(X) = \partial\mathscr{E}(X) \oplus \overline{\partial}\mathscr{E}(X). \tag{18.3}$$

Proof. (1) Since $\mathscr{E}^{1,0}(X)$ and $\mathscr{E}^{0,1}(X)$ are clearly orthogonal also $\partial\mathscr{E}(X) \perp \overline{\partial}\mathscr{E}(X)$, $\overline{\partial}\mathscr{E}(X) \perp \mathcal{O}^1(X)$, and $\partial\mathscr{E}(X) \perp \overline{\mathcal{O}}^1(X)$.

Let us prove $\partial\mathscr{E}(X)\perp\mathcal{O}^1(X)$. Let $f\in\mathscr{E}(X)$ and $\omega\in\mathcal{O}^1(X)$. By Lemma 18.1 and Proposition 8.2,

$$\omega\wedge *\partial f = i\omega\wedge\overline{\partial}\,\overline{f} = i\omega\wedge d\overline{f} = -id(\overline{f}\omega)$$

and by Theorem 8.11,

$$\int_X \omega\wedge *\partial f = -i\int_X d(\overline{f}\omega) = 0.$$

For $\overline{\partial}\mathscr{E}(X)\perp\overline{\mathcal{O}}^1(X)$, use $\overline{\omega}\wedge *\overline{\partial} f = -i\omega\wedge\partial\,\overline{f} = -i\omega\wedge d\overline{f} = id(\overline{f}\omega)$.

(2) Let $f,g\in\mathscr{E}(X)$. Then, by Lemma 18.1,

$$df\wedge *(*dg) = -df\wedge dg = -d(f\,dg).$$

Again by Theorem 8.11, $\langle df, *dg\rangle = 0$. For the identity (18.3) observe that, by Lemma 18.1,

$$df + *dg = \partial f + \overline{\partial} f + *(\partial g+\overline{\partial} g) = \partial(f-i\overline{g}) + \overline{\partial}(f+i\overline{g}). \qquad \square$$

Corollary 18.5. *Let X be a compact Riemann surface. Every exact harmonic 1-form on X vanishes. Every harmonic function on X is constant.*

Proof. By Lemma 18.4, $d\mathscr{E}(X)$ is orthogonal to $\mathscr{E}^1_{\mathrm{har}}(X)=\mathcal{O}^1(X)\oplus\overline{\mathcal{O}}^1(X)$. $\square$

Corollary 18.6. *Let X be a compact Riemann surface.*

(1) *Let $\sigma\in\mathscr{E}^1_{\mathrm{har}}(X)$. If $\int_\gamma\sigma=0$ for every closed curve γ in X, then $\sigma=0$.*
(2) *Let $\omega\in\mathcal{O}^1(X)$. If $\mathrm{Re}\int_\gamma\omega=0$ for every closed curve γ in X, then $\omega=0$.*

Proof. Corollary 9.4 implies that both σ and $\mathrm{Re}\,\omega$ are exact. Then the assertions follow from Theorem 18.3 and Corollary 18.5. $\square$

18.3. The Hodge–deRham theorem.

Theorem 18.7. *Let X be a compact Riemann surface. We have orthogonal decompositions*

$$\mathscr{E}^{0,1}(X) = \overline{\partial}\mathscr{E}(X)\oplus\overline{\mathcal{O}}^1(X) \tag{18.4}$$

and

$$\mathscr{E}^1(X) = *d\mathscr{E}(X)\oplus d\mathscr{E}(X)\oplus\mathscr{E}^1_{\mathrm{har}}(X). \tag{18.5}$$

Moreover,

$$\ker\big(d:\mathscr{E}^1(X)\to\mathscr{E}^2(X)\big) = d\mathscr{E}(X)\oplus\mathscr{E}^1_{\mathrm{har}}(X) \tag{18.6}$$

and

$$H^1(X,\mathbb{C})\cong \mathrm{Rh}^1(X)\cong\mathscr{E}^1_{\mathrm{har}}(X) \tag{18.7}$$

Proof. By Dolbeault's theorem 11.11, $\dim\big(\mathscr{E}^{0,1}(X)/\overline{\partial}\mathscr{E}(X)\big) = \dim H^1(X,\mathcal{O}) = g$ and, by (14.7), $\dim\overline{\mathcal{O}}^1(X)=\dim\mathcal{O}^1(X)=g$. So (18.4) follows from Lemma 18.4.

Applying complex conjugation to (18.4) yields $\mathscr{E}^{1,0}(X)=\partial\mathscr{E}(X)\oplus\mathcal{O}^1(X)$. Together with (18.3) this gives (18.5).

For (18.6), let $\mathscr{Z}(X) := \ker\big(d:\mathscr{E}^1(X)\to\mathscr{E}^2(X)\big)$. The inclusion $d\mathscr{E}(X)\oplus\mathscr{E}^1_{\mathrm{har}}(X)\subseteq\mathscr{Z}(X)$ is clear. For the other inclusion it suffices, by (18.5), to show $\mathscr{Z}(X)\perp *d\mathscr{E}(X)$. To this end let $\omega\in\mathscr{Z}(X)$ and $f\in\mathscr{E}(X)$. Then $\omega\wedge *(*df) = -\omega\wedge df = d(f\omega)$ and hence $\langle\omega, *df\rangle = \int_X d(f\omega)=0$, by Theorem 8.11.

Finally, (18.7) follows from deRham's theorem 11.12 and (18.6). $\square$

Corollary 18.8. *Let X be a compact Riemann surface.*

(1) *Let $\sigma\in\mathscr{E}^{0,1}(X)$. Then $\overline{\partial} f=\sigma$ has a solution $f\in\mathscr{E}(X)$ if and only if $\sigma\perp\overline{\mathcal{O}}^1(X)$.*

(2) *A* 1*-form* $\sigma \in \mathscr{E}^1(X)$ *is exact if and only if* $\int_X \sigma \wedge \omega = 0$ *for every closed* $\omega \in \mathscr{E}^1(X)$.

Proof. (1) follows from (18.4).

(2) The condition is equivalent to $\langle \omega, *\sigma \rangle = 0$ for every closed $\omega \in \mathscr{E}^1(X)$. By (18.5) and (18.6), this means that $*\sigma \in *d\mathscr{E}(X)$, or equivalently, $\sigma \in d\mathscr{E}(X)$. □

18.4. The genus is a topological invariant.

Corollary 18.9. *The genus of a compact Riemann surface* X *is a topological invariant.*

Proof. The sheaf $\mathbb{C}$ of locally constant complex valued functions on X depends only on the topology of X. Hence the first Betti number $b_1(X) := \dim H^1(X, \mathbb{C})$ is a topological invariant. By (18.7) and (18.2), $b_1(X) = 2g$. □

Remark 18.10. Every Riemann surface is a connected orientable two-dimensional smooth manifold; orientability follows from the fact that a holomorphic map between two subsets of the complex plane is orientation preserving.

There is a **topological classification of connected orientable compact two-dimensional manifolds** which depends only on the first Betti number $b_1(X) := \dim H^1(X, \mathbb{C})$ (see e.g. [**10**]): *every such surface* X *with* $b_1(X) = 2g$ *is homeomorphic to a sphere with* g *handles.* The number of handles is called the **topological genus** of the surface.

The dimension of $H^1(X, \mathcal{O})$ is sometimes referred to as the **arithmetic genus**; this is the definition we introduced in subsection 12.2. As a consequence of the Serre duality theorem 14.5 we found that the dimension of $H^1(X, \mathcal{O})$ equals the dimension of $\mathcal{O}^1(X)$. The latter is a priori an analytic invariant which depends very much on the complex structure, whence $\dim \mathcal{O}^1(X)$ is often called the **analytic genus** of X. *All three genera, the topological, the arithmetic, and the analytic genus, of a compact Riemann surface are equal.* In higher dimension this result generalizes to the so-called Hirzebruch–Riemann–Roch theorem.

Note that for every genus $g \geq 1$ there are Riemann surfaces which are homeomorphic but not biholomorphic; see e.g. [**1**] for a characterization of the holomorphic equivalence classes of Riemann surfaces of given genus.

19. Functions and forms with prescribed principal parts

Mittag–Leffler's theorem asserts that in the complex plane there always exists a meromorphic function with suitably prescribed principal parts. This is not always true on compact Riemann surfaces. In this section we explore necessary and sufficient conditions for a solution of the Mittag–Leffler problem based on the Serre duality theorem 14.5.

19.1. Mittag–Leffler distributions of meromorphic functions. In analogy to subsection 14.1 we define to Mittag–Leffler distributions of functions (instead of 1-forms): Let X be a Riemann surface. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. A cochain $\mu = (f_i) \in C^0(\mathfrak{U}, \mathscr{M})$ is called a **Mittag–Leffler distribution** if $\delta\mu \in Z^1(\mathfrak{U}, \mathcal{O})$, i.e., the differences $f_j - f_i$ are holomorphic on $U_i \cap U_j$. Then f_i and f_j have the same principal parts on the intersection of their domains. We denote by $[\delta\mu] \in H^1(\mathfrak{U}, \mathcal{O})$ the cohomology class of $\delta\mu$.

By a **solution** of μ we mean a meromorphic function $f \in \mathscr{M}(X)$ which has the same principal parts as μ, i.e., $f|_{U_i} - f_i \in \mathcal{O}(U_i)$ for all $i \in I$. Two solutions f_1, f_2 of μ differ by an additive constant, since $f_1 - f_2$ is holomorphic on X.

Proposition 19.1. *A Mittag–Leffler distribution μ has a solution if and only if $[\delta\mu] = 0$ in $H^1(\mathfrak{U}, \mathcal{O})$.*

Proof. Suppose that $f \in \mathscr{M}(X)$ is a solution of $\mu = (f_i)$. Then $g_i := f_i - f \in \mathcal{O}(U_i)$, and on $U_i \cap U_j$, we have $f_j - f_i = g_j - g_i$. So $\delta\mu = (f_j - f_i) \in B^1(\mathfrak{U}, \mathcal{O})$, i.e., $[\delta\mu] = 0$.

Conversely, assume that $\delta\mu = (f_j - f_i) \in B^1(\mathfrak{U}, \mathcal{O})$. Then there exists $(g_i) \in C^0(\mathfrak{U}, \mathcal{O})$ with $f_j - f_i = g_j - g_i$ on $U_i \cap U_j$. So there is $f \in \mathscr{M}(X)$ with $f|_{U_i} = f_i - g_i$, i.e., f is a solution of μ. □

Let X be a compact Riemann surface. Then $H^1(X, \mathscr{M}) = 0$, by Corollary 16.2. So for any $\xi \in H^1(X, \mathcal{O})$ there exists a Mittag–Leffler distribution $\mu \in C^0(\mathfrak{U}, \mathscr{M})$ with $\xi = [\delta\mu]$, for a suitable cover $\mathfrak{U}$. It follows that *on every compact Riemann surface of genus $g \geq 1$ there are Mittag–Leffler problems which have no solution.* On the other hand, $H^1(\widehat{\mathbb{C}}, \mathcal{O}) = 0$, by Theorem 10.11, and thus every Mittag–Leffler problem has a solution.

19.2. Criterion for solvability. If $\mu \in C^0(\mathfrak{U}, \mathscr{M})$ is a Mittag–Leffler distribution of meromorphic functions and $\omega \in \mathcal{O}^1(X)$ is any holomorphic 1-form on X, then the product $\omega\mu \in C^0(\mathfrak{U}, \mathscr{M}^1)$ is a Mittag–Leffler distribution of meromorphic 1-forms. By subsection 14.1, the residue $\operatorname{res}(\omega\mu)$ is defined.

Theorem 19.2. *Let X be a compact Riemann surface. Let $\mu \in C^0(\mathfrak{U}, \mathscr{M})$ be a Mittag–Leffler distribution of meromorphic functions. Then μ has a solution if and only if $\operatorname{res}(\omega\mu) = 0$ for every $\omega \in \mathcal{O}^1(X)$.*

Proof. $[\delta\mu] \in H^1(\mathfrak{U}, \mathcal{O})$ vanishes if and only if $\lambda([\delta\mu]) = 0$ for every $\lambda \in H^1(\mathfrak{U}, \mathcal{O})^*$. By the Serre duality theorem 14.5, this is the case if and only if $\langle \omega, [\delta\mu] \rangle = 0$ for every $\omega \in \mathcal{O}^1(X)$. By Theorem 14.1, $\langle \omega, [\delta\mu] \rangle = \operatorname{res}(\omega[\delta\mu]) = \operatorname{res}(\omega\mu)$. The theorem follows from Proposition 19.1. □

Clearly, $\operatorname{res}(\omega\mu) = 0$ for every $\omega \in \mathcal{O}^1(X)$ if and only if $\operatorname{res}(\omega_k\mu) = 0$ on a basis $\omega_1, \ldots, \omega_g$ of $\mathcal{O}^1(X)$.

Example 19.3 (complex tori, VI)**.** Let $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ be a lattice. Let $P = \{t_1\lambda_1 + t_2\lambda_2 : t_1, t_2 \in [0, 1)\}$. Suppose that at the points $a_1, \ldots, a_n \in P$, principal parts

$$\sum_{k=-r_j}^{-1} c_{j,k}(z - a_j)^k, \quad j = 1, \ldots, n,$$

are prescribed. *Then there exists an elliptic function with respect to Λ and having poles with the prescribed principal parts at the points $a_1, \ldots, a_n$ if and only if*

$$\sum_{j=1}^{n} c_{j,-1} = 0.$$

Indeed, the principal parts give rise to a Mittag–Leffler distribution μ on $\mathbb{C}/\Lambda$. The 1-form ω in $\mathbb{C}/\Lambda$ induced by dz on $\mathbb{C}$ (cf. Example 9.3) is a basis of $\mathcal{O}^1(\mathbb{C}/\Lambda)$. So the statement follows from Theorem 19.2.

19.3. Weierstrass points. We shall use the criterion in Theorem 19.2 to find conditions for the existence of functions on a compact Riemann surface which are holomorphic but at one point and have a pole of order $\leq g$ at that point. For instance, by Example 19.3, an elliptic function cannot have precisely one pole of order 1 in any period parallelogram.

Let X be a compact Riemann surface with genus g and let $p \in X$. Suppose that $\omega_1, \ldots, \omega_g$ is a basis of $\mathcal{O}^1(X)$. Let (U, z) be a coordinate neighborhood of p with $z(p) = 0$. Then we may expand ω_k about p:

$$\omega_k = \sum_{j=0}^{\infty} a_{kj} z^j \, dz, \quad k = 1, \ldots, g.$$

We look for a function f which has a principal part at p of the form

$$h = \sum_{j=0}^{g-1} \frac{c_j}{z^{j+1}}, \qquad \text{where at least one } c_j \neq 0.$$

Thus f is a solution of the Mittag–Leffler distribution $\mu = (h, 0) \in C^0(\mathfrak{U}, \mathscr{M})$, where $\mathfrak{U} = (U, X \setminus \{p\})$. Now

$$\operatorname{res}(\omega_k \mu) = \operatorname{res}_p(\omega_k h) = \sum_{j=0}^{g-1} a_{kj} c_j.$$

By Theorem 19.2, the solution f exists if and only if the system of linear equations $\operatorname{res}_p(\omega_k h) = 0$, $k = 1, \ldots, g$, has a non-trivial solution $c_0, \ldots, c_{g-1}$. This is the case if and only if

$$\det(a_{kj}) = 0. \tag{19.1}$$

We will express this condition in terms of the Wronskian determinant. Let $f_1, \ldots, f_g$ be holomorphic functions in a domain $U \subseteq \mathbb{C}$. The **Wronskian determinant** is defined by

$$W(f_1, \ldots, f_g) := \det \begin{pmatrix} f_1 & f_2 & \cdots & f_g \\ f_1' & f_2' & \cdots & f_g' \\ \vdots & \vdots & & \vdots \\ f_1^{(g-1)} & f_2^{(g-1)} & \cdots & f_g^{(g-1)} \end{pmatrix}$$

If $f_1, \ldots, f_g$ are linearly independent over $\mathbb{C}$, then $W(f_1, \ldots, f_g) \not\equiv 0$ (Exercise!).

Let X be a compact Riemann surface of genus $g \geq 1$ and let $\omega_1, \ldots, \omega_g$ be a basis of $\mathcal{O}^1(X)$. In a coordinate chart (U, z) we may write $\omega_k = f_k \, dz$. We define

$$W_z(\omega_1, \ldots, \omega_g) := W(f_1, \ldots, f_g)$$

with derivatives taken with respect to z.

Lemma 19.4. *Let (U, z) and (V, w) be two coordinate charts on X. On $U \cap V$ we have*

$$W_z(\omega_1, \ldots, \omega_g) = \Big(\frac{dw}{dz}\Big)^{\frac{g(g+1)}{2}} W_w(\omega_1, \ldots, \omega_g).$$

Proof. On $U \cap V$, we may write $\omega_k = f_k \, dz = g_k \, dw$. Then $f_k = g_k(dw/dz)$ and, by induction on m,

$$\frac{d^m f_k}{dz^m} = \Big(\frac{dw}{dz}\Big)^{m+1} \frac{d^m g_k}{dw^m} + \sum_{j=0}^{m-1} \varphi_{mj} \frac{d^j g_k}{dw^j},$$

where φ_{mj} are holomorphic functions on $U \cap V$ independent of k. Hence

$$\det \Big(\frac{d^m f_k}{dz^m}\Big)_{m=0,k=1}^{g-1,g} = \det \Big(\Big(\frac{dw}{dz}\Big)^{m+1} \frac{d^m g_k}{dw^m}\Big)_{m=0,k=1}^{g-1,g}$$

which implies the lemma. $\square$

Observe furthermore that if $\tilde{\omega}_1, \dots, \tilde{\omega}_g$ is another basis of $\mathcal{O}^1(X)$, then $(\omega_1, \dots, \omega_g) = C(\tilde{\omega}_1, \dots, \tilde{\omega}_g)$ for a matrix C with $\det(C) \neq 0$, and hence

$$W_z(\omega_1, \dots, \omega_g) = \det(C)\, W_z(\tilde{\omega}_1, \dots, \tilde{\omega}_g).$$

We say that a point $p \in X$ is a **Weierstrass point** if for a basis $\omega_1, \dots, \omega_g$ of $\mathcal{O}^1(X)$ and a coordinate neighborhood (U, z) of p, the Wronskian determinant $W_z(\omega_1, \dots, \omega_g)$ vanishes at p. The order of this zero is called the **weight** of the Weierstrass point. By definition, a Riemann surface of genus 0 does not have any Weierstrass points. This definition is meaningful by the lemma and the observation above.

Coming back to the arguments at the beginning of this section, it is clear that (19.1) is equivalent to

$$W_z(\omega_1, \dots, \omega_g)(p) = 0.$$

So we have proved the following theorem.

Theorem 19.5. *Let X be a compact Riemann surface with genus g and let $p \in X$. There exists a non-constant meromorphic function $f \in \mathscr{M}(X)$ which has a pole of order $\leq g$ at p and is holomorphic on $X \setminus \{p\}$ if and only if p is a Weierstrass point.*

We can even say how many Weierstrass points there are.

Theorem 19.6. *Let X be a compact Riemann surface with genus g. The number of Weierstrass points, counted according to their weights, is $(g-1)g(g+1)$.*

Proof. Let (U_i, z_i) be a cover of X by coordinate charts. On $U_i \cap U_j$, the function $\psi_{ij} := dz_j/dz_i$ is holomorphic and non-vanishing. Fix a basis $\omega_1, \dots, \omega_g$ of $\mathcal{O}^1(X)$ and let

$$W_i := W_{z_i}(\omega_1, \dots, \omega_g) \in \mathcal{O}(U_i).$$

By Lemma 19.4, we have

$$W_i = \psi_{ij}^{\frac{g(g+1)}{2}} W_j \quad \text{ on } U_i \cap U_j.$$

Let us set $D(x) := \operatorname{ord}_x(W_i)$ for $x \in U_i$. This defines a divisor D on X such that $\deg D$ is the number of Weierstrass points, counted according to their weights.

Let D_1 be the divisor of ω_1. By Corollary 14.8, $\deg D_1 = 2g - 2$. If we write $\omega_1 = f_{1i}\, dz_i$ on U_i, then $D_1(x) = \operatorname{ord}_x(f_{1i})$ for all $x \in U_i$. Since $f_{1i} = \psi_{ij} f_{1j}$ on $U_i \cap U_j$, we find

$$f_{1i}^{-\frac{g(g+1)}{2}} W_i = f_{1j}^{-\frac{g(g+1)}{2}} W_j \quad \text{ on } U_i \cap U_j.$$

Thus there exists a meromorphic function $f \in \mathscr{M}(X)$ with $f|_{U_i} = f_{1i}^{-\frac{g(g+1)}{2}} W_i$. The divisor of f satisfies $(f) = D - \frac{g(g+1)}{2} D_1$. Since $\deg(f) = 0$ (cf. subsection 13.2), we obtain

$$\deg D = \frac{g(g+1)}{2} \deg D_1 = (g-1)g(g+1).$$

The proof is complete. □

Corollary 19.7. *Every compact Riemann surface X for genus $g \geq 2$ admits a holomorphic covering map $f : X \to \widehat{\mathbb{C}}$ having at most g sheets. In particular, every compact Riemann surface of genus 2 is hyperelliptic.*

Proof. By Theorem 19.6, there exists a non-constant meromorphic function $f \in \mathscr{M}(X)$ with a single pole of order $\leq g$. Then f is the required covering map; it assumes ∞ with multiplicity $\leq g$ (cf. Theorem 3.19). □

Actually, any compact Riemann surface of genus $g \geq 2$ admits a covering of $\widehat{\mathbb{C}}$ with at most $(g+3)/2$ sheets; see [**11**].

19.4. Differential forms with prescribed principal parts. Let X be a Riemann surface, $\mathfrak{U} = (U_i)_{i \in I}$ an open cover of X, and $\mu = (\omega_i) \in C^0(\mathfrak{U}, \mathscr{M}^1)$ a Mittag–Leffler distribution of meromorphic 1-forms on X. A **solution** of μ is a meromorphic 1-form $\omega \in \mathscr{M}^1(X)$ which has the same principal parts as μ, i.e., $\omega|_{U_i} - \omega_i \in \mathcal{O}(U_i)$ for all $i \in I$.

Similarly as Proposition 19.1 one proves:

Proposition 19.8. *A Mittag–Leffler distribution $\mu \in C^0(\mathfrak{U}, \mathscr{M}^1)$ has a solution if and only if $[\delta\mu] = 0$ in $H^1(\mathfrak{U}, \mathcal{O}^1)$.*

Theorem 19.9. *Let X be a compact Riemann surface. A Mittag–Leffler distribution $\mu \in C^0(\mathfrak{U}, \mathscr{M}^1)$ on X has a solution if and only if* $\operatorname{res}(\mu) = 0$.

Proof. By Theorem 14.1, $\operatorname{res}(\mu) = \operatorname{res}([\delta\mu])$ and, by Corollary 14.7, the map $\operatorname{res} : H^1(\mathfrak{U}, \mathcal{O}^1) \to \mathbb{C}$ is an isomorphism. Thus, $\operatorname{res}(\mu) = 0$ if and only if $[\delta\mu] = 0$. □

Let X be a compact Riemann surface. Theorem 19.9 implies:

(1) For every $p \in X$ and every $n \geq 2$ there exists a meromorphic 1-form on X which has a pole of order n at p and is otherwise holomorphic. It is called an **elementary differential of second kind**.
(2) For any two points $p_1, p_2 \in X$ there exists a meromorphic 1-form on X which has poles of first order at p_1 and p_2 with residues 1 and -1, respectively, and is otherwise holomorphic. It is called an **elementary differential of third kind**.

1-forms that are everywhere holomorphic are called **elementary differential of first kind**.

20. Abel's theorem

In the complex plane the Weierstrass theorem guarantees the existence of a meromorphic function with prescribed zeros and poles. We already know that on a compact Riemann surface the total order of the zeros must equal the total order of the poles. For Riemann surfaces with genus $g \geq 1$ this condition is not sufficient. In Abel's theorem we shall find necessary and sufficient conditions for the existence of such functions. To prescribe zeros and poles with their orders is to prescribe the divisor of the function. So, in other words, we will give necessary and sufficient conditions for a divisor to be principal.

20.1. Meromorphic functions with prescribed divisors. Let X be a compact Riemann surface and $D \in \operatorname{Div}(X)$. We say that $f \in \mathscr{M}(X)$ is a **solution** of the divisor D if $(f) = D$. A necessary condition for this is that $\deg D = 0$.

Let $X_D := \{x \in X : D(x) \geq 0\}$. A **weak solution** of D is a function $f \in \mathscr{E}(X_D)$ such that for each $a \in X$ there is a coordinate neighborhood (U, z) with $z(a) = 0$ and a function $\psi \in \mathscr{E}(U)$ with $\psi(a) \neq 0$ such that

$$f = z^k \psi \quad \text{on } U \cap X_D, \quad \text{where } k = D(a). \tag{20.1}$$

Then a weak solution f of D is a solution of D if and only if $f \in \mathcal{O}(X_D)$. If f, g are two weak solutions of D, then there exists a non-vanishing function $\varphi \in \mathscr{E}(X)$ such that $f = \varphi g$.

If f_i is a weak solution of D_i, where $i = 1, 2$, then $f_1 f_2$ is a weak solution of $D_1 + D_2$ and f_1/f_2 is a weak solution of $D_1 - D_2$. It may happen that, for instance,

$D_1(a) + D_2(a) \geq 0$ but $D_1(a) < 0$ or $D_2(a) < 0$. Then formally $f_1 f_2$ is not defined at a, but it extends by continuity.

Suppose that f is a weak solution of D. The **logarithmic derivative** df/f is a smooth 1-form on the complement of the **support** $\operatorname{supp} D := \{x \in X : D(x) \neq 0\}$ of the divisor D. For $a \in \operatorname{supp} D$ and $k = D(a)$, (20.1) implies

$$\frac{df}{f} = k\frac{dz}{z} + \frac{d\psi}{\psi}, \tag{20.2}$$

where $d\psi/\psi$ is smooth in a neighborhood of a. For each 1-form $\sigma \in \mathscr{E}^1(X)$ with compact support the integral

$$\int_X \frac{df}{f} \wedge \sigma$$

exists; this can be easily checked in polar coordinates. Note that the 1-form $\overline{\partial} f/f$ is smooth on all of X, since (20.1) implies $\overline{\partial} f/f = \overline{\partial}\psi/\psi$.

Lemma 20.1. *Let X be a Riemann surface and let D be a divisor on X with* $\operatorname{supp} D = \{a_1, \ldots, a_n\}$. *Let f be a weak solution of D. Then for every $g \in \mathscr{E}(X)$ with compact support, we have*

$$\frac{1}{2\pi i}\int_X \frac{df}{f} \wedge dg = \sum_{j=1}^n D(a_j) g(a_j)$$

Proof. The exist disjoint coordinate neighborhoods (U_j, z_j) of the points a_j with $z_j(a_j) = 0$ such that on U_j

$$f = z_j^{k_j}\psi_j \quad \text{for non-vanishing } \psi_j \in \mathscr{E}(U_j) \text{ and } k_j := D(a_j).$$

We may assume that $z_j(U_j) = \mathbb{D} \subseteq \mathbb{C}$ for all j.

Let $0 < r_1 < r_2 < 1$. Choose functions $\varphi_j \in \mathscr{E}(X)$ with $\operatorname{supp}\varphi_j \subseteq \{|z_j| < r_2\}$ and $\varphi_j|_{\{|z_j| \leq r_1\}} = 1$. Set $g_j := \varphi_j g$, $j = 1, \ldots, n$, and $g_0 := g - (g_1 + \cdots + g_n)$. Then g_0 has compact support in $Y = X \setminus \{a_1, \ldots, a_n\}$. Thus, by Theorem 8.11,

$$\int_X \frac{df}{f} \wedge dg_0 = -\int_Y d\Big(g_0\frac{df}{f}\Big) = 0.$$

It follows that

$$\int_X \frac{df}{f} \wedge dg = \sum_{j=1}^n \int_{U_j} \frac{df}{f} \wedge dg_j = \sum_{j=1}^n k_j \int_{U_j} \frac{dz_j}{z_j} \wedge dg_j,$$

using (20.2) and again Theorem 8.11. By Stokes' theorem 8.10,

$$\begin{aligned}\int_{U_j} \frac{dz_j}{z_j} \wedge dg_j &= -\lim_{\epsilon\downarrow 0}\int_{\epsilon \leq |z_j| \leq r_2} d\Big(g_j\frac{dz_j}{z_j}\Big) \\ &= \lim_{\epsilon\downarrow 0}\int_{|z_j| = \epsilon} g_j\frac{dz_j}{z_j} = 2\pi i g_j(a_j) = 2\pi i g(a_j).\end{aligned}$$

The proof is complete. □

20.2. Homology. A **1-chain** on a Riemann surface X is a formal finite linear combination of curves $\gamma_j : [0,1] \to X$ with integer coefficients,

$$\gamma = \sum_{j=1}^k n_j\gamma_j, \quad n_j \in \mathbb{Z}.$$

The integral along γ of a closed 1-form $\omega \in \mathscr{E}^1(X)$ is defined by

$$\int_\gamma \omega := \sum_{j=1}^k n_j \int_{\gamma_j} \omega.$$

Let $C_1(X)$ denote the set of all 1-chains which is an abelian group in a natural way. We define the following boundary operator

$$\partial : C_1(X) \to \mathrm{Div}(X).$$

If $\gamma : [0,1] \to X$ is a closed curve, set $\partial\gamma = 0$. If γ is a curve which is not closed, let $\partial\gamma$ be the divisor which is 1 at $\gamma(1)$, -1 at $\gamma(0)$, and 0 everywhere else. For an arbitrary 1-chain $\gamma = \sum_{j=1}^k n_j\gamma_j$ set $\partial\gamma := \sum_{j=1}^k n_j \partial\gamma_j$. Then

$$\deg(\partial\gamma) = 0 \quad \text{for all } \gamma \in C_1(X).$$

On a compact Riemann surface X, for any divisor D with $\deg D = 0$ there exists a 1-chain γ such that $\partial\gamma = D$. Indeed, any divisor D with zero degree can be written as $D = D_1 + \cdots + D_k$, where each D_j is 1 at some point b_j, -1 at some point a_j, and 0 everywhere else. Then it suffices to choose curves γ_j from a_j to b_j and take $\gamma = \gamma_1 + \cdots + \gamma_k$.

The kernel $Z_1(X) := \ker(\partial : C_1(X) \to \mathrm{Div}(X))$ is called the group of **1-cycles**. Two cycles $\gamma_1, \gamma_2 \in Z_1(X)$ are said to be **homologous** if

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \quad \text{for all closed } \omega \in \mathscr{E}^1(X).$$

This defines an equivalence relation on $Z_1(X)$. The set of equivalence classes, called **homology classes**, is an additive group $H_1(X)$, the **first homology group** of X.

Since closed homotopic curves are also homologous (cf. Proposition 8.9), there is a group homomorphism $\pi_1(X) \to H_1(X)$. This map is surjective, but not in general injective, since $\pi_1(X)$ is not always abelian.

20.3. Existence of weak solutions of $\partial\gamma$.

Lemma 20.2. *Let X be a Riemann surface, $\gamma : [0,1] \to X$ a curve, and U a relatively compact open neighborhood of $\gamma([0,1])$. Then there exists a weak solution f of the divisor $\partial\gamma$ with $f|_{X\setminus U} = 1$ such that*

$$\int_\gamma \omega = \frac{1}{2\pi i}\int_X \frac{df}{f} \wedge \omega \quad \textit{for all closed } \omega \in \mathscr{E}^1(X).$$

Proof. The integral on the right-hand side exists, since $df/f = 0$ on $X \setminus U$.

Let us first assume that (U, z) is a coordinate chart in X with $z(U) = \mathbb{D}$ and $\gamma([0,1]) \subseteq U$. Let $a = \gamma(0)$ and $b = \gamma(1)$. There exists $r < 1$ such that $\gamma([0,1]) \subseteq \{|z| < r\}$. The function $z \mapsto \log(\frac{z-b}{z-a})$ has a well-defined branch in $\{r < |z| < 1\}$; see e.g. [**14**, Lemma 15.1]. Choose $\psi \in \mathscr{E}(U)$ such that $\psi|_{\{|z|\le r\}} = 1$ and $\psi|_{\{|z|\ge R\}} = 0$, where $r < R < 1$. Define $f_0 \in \mathscr{E}(U \setminus \{a\})$ by

$$f_0(z) := \begin{cases} \frac{z-b}{z-a} & \text{if } |z| \le r, \\ \exp\Big(\psi(z)\log\frac{z-b}{z-a}\Big) & \text{if } r < |z| < 1. \end{cases}$$

Then f_0 equals 1 on $\{R < |z| < 1\}$ and hence can be extended to a function $f \in \mathscr{E}(X \setminus \{a\})$ by setting $f|_{X\setminus U} = 1$. By construction, f is a weak solution of $\partial\gamma$. Let $\omega \in \mathscr{E}^1(X)$ be closed. Then ω has a primitive on U. So there exists $g \in \mathscr{E}(X)$ with compact support such that $dg = \omega$ on $\{|z| \le R\}$. By Lemma 20.1,

$$\frac{1}{2\pi i}\int_X \frac{df}{f} \wedge \omega = \frac{1}{2\pi i}\int_X \frac{df}{f} \wedge dg = g(b) - g(a) = \int_\gamma \omega.$$

In general there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and coordinate charts (U_j, z_j), $j = 1, \ldots, n$, such that $\gamma_j := \gamma|_{[t_{j-1},t_j]}$ lies in $U_j \subseteq U$ and $z_j(U_j) = \mathbb{D}$ for all j. By the special case there exist weak solutions f_j of $\partial\gamma_j$ such that $f_j|_{X\setminus U_j} = 1$ and

$$\int_{\gamma_j} \omega = \frac{1}{2\pi i}\int_X \frac{df_j}{f_j} \wedge \omega \quad \text{for all closed } \omega \in \mathscr{E}^1(X).$$

Then $f := f_1 \cdots f_n$ is as desired. □

20.4. Abel's theorem.

Theorem 20.3 (Abel's theorem)**.** *Let D be a divisor on a compact Riemann surface X with* $\deg D = 0$*. Then D has a solution if and only if there exists a 1-chain $\gamma \in C_1(X)$ with $\partial\gamma = D$ such that*

$$\int_\gamma \omega = 0 \quad \textit{for all } \omega \in \mathcal{O}^1(X). \tag{20.3}$$

It is clearly enough to check (20.3) on a basis of $\mathcal{O}^1(X)$. If $\tilde\gamma \in C_1(X)$ is an arbitrary 1-chain with $\partial\tilde\gamma = D$ then (20.3) can be formulated as follows: there exists a cycle $\alpha \in Z_1(X)$ (namely $\alpha = \tilde\gamma - \gamma$) such that

$$\int_{\tilde\gamma} \omega_i = \int_\alpha \omega_i \quad \text{for a basis } \omega_1, \ldots, \omega_g \text{ of } \mathcal{O}^1(X). \tag{20.4}$$

Proof. Let $\gamma \in C_1(X)$ with $\partial\gamma = D$ such that (20.3) holds. By Lemma 20.2, there is a weak solution f of D such that

$$\int_\gamma \omega = \frac{1}{2\pi i}\int_X \frac{df}{f} \wedge \omega \quad \text{for all closed } \omega \in \mathscr{E}^1(X).$$

By (20.3), for every $\omega \in \mathcal{O}^1(X)$,

$$0 = \int_\gamma \omega = \frac{1}{2\pi i}\int_X \frac{df}{f} \wedge \omega = \frac{1}{2\pi i}\int_X \frac{\overline{\partial} f}{f} \wedge \omega.$$

Note that $\overline{\partial} f/f \in \mathscr{E}^{0,1}(X)$, as explained before Lemma 20.1. By Corollary 18.8, there exists $g \in \mathscr{E}(X)$ such that $\overline{\partial} g = \overline{\partial} f/f$. Then $F := e^{-g} f$ is a weak solution of D and

$$\overline{\partial} F = -e^{-g} f \overline{\partial} g + e^{-g}\overline{\partial} f = 0.$$

This implies that $F \in \mathcal{O}(X_D)$ and hence it is a meromorphic solution of D.

Now let $f \in \mathscr{M}(X)$ be a solution of D. We may assume that $D \neq 0$. The function f defines an n-sheeted branched covering $f : X \to \widehat{\mathbb{C}}$ for some positive integer n. Let $a_1, \ldots, a_r \in X$ be the branch points and set $Y := \widehat{\mathbb{C}} \setminus \{f(a_1), \ldots, f(a_r)\}$. Each $y \in Y$ has an open neighborhood V such that $f^{-1}(V)$ is a disjoint union of open sets $U_1, \ldots, U_n$ and all the maps $f|_{U_j} : U_j \to V$ are biholomorphic with inverse $\varphi_j := f|_{U_j}^{-1}$. Given a holomorphic 1-form $\omega \in \mathcal{O}^1(X)$ consider the 1-form

$$\varphi_1^*\omega + \cdots + \varphi_n^*\omega$$

on V. If we repeat the same construction on an open neighborhood of an other point in Y, then on the intersection we obtain the same 1-form. We thus obtain a holomorphic 1-from $\operatorname{tr}(\omega)$ on all of $\widehat{\mathbb{C}}$, similarly as in Theorem 7.1. Since $\mathcal{O}^1(\widehat{\mathbb{C}}) = 0$ (see (14.7)), we have $\operatorname{tr}(\omega) = 0$.

Let σ be a curve in $\widehat{\mathbb{C}}$ from ∞ to 0 which lies in Y except possibly the endpoints. The preimage under f of σ consists of n curves $\gamma_1, \ldots, \gamma_n$ which join poles of f with zeros of f. Thus $\gamma = \gamma_1 + \cdots + \gamma_n$ satisfies $\partial\gamma = D$ and

$$\int_\gamma \omega = \int_\sigma \operatorname{tr}(\omega) = 0$$

for all $\omega \in \mathcal{O}^1(X)$. □

Example 20.4 (complex tori, VII)**.** Let $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ be a lattice and let $P = \{t_1\lambda_1 + t_2\lambda_2 : t_1, t_2 \in [0,1)\}$. Suppose zeros $a_1, \ldots, a_n \in P$ and poles $b_1, \ldots, b_n \in P$ are prescribed, where each zero and each pole appears as often as its multiplicity demands. We claim that *there exists a Λ-elliptic function with zeros $a_1, \ldots, a_n$ and poles $b_1, \ldots, b_n$ if and only if*

$$\sum_{k=1}^{n} (a_k - b_k) \in \Lambda.$$

For, let D be the divisor on $\mathbb{C}/\Lambda$ determined by the prescribed zeros and poles. Choose curves σ_k from b_k to a_k in $\mathbb{C}$. Then, if $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the canonical projection,

$$\gamma := \pi \circ \sigma_1 + \cdots + \pi \circ \sigma_n$$

is a 1-chain in $\mathbb{C}/\Lambda$ satisfying $\partial\gamma = D$. Let ω be the holomorphic 1-form on $\mathbb{C}/\Lambda$ induced by dz on $\mathbb{C}$ (cf. Example 9.3). Then

$$\int_\gamma \omega = \sum_{k=1}^{n} \int_{\sigma_k} dz = \sum_{k=1}^{n} (a_k - b_k).$$

The statement follows from Abel's theorem 20.3.

21. The Jacobi inversion problem

In Abel's theorem we found necessary and sufficient conditions for a divisor to be principal. In this section we will study the quotient group of divisors with degree zero modulo the subgroup of principal divisors. We shall see that this group is isomorphic to a g-dimensional torus, where g is the genus of the underlying Riemann surface.

21.1. n-dimensional lattices. Let V be an n-dimensional vector space over $\mathbb{R}$. An additive subgroup $\Lambda \subseteq V$ is called a **lattice** if there exist n linearly independent vectors $\lambda_1, \ldots, \lambda_n \in V$ such that $\Lambda = \mathbb{Z}\lambda_1 + \cdots \mathbb{Z}\lambda_n$.

Proposition 21.1. *An additive subgroup $\Lambda \subseteq V$ of a vector space V is a lattice if and only if:*

(1) *Λ is discrete.*
(2) *Λ is contained in no proper subspace of V.*

Proof. The necessity is clear. So let $\Lambda \subseteq V$ satisfy (1) and (2). We will show that there exist $n = \dim V$ linearly independent vectors $\lambda_1, \ldots, \lambda_n$ such that $\Lambda = \mathbb{Z}\lambda_1 + \cdots \mathbb{Z}\lambda_n$. We use induction on n. The statement for $n = 0$ is trivial. Let $n > 0$. By (2) there exist n linearly independent vectors $x_1, \ldots, x_n \in \Lambda$. Consider $V_1 := \operatorname{span}(x_1, \ldots, x_{n-1})$ and $\Lambda_1 := \Lambda \cap V_1$. By the induction hypothesis, there exist $n-1$ linearly independent vectors $y_1, \ldots, y_{n-1} \in \Lambda_1$ such that $\Lambda_1 = \mathbb{Z}y_1 + \cdots \mathbb{Z}y_{n-1}$.

Every $x \in \Lambda$ can be written uniquely as

$$x = c_1(x)y_1 + \cdots c_{n-1}(x)y_{n-1} + c(x)x_n, \quad c_j(x), c(x) \in \mathbb{R}.$$

Consider the compact parallelotope $P = \{t_1 y_1 + \cdots t_{n-1}y_{n-1} + t x_n : t_j, t \in [0,1]\}$. By (1), $\Lambda \cap P$ is finite. Then there exists a vector $y_n \in (\Lambda \cap P) \setminus V_1$ such that $c(y_n) = \min\{c(x) : x \in (\Lambda \cap P) \setminus V_1\}$. Clearly, $0 < c(y_n) \leq 1$.

Let us show that $\Lambda = \Lambda_1 + \mathbb{Z}y_n$. Let $x \in \Lambda$. Then there exist $k_j \in \mathbb{Z}$ such that

$$x' := x - \sum_{j=1}^{n} k_j y_j = \sum_{j=1}^{n-1} t_j y_j + t x_n,$$

where $0 \le t_j < 1$, for $j = 1, \dots, n-1$, and $0 \le t < c(y_n)$. But $x' \in \Lambda \cap P$ and hence $t = 0$. Consequently, $x' \in \Lambda_1$, whence all t_j are integers and thus zero. So $x' = 0$ and the assertion follows. □

21.2. Period lattices. Let X be a compact Riemann surface of genus $g \ge 1$ and let $\omega_1, \dots, \omega_g$ be a basis of $\mathcal{O}^1(X)$. The **period lattice** of X relative to the basis $\omega_1, \dots, \omega_g$ is the subgroup of $\mathbb{C}^g$ defined by (cf. subsection 20.2)

$$\begin{aligned}\operatorname{Per}(\omega_1, \dots, \omega_g) &:= \Big\{ \Big(\int_\alpha \omega_1, \dots, \int_\alpha \omega_g \Big) \in \mathbb{C}^g : \alpha \in \pi_1(X) \Big\} \\ &= \Big\{ \Big(\int_\alpha \omega_1, \dots, \int_\alpha \omega_g \Big) \in \mathbb{C}^g : \alpha \in H_1(X) \Big\}.\end{aligned}$$

To see that $\operatorname{Per}(\omega_1, \dots, \omega_g)$ is a lattice we need the following lemma.

Lemma 21.2. *Let X be a compact Riemann surface of genus g. There exist g distinct points $a_1, \dots, a_g \in X$ with the following property: if $\omega \in \mathcal{O}^1(X)$ vanishes at all a_j then $\omega = 0$.*

Proof. For any $a \in X$ consider the set $H_a := \{\omega \in \mathcal{O}^1(X) : \omega(a) = 0\}$. Each H_a either coincides with $\mathcal{O}^1(X)$ or has codimension one. Since $\bigcap_{a \in X} H_a = \{0\}$ and $\dim \mathcal{O}^1(X) = g$, there exist g points $a_1, \dots, a_g \in X$ such that $H_{a_1} \cap \dots \cap H_{a_g} = \{0\}$. The lemma follows. □

Proposition 21.3. $\operatorname{Per}(\omega_1, \dots, \omega_g)$ *is a lattice in* $\mathbb{C}^g \cong \mathbb{R}^{2g}$.

Proof. Let $a_1, \dots, a_g \in X$ be the points provided by Lemma 21.2. Choose simply connected coordinate neighborhoods (U_j, z_j) of a_j with $z_j(a_j) = 0$ for all $j = 1, \dots, g$. With respect to these coordinates

$$\omega_i = \varphi_{ij}\, dz_j \quad \text{ on } U_j.$$

By Lemma 21.2, the matrix $A := (\varphi_{ij}(a_j))_{1 \le i,j \le g}$ has rank g.

We define a map $F : U_1 \times \dots \times U_g \to \mathbb{C}^g$ as follows:

$$F(x) = (F_1(x), \dots, F_g(x)) \quad \text{ where } \quad F_i(x) := \sum_{j=1}^{g} \int_{a_j}^{x_j} \omega_i. \tag{21.1}$$

Here the integral $\int_{a_j}^{x_j} \omega_i$ is along any curve in U_j from a_j to x_j (U_j is simply connected!). Then F is complex differentiable with respect to $x_1, \dots, x_g$ and has Jacobian matrix $J_F(x) = (dF_i(x)/dx_j) = (\varphi_{ij}(x_j))$. So $J_F(a_1, \dots, a_g) = A$ is invertible. It follows that $W := F(U_1 \times \dots \times U_g)$ is a neighborhood of $F(a_1, \dots, a_g) = 0 \in \mathbb{C}^g$.

We claim that $\Lambda \cap W = \{0\}$, where $\Lambda := \operatorname{Per}(\omega_1, \dots, \omega_g)$, which implies that Λ is a discrete subgroup of $\mathbb{C}^g$. Suppose that there exists a point except 0 in $\Lambda \cap W$. Then there exists $x \in U_1 \times \dots \times U_g$, $x \ne a$, such that $F(x) \in \Lambda$. Renumbering if necessary, we may assume that $x_j \ne a_j$ for $1 \le j \le k$ and $x_j = a_j$ for $j > k$, where $1 \le k \le g$. By Abel's theorem 20.3 and (20.4), there exists a meromorphic function f on X with a pole of first order at a_j, $1 \le j \le k$, a zero of first order at x_j, $1 \le j \le k$, and is holomorphic otherwise (since $F(x) \in \Lambda$). Let c_j/z_j be the principal part of f at a_j; then $c_j \neq 0$ for $1 \leq j \leq k$. By the residue theorem 8.12,

$$0 = \operatorname{res}(f\omega_i) = \sum_{j=1}^{k} c_j \varphi_{ij}(a_j) \quad \text{for } i = 1, \dots, g,$$

which contradicts the fact that $A = (\varphi_{ij}(a_j))$ has rank g. The claim is proved.

To finish the proof we show that Λ is not contained in any proper real linear subspace of $\mathbb{C}^g$. Otherwise, there would be a real non-trivial linear form on $\mathbb{C}^g$ vanishing on Λ. Since every real linear form is the real part of a complex linear form, there is a non-zero vector $(c_1, \dots, c_g) \in \mathbb{C}^g$ such that

$$\operatorname{Re}\Big(\sum_{j=1}^{g} c_j \int_\alpha \omega_j\Big) = 0 \quad \text{for all } \alpha \in \pi_1(X).$$

This would imply $c_1\omega_1 + \cdots + c_g\omega_g = 0$, by Corollary 18.6, a contradiction. □

Corollary 21.4. *Let X be a compact Riemann surface of genus $g \geq 1$. Then $H_1(X) \cong \mathbb{Z}^{2g}$.*

Proof. By Proposition 21.3, there exist $2g$ closed curves $\alpha_1, \dots, \alpha_{2g}$ in X such that the vectors

$$\lambda_j = \Big(\int_{\alpha_j} \omega_1, \dots, \int_{\alpha_j} \omega_g\Big), \quad j = 1, \dots, 2g,$$

are linearly independent over $\mathbb{R}$ and

$$\operatorname{Per}(\omega_1, \dots, \omega_g) = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_{2g}.$$

It follows that the homology classes of the α_j in $H_1(X)$ are linearly independent over $\mathbb{Z}$ and generate $H_1(X)$. The statement follows. □

21.3. The Jacobi variety and the Picard group. Let X be a compact Riemann surface of genus g and let $\omega_1, \dots, \omega_g$ be a basis of $\mathcal{O}^1(X)$. Then

$$\operatorname{Jac}(X) = \mathbb{C}^g / \operatorname{Per}(\omega_1, \dots, \omega_g)$$

is called the **Jacobi variety** of X. It is an abelian group and has the structure of a g-dimensional complex torus. The definition depends on the basis $\omega_1, \dots, \omega_g$, but a different basis yields an isomorphic $\operatorname{Jac}(X)$.

Let $\operatorname{Div}_0(X) \subseteq \operatorname{Div}(X)$ be the subgroup of divisors of degree 0 and $\operatorname{Div}_p(X) \subseteq \operatorname{Div}_0(X)$ the subgroup of principal divisors. The quotient

$$\operatorname{Pic}(X) := \operatorname{Div}(X) / \operatorname{Div}_p(X)$$

is the **Picard group** of X. We shall be primarily interested in the subgroup

$$\operatorname{Pic}_0(X) := \operatorname{Div}_0(X) / \operatorname{Div}_p(X).$$

Since $\operatorname{Div}(X)/\operatorname{Div}_0(X) = \mathbb{Z}$, we have an exact sequence

$$0 \to \operatorname{Pic}_0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0.$$

Next we define a map $\Phi : \operatorname{Div}_0(X) \to \operatorname{Jac}(X)$ as follows. Let $D \in \operatorname{Div}_0(X)$ and let $\gamma \in C_1(X)$ be a chain with $\partial\gamma = D$. Then the vector

$$\Big(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g\Big) \in \mathbb{C}^g$$

is determined uniquely by D up to equivalence modulo $\operatorname{Per}(\omega_1, \dots, \omega_g)$. By definition, $\Phi(D)$ is its equivalence class. Note that Φ is a group homomorphism.

Abel's theorem 20.3 states that $\ker \Phi = \operatorname{Div}_p(X)$, whence we get an injective map

$$j : \operatorname{Pic}_0(X) \to \operatorname{Jac}(X).$$

The **Jacobi inversion problem** asks if this map is also surjective.

Theorem 21.5. *The map* $j : \mathrm{Pic}_0(X) \to \mathrm{Jac}(X)$ *is an isomorphism for every compact Riemann surface* X*.*

Proof. Let $p \in \mathrm{Jac}(X)$ be represented by $\xi \in \mathbb{C}^g$. If $N \in \mathbb{N}$ is large enough, then $N^{-1}\xi$ lies in the image of the map F from (21.1) (in fact, we saw that the image of F is a neighborhood of 0 in $\mathbb{C}^g$). So there exist points $a_j, x_j \in X$ and curves γ_j from a_j to x_j such that, for $\gamma = \gamma_1 + \cdots + \gamma_g$,

$$\Big(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g\Big) = \frac{1}{N}\xi.$$

Thus, for the divisor $D = \partial\gamma$,

$$\Phi(D) = \frac{1}{N}\xi \mod \mathrm{Per}(\omega_1, \dots, \omega_g).$$

If θ is the element of $\mathrm{Pic}_0(X)$ represented by the divisor ND, then $j(\theta) = p$. □

21.4. A sharper version. Let X be a compact Riemann surface of genus g and let $a_1, \dots, a_g \in X$ be arbitrarily chosen points. We define a map $\psi : X^g \to \mathrm{Pic}_0(X)$ as follows. Let D_x, for $x \in X$, be the divisor defined by $D_x(x) = 1$ and $D_x(y) = 0$ if $y \neq x$. For $(x_1, \dots, x_g) \in X^g$ set

$$\psi(x_1, \dots, x_g) := \sum_{j=1}^{g} (D_{x_j} - D_{a_j}) \mod \mathrm{Div}_p(X).$$

Let $J := j \circ \psi : X^g \to \mathrm{Jac}(X)$. Then

$$J(x_1, \dots, x_g) = \Big(\sum_{j=1}^{g} \int_{a_j}^{x_j} \omega_1, \dots, \sum_{j=1}^{g} \int_{a_j}^{x_j} \omega_g\Big) \mod \mathrm{Per}(\omega_1, \dots, \omega_g).$$

Theorem 21.6. *The map* $J : X^g \to \mathrm{Jac}(X)$ *is surjective.*

Proof. By Theorem 21.5, it suffices to prove that $\psi : X^g \to \mathrm{Pic}_0(X)$ is surjective. This means that every divisor $D \in \mathrm{Div}_0(X)$ is equivalent modulo $\mathrm{Div}_p(X)$ to a divisor of the form $\sum_{j=1}^{g}(D_{x_j} - D_{a_j})$ for $(x_1, \dots, x_g) \in X^g$.

Let $D' := D + D_{a_1} + \cdots + D_{a_g}$. Then $\deg D' = g$. By the Riemann–Roch theorem 13.4, $\dim H^0(X, \mathscr{L}_{D'}) \geq 1$, and hence there exists a non-trivial meromorphic function f on X with $D'' := (f) + D' \geq 0$. Since $\deg D'' = g$, there exist points $x_1, \dots, x_g \in X$ with $D'' = D_{x_1} + \cdots + D_{x_g}$. Then

$$\sum_{j=1}^{g} (D_{x_j} - D_{a_j}) = D'' - D' + D = (f) + D$$

as desired. □

Remark 21.7. It is obvious that $J(x_1, \dots, x_g)$ is invariant under permutations of $x_1, \dots, x_g$. Thus J induces a map $S^gX \to \mathrm{Jac}(X)$, where S^gX is the g-fold symmetric product of X. It carries the structure of a compact complex g-dimensional manifold and it turns out that the map $S^gX \to \mathrm{Jac}(X)$ is holomorphic. It is not bijective, but it is bimeromorphic, i.e., it induces an isomorphism between the fields of meromorphic functions of $\mathrm{Jac}(X)$ and S^gX; see [**5**].

21.5. Riemann surfaces of genus one.

Theorem 21.8. *The map* $J : X \to \mathrm{Jac}(X)$ *is an isomorphism for every compact Riemann surface of genus one.*

Proof. Let $\omega \in \mathcal{O}^1(X)$ be non-trivial. Let $a \in X$. For all $x \in X$, we have

$$J(x) = \int_a^x \omega \mod \mathrm{Per}(\omega).$$

Clearly, J is holomorphic and surjective (by Theorem 21.6 or Corollary 1.10). Suppose that there exists $y \neq x$ such that $J(y) = J(x)$. Then there exists a 1-cycle $\alpha \in Z_1(X)$ with

$$\int_a^y \omega = \int_a^x \omega + \int_\alpha \omega.$$

By Abel's theorem 20.3, this would imply the existence of a meromorphic function f on X having a single pole of order one. In that case X would be isomorphic to $\widehat{\mathbb{C}}$, a contradiction. □

Corollary 21.9. *The Riemann surfaces of genus one are precisely the complex tori* $\mathbb{C}/\Lambda$.

Proof. Theorem 21.8 and Corollary 14.9. □

CHAPTER 6

Non-compact Riemann surfaces

The function theory on non-compact Riemann surfaces has many similarities with the one on regions of the complex plane. We shall see that there are analogues of Runge's theorem, the Mittag–Leffler theorem, Weierstrass' theorem, and the Riemann mapping theorem.

22. The Dirichlet problem

In this section we consider the solution of the Dirichlet problem on Riemann surfaces. We assume familiarity with the Dirichlet problem in the complex plane and its solution by Perron's method. The extension to Riemann surfaces requires very little additional effort. For this reason we will most of the time just state the results; full details may be found in [**4**].

22.1. Harmonic functions and the Dirichlet problem. Let Y be an open subset of a Riemann surface X. Then $u \in \mathscr{E}(Y)$ is **harmonic** if $\partial\overline{\partial}u = 0$. With respect to a local coordinate $z = x + iy$ this holds if and only if

$$\Delta u = (\partial_x^2 + \partial_y^2)u = 4\partial_z\partial_{\overline{z}}u = 0.$$

Every real-valued harmonic function u on a simply connected, connected open subset Y of X is the real part of a holomorphic function $f \in \mathcal{O}(Y)$. Indeed, by Theorem 18.3, $du = \mathrm{Re}(dg)$ for some $g \in \mathcal{O}(Y)$ and so $u = \mathrm{Re}(g) + \text{const}$.

This allows to deduce the maximum principle for harmonic functions from the maximum principle for holomorphic functions: if a harmonic function $u : Y \to \mathbb{R}$ attains its maximum at a point of the connected open set $Y \subseteq X$, then u is constant.

The **Dirichlet problem** on a Riemann surface X is the following. Let Y be an open subset of X and $f : \partial Y \to \mathbb{R}$ a continuous function. Find a continuous function $u : \overline{Y} \to \mathbb{R}$ which is harmonic on Y and satisfies $u|_{\partial Y} = f$. Suppose that $\overline{Y}$ is compact and $\partial Y \neq \emptyset$. The maximum principle implies that, if a solution exists, then it is unique.

22.2. Harmonic functions on domains in $\mathbb{C}$. For the disk $D_R(0) \subseteq \mathbb{C}$ the Dirichlet problem is solved by the Poisson integral:

Theorem 22.1. *Let $f : \partial D_R(0) \to \mathbb{R}$ be continuous. Then the function defined by*

$$u(z) := \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\frac{R^2 - |z|^2}{|Re^{it} - z|^2}\,dt, \qquad \text{for } z \in D_R(0), \tag{22.1}$$

and $u(z) := f(z)$, for $z \in \partial D_R(0)$, is continuous on $\overline{D}_R(0)$ and harmonic in $D_R(0)$.

Let us state some further results; proofs may be found e.g. in [**14**].

Proposition 22.2 (mean value property)**.** *Let $u : U \to \mathbb{R}$ be harmonic on a domain $U \subseteq \mathbb{C}$, and let $\overline{D}_r(a) \subseteq U$. Then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\,dt.$$

Conversely, we have the following

Theorem 22.3. *Let $U \subseteq \mathbb{C}$ be a domain, and let $f : U \to \mathbb{R}$ be continuous with the following property: for each $a \in U$ there is $r_a > 0$ such that $\overline{D}_{r_a}(a) \subseteq U$ and for every $0 < r < r_a$*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt.$$

Then f is harmonic.

Corollary 22.4. *If $u_n : U \to \mathbb{R}$ is a sequence of harmonic functions which converges uniformly on compact set to $u : U \to \mathbb{R}$, then u is harmonic.*

Theorem 22.5 (Harnack's principle)**.** *Let $u_1 \leq u_2 \leq \cdots$ be harmonic functions on a region $U \subseteq \mathbb{C}$. Then either $u_n \to \infty$ uniformly on compact sets or there is a harmonic function u on U and $u_n \to u$ uniformly on compact sets.*

22.3. Solution of the Dirichlet problem. Let X be a Riemann surface. Harmonicity of a function remains invariant under biholomorphic maps. Thus the Dirichlet problem can be solved on all domains $D \subseteq X$ which are relatively compact and contained in a chart (U, z) so that $z(D) \subseteq \mathbb{C}$ is a disk.

Let $Y \subseteq X$ be an open subset. We denote by $\mathrm{Reg}(Y)$ the set of all subdomains $D \Subset Y$ such that the Dirichlet problem can be solved on D for all continuous boundary values $f : \partial D \to \mathbb{R}$.

For $u \in C(Y, \mathbb{R})$ and $D \in \mathrm{Reg}(Y)$ let $P_D u$ be the continuous function on Y which coincides with u on $Y \setminus D$ and solves the Dirichlet problem on $\overline{D}$ for the boundary values $u|_{\partial D}$. Note that a function $u \in C(Y, \mathbb{R})$ is harmonic if and only if $P_D u = u$ for all $D \in \mathrm{Reg}(Y)$. A function $u \in C(Y, \mathbb{R})$ is said to be **subharmonic** if $P_D u \geq u$ for all $D \in \mathrm{Reg}(Y)$.

A point $x \in \partial Y$ is called **regular** or a **peak point** if there is an open neighborhood U of x in X and a function $\beta \in C(\overline{Y} \cap U, \mathbb{R})$ such that:

(1) $\beta|_{Y \cap U}$ is subharmonic.
(2) $\beta(x) = 0$ and $\beta(y) < 0$ for all $y \in (\overline{Y} \cap U) \setminus \{x\}$.

Then β is called a **peaking function**. If all boundary points of Y are peak points, then we say that Y has **regular boundary**. For later reference we observe:

Lemma 22.6. *If $x \in \partial Y$ is a peak point of Y and Y_1 is an open subset of Y with $x \in \partial Y_1$, then x is a peak point of Y_1. In particular, if Y has regular boundary, then so does every connected component of Y.*

Proof. This is clear by the definition of peak point. □

Theorem 22.7 (solution of the Dirichlet problem)**.** *Let Y be an open subset of a Riemann surface X such that all boundary points of Y are peak points. Then for every continuous bounded function $f : \partial Y \to \mathbb{R}$ the Dirichlet problem on Y has a solution.*

Let us give a simple geometric condition which implies that a boundary point is a peak point. Since being a peak point is a local condition invariant under biholomorphic maps, we can formulate this condition for $Y \subseteq \mathbb{C}$. See also [**14**, Theorem 29.9].

Theorem 22.8. *Let $Y \subseteq \mathbb{C}$ be a domain. A point $a \in \partial Y$ is a peak point if there is a disk D such that $a \in \partial D$ and $\overline{D} \cap Y = \emptyset$.*

Proof. Let $D = D_r(c)$ and let $b = (a + c)/2$.

$$\beta(z) := \log \frac{r}{2} - \log |z - b|$$

defines a peaking function at a. □

23. Radó's theorem

We prove Radó's theorem that every Riemann surface has a countable topology. (Note that this is trivial for compact Riemann surfaces.)

Lemma 23.1. *Let X, Y be topological spaces and let $f : X \to Y$ be continuous, open, and surjective. If X has a countable topology, then so does Y.*

Proof. Let $\mathfrak{U}$ be a countable basis for the topology on X. We claim that the countable family $\mathfrak{V} := \{f(U) : U \in \mathfrak{U}\}$ of open sets in Y is a basis for the topology on Y. Let W be an open subset of Y and $y \in W$. We must show that there exists $V \in \mathfrak{V}$ such that $y \in V \subseteq W$. There is $x \in X$ with $f(x) = y$ and $f^{-1}(W)$ is an open neighborhood of x. So there exists $U \in \mathfrak{U}$ with $x \in U \subseteq f^{-1}(W)$. Then $V := f(U)$ satisfies $y \in V \subseteq W$. □

Lemma 23.2 (Poincaré–Volterra). *Let X be a connected manifold, Y a Hausdorff space with countable topology, and let $f : X \to Y$ be a continuous discrete map. Then X has a countable topology.*

Proof. Let $\mathfrak{V}$ be a countable basis for the topology of Y. Let $\mathfrak{U}$ be the collection of all open subsets U of X with the following properties:

(1) U has a countable topology.
(2) U is the connected component of a set $f^{-1}(V)$ with $V \in \mathfrak{V}$.

Then $\mathfrak{U}$ is a basis for the topology of X. For, let D be open in X with $x \in D$. We must show that there is $U \in \mathfrak{U}$ with $x \in U \subseteq D$. Since f is discrete, we find a relatively compact open neighborhood $W \subseteq D$ of x such that $\partial W \cap f^{-1}(f(x)) = \emptyset$. Then $f(\partial W)$ is compact, hence closed, and does not contain $f(x)$. So there exists $V \in \mathfrak{V}$ with $f(x) \in V$ and $V \cap f(\partial W) = \emptyset$. Let U be the connected component of $f^{-1}(V)$ which contains x. Since U does not meet ∂W, we have $U \subseteq W$. Then U has a countable topology. Thus, $U \in \mathfrak{U}$, and the claim is proved.

In order to see that $\mathfrak{U}$ is countable, we first check that for every $U_0 \in \mathfrak{U}$ there are at most countably many $U \in \mathfrak{U}$ with $U_0 \cap U \neq \emptyset$. For each $V \in \mathfrak{V}$ the connected components of $f^{-1}(V)$ are disjoint. Since U_0 has countable topology, U_0 can only meet countably many of these components. So the assertion follows from the fact that also $\mathfrak{V}$ is countable.

Let us now show that $\mathfrak{U}$ is countable. Fix $U_0 \in \mathfrak{U}$. For all $n \in \mathbb{N}$ let $\mathfrak{U}_n$ be the collection of all $U \in \mathfrak{U}$ such that there exist $U_1, \ldots, U_n \in \mathfrak{U}$ with $U_n = U$ and $U_{k-1} \cap U_k \neq \emptyset$ for all $k = 1, \ldots, n$. Since X is connected, $\mathfrak{U} = \bigcup_{n \in \mathbb{N}} \mathfrak{U}_n$. Each $\mathfrak{U}_n$ is countable which can be seen by induction using the observation of the previous paragraph. □

Theorem 23.3 (Radó's theorem). *Every Riemann surface has a countable topology.*

Proof. Let U be a coordinate neighborhood in X. Let K_0 and K_1 be two disjoint compact disks in U. Set $Y := X \setminus (K_0 \cup K_1)$. By Theorem 22.7 and Theorem 22.8, there exists a continuous function $u : \overline{Y} \to \mathbb{R}$ which is harmonic on Y and is 0 on ∂K_0 and 1 on ∂K_1. Then $\omega := \partial u$ is a non-trivial holomorphic 1-form on Y. Let f be a holomorphic primitive of $p^*\omega$ on the universal covering $p : \widetilde{Y} \to Y$ which exists by Corollary 8.6. The map $f : \widetilde{Y} \to \mathbb{C}$ (being non-constant) satisfies the assumption of Lemma 23.2 (cf. Lemma 3.1). Thus $\widetilde{Y}$ has countable topology, and, by Lemma 23.1, so does Y. Then also the topology of $X = Y \cup U$ is countable. □

24. Weyl's lemma

In this section we assume some familiarity with basic distribution theory.

Lemma 24.1 (Weyl's lemma). *Let $U \subseteq \mathbb{C}$ be a domain. Let u be a distribution on U with $\Delta u = 0$. Then u is a smooth function.*

This means the following: if the distribution $u \in \mathscr{D}'(U)$ satisfies $u(\Delta\varphi) = 0$ for all $\varphi \in \mathscr{D}(U)$, then there exists a function $h \in \mathscr{E}(U)$ with $\Delta h = 0$ and

$$u(f) = \int_U f(z)h(z)\,dxdy \quad \text{for all } f \in \mathscr{D}(U).$$

Recall that $\mathscr{D}(U)$ denotes the space of smooth functions with compact support in U.

Proof. Let $\epsilon > 0$ and $U_\epsilon := \{z \in U : \overline{D}_\epsilon(z) \subseteq U\}$. Let ρ be a rotation invariant smooth function with support in $\mathbb{D}$ and $\int_{\mathbb{C}} \rho\,dxdy = 1$, and set $\rho_\epsilon(z) = \epsilon^{-2}\rho(\epsilon^{-1}z)$. Then, for $z \in U_\epsilon$, the functions $\zeta \mapsto \rho_\epsilon(\zeta - z)$ has support in U. Consider

$$h(z) := u(\rho_\epsilon(\zeta - z)).$$

Then it is not hard to see (cf. [**4**, Lemma 24.5]) that $h \in \mathscr{E}(U_\epsilon)$. We will prove that for each smooth function f with support in U_ϵ we have

$$u(f) = \int_{U_\epsilon} f(z)h(z)\,dxdy. \tag{24.1}$$

Since $\epsilon > 0$ was arbitrary, this will imply the statement.

For $z \in U_\epsilon$ consider $\rho_\epsilon * f(\zeta) = \int \rho_\epsilon(\zeta - z)f(z)\,dxdy$. The function $\rho_\epsilon * f$ has support in U. We have (cf. [**4**, Lemma 24.6])

$$u(\rho_\epsilon * f) = u\Big(\int_U \rho_\epsilon(\zeta - z)f(z)\,dxdy\Big) = \int_{U_\epsilon} h(z)f(z)\,dxdy. \tag{24.2}$$

By Theorem 10.9, there is a function $\psi \in \mathscr{E}(\mathbb{C})$ with $\Delta\psi = f$. Note that ψ is harmonic on $V := \mathbb{C} \setminus \operatorname{supp}(f)$. We claim that $\rho_\epsilon * \psi = \psi$ on $V_\epsilon := \{z \in V : \overline{D}_\epsilon(z) \subseteq V\}$. Indeed, by the mean value property

$$\psi(z) = \frac{1}{2\pi}\int_0^{2\pi} \psi(z + re^{it})\,dt, \quad \text{for } r < \epsilon,$$

and hence

$$\begin{aligned}
\rho_\epsilon * \psi(z) &= \int_{|\zeta|<\epsilon} \rho_\epsilon(\zeta)\psi(z+\zeta)\,d\xi d\eta \\
&= \int_0^\epsilon \int_0^{2\pi} \rho_\epsilon(r)\psi(z + re^{it})r\,dt\,dr \\
&= 2\pi\psi(z)\int_0^\epsilon \rho_\epsilon(r)r\,dr = \psi(z).
\end{aligned}$$

Then $\varphi := \psi - \rho_\epsilon * \psi$ has compact support in U and

$$\Delta\varphi = \Delta\psi - \rho_\epsilon * \Delta\psi = f - \rho_\epsilon * f.$$

Since $\Delta u = 0$, we have $u(\Delta\varphi) = 0$, whence

$$u(f) = u(\rho_\epsilon * f + \Delta\varphi) = u(\rho_\epsilon * f).$$

Together with (24.2) this implies (24.1) and hence the assertion. □

Corollary 24.2. *Let u be a distribution on a domain $U \subseteq \mathbb{C}$ satisfying $\partial_{\overline{z}} u = 0$. Then $u \in \mathcal{O}(U)$.*

Proof. This follows from Weyl's lemma 24.1 thanks to $\Delta = 4\partial_z \partial_{\overline{z}}$. □

25. The Runge approximation theorem

This section is devoted to a version of the Runge approximation theorem on Riemann surfaces due to Behnke and Stein [**2**]. We present Malgrange's proof [**9**] which is based on Weyl's lemma.

25.1. Exhaustion by Runge regions. Let X be a Riemann surface. For any subset $Y \subseteq X$ we denote by $\widehat{Y}$ the union of Y with all relatively compact connected components of $X \setminus Y$. We say that an open subset Y of X is **Runge** if $Y = \widehat{Y}$, i.e., none of the connected components of $X \setminus Y$ is compact. By a **Runge region** we mean a connected open Runge set.

Clearly, $\widehat{Y}$ depends on the ambient Riemann surface X which will always be clear from the context.

Lemma 25.1. *Let X be a Riemann surface. We have:*

(1) $\widehat{\widehat{Y}} = \widehat{Y}$ *for all* $Y \subseteq X$.
(2) $Y_1 \subseteq Y_2 \subseteq X$ *implies* $\widehat{Y_1} \subseteq \widehat{Y_2}$.
(3) *If* $Y \subseteq X$ *is closed, then* $\widehat{Y}$ *is closed.*
(4) *If* $Y \subseteq X$ *is compact, then* $\widehat{Y}$ *is compact.*

Proof. (1) and (2) are checked easily. Let C_j, $j \in J$, be the connected components of $X \setminus Y$. Since $X \setminus Y$ is open and X is a manifold, all C_j are open. Then

$$X \setminus \widehat{Y} = \bigcup \{C_j : C_j \text{ is not relatively compact}\}$$

is open. This shows (3).

(4) Suppose $Y \neq \emptyset$. Let U be a relatively compact neighborhood of Y. We claim that every C_j intersects $\overline{U}$. Otherwise, if $C_j \subseteq X \setminus \overline{U}$ then $\overline{C}_j \subseteq X \setminus U \subseteq X \setminus Y$ which implies $C_j = \overline{C}_j$. That means that C_j is open and closed, in contradiction to connectedness of X.

Since ∂U is compact and is covered by the disjoint open C_j, only finitely many C_j meet ∂U. Consider the collection of relatively compact C_j and let $C_{j_1}, \ldots, C_{j_m}$ those among them which meet ∂U (all others are contained in U by the claim). Then $\widehat{Y} \subseteq U \cup C_{j_1} \cup \cdots \cup C_{j_m}$ is relatively compact, and thus compact, by (3). □

Proposition 25.2 (compact exhaustion)**.** *Let X be a non-compact Riemann surface. There exists a sequence K_j, $j \in \mathbb{N}$, of compact subsets of X such that*

(1) $K_j = \widehat{K}_j$ *for all* j,
(2) $K_{j-1} \subseteq \operatorname{int} K_j$ *for all* $j \geq 1$,
(3) $X = \bigcup_{j=0}^{\infty} K_j$.

Proof. There is a sequence of compact subsets $K'_0 \subseteq K'_1 \subseteq \cdots$ which cover X, since X has countable topology by Radó's theorem 23.3. Set $K_0 := \widehat{K}'_0$. Suppose that $K_1, \ldots, K_m$ satisfying (1) and (2) have already been constructed. Choose a compact set L with $K'_m \cup K_m \subseteq \operatorname{int} L$. Set $K_{m+1} := \widehat{L}$. In this way we obtain a sequence K_j, $j \in \mathbb{N}$, with the desired properties. □

Lemma 25.3. *Let K_1, K_2 be compact subsets of a Riemann surface X satisfying $K_1 \subseteq \operatorname{int} K_2$ and $K_2 = \widehat{K}_2$. Then there exists an open Runge set Y with regular boundary such that $K_1 \subseteq Y \subseteq K_2$.*

Proof. For every $x \in \partial K_2$ there is a coordinate neighborhood U of x which does not meet K_1, and an open disk $D \ni x$ with $\overline{D} \subseteq U$. We may cover ∂K_2 by finitely many such disks, $D_1, \ldots, D_m$. Then $Y := K_2 \setminus (\overline{D}_1 \cup \cdots \cup \overline{D}_m)$ is open and $K_1 \subseteq Y \subseteq K_2$. The connected components C_j, $j \in J$, of $X \setminus K_2$ are not relatively compact. Every D_j meets at least one C_j. It follows that no connected component of $X \setminus Y$ is relatively compact (since each D_j is connected), i.e., Y is Runge. All the boundary points of Y are peak points, by Theorem 22.8. □

Lemma 25.4. *Let Y be a Runge open subset of a Riemann surface X. Then every connected component of Y is Runge.*

Proof. Let Y_i, $i \in I$, be the connected components of Y. All the Y_i are open, since Y is open and X is a manifold. Let A_k, $k \in K$, be the connected components of $A := X \setminus Y$. Then the A_k are closed, but not compact, since Y is Runge.

We have $\overline{Y}_i \cap A \neq \emptyset$ for all $i \in I$. For, otherwise $\overline{Y}_i \subseteq Y$, and hence $\overline{Y}_i = Y_i$, contradicting the fact that X is connected.

Let C be a connected component of $X \setminus Y_i$. Then $C \cap A \neq \emptyset$. For, otherwise $C \cap Y_j \neq \emptyset$ for some $j \neq i$, and thus $\overline{Y}_j \subseteq C$, since C is closed and Y_j is connected. But then $C \cap A \neq \emptyset$, by the previous paragraph.

Now C meets at least one A_k and thus $A_k \subseteq C$. It follows that C cannot be compact. Since C was arbitrary, Y_i is Runge. □

Theorem 25.5 (exhaustion by Runge regions)**.** *Let X be a non-compact Riemann surface. Then there is a sequence $Y_0 \Subset Y_1 \Subset \cdots$ of relatively compact Runge regions with regular boundary such that $X = \bigcup_{j \in \mathbb{N}} Y_j$.*

Proof. We will show that for every compact set $K \subseteq X$ there is a Runge region Y with regular boundary such that $K \subseteq Y \Subset X$. This implies the theorem.

We can find a connected compact set K_1 and a compact set K_2 such that $K \subseteq K_1 \subseteq \operatorname{int} K_2$. By Lemma 25.3, there is a Runge open set Y_1 with regular boundary such that $K_1 \subseteq Y_1 \subseteq \widehat{K}_2$. The connected component Y of Y_1 which contains K_1 is Runge, by Lemma 25.4, and has regular boundary, by Lemma 22.6. This proves the claim and the theorem. □

25.2. The Fréchet space of smooth functions and continuous linear functionals. Let X be a Riemann surface and let $Y \subseteq X$ be an open subset. Choose a countable family of compact sets $K_j \subseteq Y$, $j \in J$, such that each K_j is contained in some coordinate chart (U_j, z_j) and the union of the interiors of the K_j is Y. We endow $\mathscr{E}(Y)$ with the topology generated by the following family of seminorms:

$$p_{j\alpha}(f) := \sup_{x \in K_j} |\partial_j^\alpha f(x)|, \quad j \in J, \ \alpha \in \mathbb{N}^2,$$

where $\partial_j^\alpha = \partial_{x_j}^{\alpha_1} \partial_{y_j}^{\alpha_2}$ is the differential operator with respect to $z_j = x_j + iy_j$; this topology is independent of the choice of the K_j and (U_j, z_j). It makes $\mathscr{E}(Y)$ into a Fréchet space. On $\mathcal{O}(Y) \subseteq \mathscr{E}(Y)$ it induces the topology of uniform convergence on compact sets.

In a similar way we obtain the Fréchet space $\mathscr{E}^{0,1}(Y)$.

Lemma 25.6. *Every continuous linear $u : \mathscr{E}(Y) \to \mathbb{C}$ has compact support, i.e., there is a compact $K \subseteq Y$ such that $u(f) = 0$ for all $f \in \mathscr{E}(Y)$ with $\operatorname{supp}(f) \subseteq Y \setminus K$. Every continuous linear $u : \mathscr{E}^{0,1}(Y) \to \mathbb{C}$ has compact support.*

Proof. By continuity, there is a neighborhood U of 0 in $\mathscr{E}(Y)$ such that $|u(f)| < 1$ for $f \in U$. Thus there exist $\epsilon > 0$, $j_1, \dots, j_m \in J$ and $\alpha_1, \dots, \alpha_m \in \mathbb{N}^2$ such that

$$U_{j_1,\alpha_1}(\epsilon) \cap \cdots \cap U_{j_m,\alpha_m}(\epsilon) \subseteq U,$$

where $U_{j,\alpha}(\epsilon) := \{f \in \mathscr{E}(Y) : p_{j\alpha}(f) < \epsilon\}$. Set $K := K_{j_1} \cup \cdots \cup K_{j_m}$, where K_j is the compact set in the definition of $p_{j\alpha}$. Let $f \in \mathscr{E}(Y)$ have $\operatorname{supp}(f) \subseteq Y \setminus K$. Then, for all $t > 0$,

$$p_{j_1\alpha_1}(tf) = \cdots = p_{j_m\alpha_m}(tf) = 0$$

whence $tf \in U$ and so $|u(f)| < 1/t$. This is possible only if $u(f) = 0$. The proof for $\mathscr{E}^{0,1}(Y)$ is similar. □

Lemma 25.7. *Let Y be an open subset of a Riemann surface X. Let $u : \mathscr{E}^{0,1}(X) \to \mathbb{C}$ be a continuous linear map such that $u(\overline{\partial} g) = 0$ for every $g \in \mathscr{E}(X)$ with $\operatorname{supp}(g) \Subset Y$. Then there exists a holomorphic 1-form $\sigma \in \mathcal{O}^1(X)$ such that*

$$u(\omega) = \int_Y \sigma \wedge \omega, \quad \text{for all } \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \Subset Y.$$

Proof. Let (U, z) be a chart on X contained in Y. We identify U with $z(U) \subseteq \mathbb{C}$. For $\varphi \in \mathscr{D}(U)$ we denote by $\tilde{\varphi}$ the 1-form in $\mathscr{E}^{0,1}(X)$ which equals $\varphi\, d\overline{z}$ on U and zero on $X \setminus U$. Then $u_U : \mathscr{D}(U) \to \mathbb{C}$ with $u_U(\varphi) := u(\tilde{\varphi})$ is a distribution on U such that $\partial_{\overline{z}} u_U = 0$ (indeed, $u_U(\partial_{\overline{z}} g) = u(\overline{\partial} g) = 0$ for all $g \in \mathscr{D}(U)$). So, by Corollary 24.2, there is a holomorphic $h \in \mathcal{O}(U)$ such that

$$u(\tilde{\varphi}) = \int_U h(z)\varphi(z)\, dz \wedge d\overline{z} \quad \text{for all } \varphi \in \mathscr{D}(U).$$

Hence for $\sigma_U := h\, dz \in \mathcal{O}^1(U)$ we obtain

$$u(\omega) = \int_U \sigma_U \wedge \omega, \quad \text{for all } \omega \in \mathscr{E}^{0,1}(U) \text{ with } \operatorname{supp}(\omega) \Subset U.$$

We may repeat this construction for another chart U' and obtain $\sigma_{U'} \in \mathcal{O}^1(U')$. Then

$$\int_U \sigma_U \wedge \omega = \int_{U'} \sigma_{U'} \wedge \omega \quad \text{for all } \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \Subset U \cap U',$$

whence $\sigma_U = \sigma_{U'}$ on $U \cap U'$. So there exists a holomorphic 1-form $\sigma \in \mathcal{O}^1(Y)$ such that

$$u(\omega) = \int_Y \sigma \wedge \omega,$$

for all $\omega \in \mathscr{E}^{0,1}(X)$ which are compactly supported in a chart lying in Y. Using a partition of unity we may write an arbitrary $\omega \in \mathscr{E}^{0,1}(X)$ with $\operatorname{supp}(\omega) \Subset Y$ in the form $\omega = \omega_1 + \cdots + \omega_n$, where each ω_i satisfies the above. Then the statement follows from linearity. □

25.3. The Runge approximation theorem. We will use the following consequence of the Hahn–Banach theorem.

Lemma 25.8. *Let G be a locally convex space and let $E \subseteq F \subseteq G$ be linear subspaces. If every continuous linear functional $\ell : G \to \mathbb{C}$ which vanishes on E also vanishes on F, then E is dense in F.*

Proof. If E is not dense in F, then there exists $x_0 \in F \setminus \overline{E}$. Consider $\overline{E} \oplus \mathbb{C}x_0$ and the continuous linear functional $\ell_0 : \overline{E} \oplus \mathbb{C}x_0 \to \mathbb{C}$ defined by $\ell_0(x + \lambda x_0) = \lambda$. By the Hahn–Banach theorem, ℓ_0 extends to a continuous linear functional $\ell : G \to \mathbb{C}$ which vanishes on E but not on F. □

Proposition 25.9. *Let X be a non-compact Riemann surface. Let $Y \Subset X$ be a relatively compact open Runge subset. Then, for every open Y' with $Y \Subset Y' \Subset X$, the image of the restriction map $\mathcal{O}(Y') \to \mathcal{O}(Y)$ is dense.*

Proof. Let $\rho : \mathscr{E}(Y') \to \mathscr{E}(Y)$ denote the restriction map. In order to show that $\rho(\mathcal{O}(Y'))$ is dense in $\mathcal{O}(Y)$ it suffices, by Lemma 25.8, to prove the following: *If $v : \mathscr{E}(Y) \to \mathbb{C}$ is a continuous linear functional with $v|_{\rho(\mathcal{O}(Y'))} = 0$, then $v|_{\mathcal{O}(Y)} = 0$.*

Let such v be fixed. Recall that, by Corollary 12.11, for each $\omega \in \mathscr{E}^{0,1}(X)$ there is $f \in \mathscr{E}(Y')$ such that $\overline{\partial} f = \omega|_{Y'}$. This induces a linear map $u : \mathscr{E}^{0,1}(X) \to \mathbb{C}$ by setting $u(\omega) := v(f|_Y)$. In fact, this definition does not depend on the choice of f: if also $\overline{\partial} g = \omega|_{Y'}$, then $f - g \in \mathcal{O}(Y')$ and hence $v((f-g)|_Y) = 0$.

We claim that u is continuous. To this end consider

$$V := \{(\omega, f) \in \mathscr{E}^{0,1}(X) \times \mathscr{E}(Y') : \overline{\partial} f = \omega|_{Y'}\}$$

which is a closed linear subspace of $\mathscr{E}^{0,1}(X) \times \mathscr{E}(Y')$ and hence a Fréchet space, since $\overline{\partial} : \mathscr{E}(Y') \to \mathscr{E}^{0,1}(Y')$ is continuous. The projection $\mathrm{pr}_1 : V \to \mathscr{E}^{0,1}(X)$ is surjective and thus open, by the open mapping theorem (e.g. [**12**]). So continuity of u follows from the following commutative diagram.

$$\begin{array}{ccc} V & \xrightarrow{\rho \circ \mathrm{pr}_2} & \mathscr{E}(Y) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle v} \\ \mathscr{E}^{0,1}(X) & \xrightarrow{u} & \mathbb{C} \end{array}$$

By Lemma 25.6, $v : \mathscr{E}(Y) \to \mathbb{C}$ and $u : \mathscr{E}^{0,1}(X) \to \mathbb{C}$ have compact support, i.e., there exist compact sets $K \subseteq Y$ and $L \subseteq X$ such that

$$v(f) = 0 \quad \text{for all } f \in \mathscr{E}(Y) \text{ with } \operatorname{supp}(f) \subseteq Y \setminus K, \tag{25.1}$$

$$u(\omega) = 0 \quad \text{for all } \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \subseteq X \setminus L. \tag{25.2}$$

If $g \in \mathscr{E}(X)$ satisfies $\operatorname{supp}(g) \subseteq X \setminus K$, then $u(\overline{\partial} g) = v(g|_Y) = 0$. By Lemma 25.7, there exits a holomorphic 1-form $\sigma \in \mathcal{O}^1(X \setminus K)$ such that

$$u(\omega) = \int_{X \setminus K} \sigma \wedge \omega \quad \text{for all } \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \Subset X \setminus K.$$

By (25.2), $\sigma|_{X \setminus (K \cup L)} = 0$. Any connected component of $X \setminus \widehat{K}$, not being relatively compact, must meet $X \setminus (K \cup L)$. Thus $\sigma|_{X \setminus \widehat{K}} = 0$, by the identity theorem, and consequently

$$u(\omega) = 0 \quad \text{for all } \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \Subset X \setminus \widehat{K}. \tag{25.3}$$

To finish the proof let $f \in \mathcal{O}(Y)$. We will show that $v(f) = 0$. Since Y is Runge, $\widehat{K} \subseteq Y$, by Lemma 25.1. Thus there is a function $g \in \mathscr{E}(X)$ with $f = g$ in a neighborhood of $\widehat{K}$ and $\operatorname{supp}(g) \Subset Y$. Then $v(f) = v(g|_Y) = u(\overline{\partial} g)$, by (25.1). Since g is holomorphic in a neighborhood of $\widehat{K}$, we have $\operatorname{supp}(\overline{\partial} g) \Subset X \setminus \widehat{K}$. Thus $v(f) = u(\overline{\partial} g) = 0$, by (25.3). ☐

Theorem 25.10 (Runge approximation theorem)**.** *Let X be a non-compact Riemann surface. Let Y be an open subset such that $X \setminus Y$ has no compact connected component. Then every holomorphic function on Y can be approximated uniformly on compact subsets of Y by holomorphic functions on X.*

Proof. We may assume that Y is relatively compact in X. Let $f \in \mathcal{O}(Y)$, a compact set $K \subseteq Y$, and $\epsilon > 0$ be given. There exists an exhaustion $Y_1 \Subset Y_2 \cdots$ of X by

Runge regions, by Theorem 25.5, where $Y_0 := Y \Subset Y_1$. Proposition 25.9 provides iteratively a sequence of functions $f_n \in \mathcal{O}(Y_n)$ such that $|f_1 - f|_K < 2^{-1}\epsilon$ and

$$|f_n - f_{n-1}|_{\overline{Y}_{n-2}} < 2^{-n}\epsilon, \quad n \geq 2.$$

For every $k \in \mathbb{N}$, the sequence $(f_n)_{n>k}$ converges uniformly on Y_k. Thus there exists $F \in \mathcal{O}(X)$ such that, on Y_k, F is the limit of $(f_n)_{n>k}$. By construction, $|F - f|_K < \epsilon$. □

25.4. Solution of the inhomogeneous Cauchy–Riemann equation.

Corollary 25.11. *Let X be a non-compact Riemann surface. Then for every $\omega \in \mathscr{E}^{0,1}(X)$ there exists a function $f \in \mathscr{E}(X)$ with $\overline{\partial} f = \omega$.*

Proof. By Corollary 12.11, for each relatively compact open $Y \Subset X$ there is $g \in \mathscr{E}(Y)$ such that $\overline{\partial} g = \omega|_Y$. Let $Y_0 \Subset Y_1 \Subset Y_2 \cdots$ be an exhaustion of X by Runge regions, which exists by Theorem 25.5. We claim that there exist functions $f_n \in \mathscr{E}(Y_n)$ such that $\overline{\partial} f_n = \omega|_{Y_n}$ and $|f_{n+1} - f_n|_{Y_{n-1}} \leq 2^{-n}$.

Choose any $f_0 \in \mathscr{E}(Y_0)$ such that $\overline{\partial} f_0 = \omega|_{Y_0}$. Suppose that suitable $f_0, \ldots, f_n$ have been constructed. There exists $g \in \mathscr{E}(Y_{n+1})$ such that $\overline{\partial} g = \omega|_{Y_{n+1}}$. Then $g - f_n$ is holomorphic on Y_n. By the Runge approximation theorem 25.10, there exists $h \in \mathcal{O}(Y_{n+1})$ such that

$$|(g - f_n) - h|_{Y_{n-1}} \leq 2^{-n}.$$

Setting $f_{n+1} := g - h$ we have $\overline{\partial} f_{n+1} = \overline{\partial} g = \omega|_{Y_{n+1}}$ and $|f_{n+1} - f_n|_{Y_{n-1}} \leq 2^{-n}$. The claim is proved.

We define

$$f := f_n + \sum_{k \geq n} (f_{k+1} - f_k) \quad \text{ on } Y_n.$$

The series converges uniformly on Y_{n-1} to a holomorphic function F_n. Thus f is smooth on Y_{n-1} for every n and hence $f \in \mathscr{E}(X)$. Moreover,

$$\overline{\partial} f = \overline{\partial} f_n = \omega \quad \text{ on } Y_n$$

for all n. Thus $\overline{\partial} f = \omega$ on X. □

26. The Mittag–Leffler and Weierstrass theorem

We come back to the problem of constructing meromorphic functions with prescribed principal parts, respectively, prescribed zeros and poles of given orders. In the complex plane this is the content of the Mittag–Leffler and the Weierstrass theorem. The analogues of these theorems hold on non-compact Riemann surfaces without any restriction (in contrast to compact Riemann surfaces). They were first proved by Florack [**3**] building on the methods of [**2**]. The respective analogues in several complex variables are the first and second Cousin problems.

26.1. The Mittag–Leffler theorem.

Theorem 26.1. *For any non-compact Riemann surface X we have $H^1(X, \mathcal{O}) = 0$.*

Proof. By Dolbeault's theorem 11.11, $H^1(X, \mathcal{O}) \cong \mathscr{E}^{0,1}(X)/\overline{\partial}\mathscr{E}(X)$. So Corollary 25.11 implies the statement. □

Remark 26.2. We remark that this result is a special case of Theorem B of Cartan–Serre which holds on all Stein manifolds; cf. [**7**].

Corollary 26.3 (Mittag–Leffler theorem)**.** *On a non-compact Riemann surface every Mittag–Leffler distribution has a solution.*

Proof. Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of X. Recall that a cochain $\mu = (f_i) \in C^0(\mathfrak{U}, \mathscr{M})$ is a Mittag–Leffler distribution if the differences $f_j - f_i$ are holomorphic on $U_i \cap U_j$, i.e., f_i and f_j have the same principal parts. A solution of μ is a meromorphic function $f \in \mathscr{M}(X)$ which has the same principal parts as μ, i.e., $f|_{U_i} - f_i \in \mathcal{O}(U_i)$ for all $i \in I$. By Proposition 19.1, μ has a solution if and only if the cocycle $f_{ij} := f_j - f_i \in \mathcal{O}(U_i \cap U_j)$ is a coboundary. By Theorem 26.1, this is always the case. □

26.2. The Weierstrass theorem. Let $D \in \mathrm{Div}(X)$ be a divisor on the Riemann surface X. We are looking for a meromorphic function f such that $(f) = D$, i.e., a solution of D. We start with the existence of weak solutions (as defined in subsection 20.1)

Lemma 26.4. *Every divisor D on a non-compact Riemann surface X has a weak solution.*

Proof. Let $K_1, K_2, \ldots$ be an exhaustion of X compact sets with the properties given in Proposition 25.2.

We claim that the following holds. *If $a_0 \in X \setminus K_j$ and $A_0 \in \mathrm{Div}(X)$ is 1 at a_0 and zero otherwise, then A_0 has a weak solution φ satisfying $\varphi|_{K_j} = 1$.* Since $K_j = \widehat{K}_j$, the point a_0 lies in a connected component U of $X \setminus K_j$ which is not relatively compact. We may conclude that there is a point $a_1 \in U \setminus K_{j+1}$ and a curve γ_0 in U from a_1 to a_0. There is a weak solution φ_0 of the divisor $\partial\gamma_0$ with $\varphi_0|_{K_j} = 1$, by Lemma 20.2. We may repeat this construction and obtain a sequence of points $a_k \in X \setminus K_{j+k}$, $k \in \mathbb{N}$, curves γ_k in $X \setminus K_{j+k}$ from a_{k+1} to a_k, and weak solutions of the divisors $\partial\gamma_k$ with $\varphi_k|_{K_{j+k}} = 1$. Let A_k be the divisor which is 1 at a_k and zero otherwise. Then $\partial\gamma_k = A_k - A_{k+1}$ and the product $\varphi_0\varphi_1 \cdots \varphi_n$ is a weak solution of the divisor $A_0 - A_{n+1}$. The infinite product $\prod_{k=0}^\infty \varphi_k$ converges, since on every compact subset of X there are only finitely many factors that are not identically 1, and it is the desired weak solution. The claim is proved.

Let $D \in \mathrm{Div}(X)$. For $j \in \mathbb{N}$ set

$$D_j(x) := \begin{cases} D(x) & \text{if } x \in K_{j+1} \setminus K_j, \\ 0 & \text{if } x \notin K_{j+1} \setminus K_j, \end{cases}$$

where $K_0 := \emptyset$. Then $D = \sum_{j=0}^\infty D_j$. Each D_j is non-zero only at finitely many points. By the claim, there is a weak solution φ_j of D_j with $\varphi_j|_{K_j} = 1$. So $\varphi := \prod_{j=0}^\infty \varphi_j$ is a weak solution of D. □

Theorem 26.5 (Weierstrass theorem)**.** *Every divisor D on a non-compact Riemann surface X has a solution.*

Proof. The problem has a solution locally. Thus there is an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of X and meromorphic functions $f_i \in \mathscr{M}^*(U_i)$ such that $(f_i) = D$ on U_i. We may assume that all U_i are simply connected. We have $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$ for all $i, j \in I$, since, on $U_i \cap U_j$, f_i and f_j have the same zeros and poles. By Lemma 26.4, D has a weak solution φ. On U_i we have $\varphi = \varphi_i f_i$, where $\varphi_i \in \mathscr{E}(U_i)$ has no zeros. Since U_i is simply connected, there is a function $\psi_i \in \mathscr{E}(U_i)$ such that $\varphi_i = e^{\psi_i}$. So on $U_i \cap U_j$ we have

$$e^{\psi_j - \psi_i} = \frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j),$$

and thus $\psi_{ij} := \psi_j - \psi_i \in \mathcal{O}(U_i \cap U_j)$. Clearly, $(\psi_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$. Since $H^1(X, \mathcal{O}) = 0$, by Theorem 26.1, there exist $g_i \in \mathcal{O}(U_i)$ with

$$\psi_{ij} = \psi_j - \psi_i = g_j - g_i \quad \text{on } U_i \cap U_j.$$

This implies $e^{g_j} f_j = e^{g_i} f_i$ on $U_i \cap U_j$. Hence there is a meromorphic function f on X with $f = e^{g_i} f_i$ on U_i for all i. Clearly, $(f) = D$. □

Corollary 26.6. *Let X be a non-compact Riemann surface. There exists a holomorphic 1-form $\omega \in \mathcal{O}^1(X)$ which vanishes nowhere.*

Proof. Let g be a non-constant meromorphic function on X and let $f \in \mathscr{M}^*(X)$ be a function with divisor $-(dg)$ which exists by the Weierstrass theorem 26.5. Then $\omega := f\,dg$ is the desired 1-form. □

26.3. Non-compact Riemann surfaces are Stein. Let X be a Riemann surface. That X is **Stein** means the following:

(1) The holomorphic functions separate points, i.e., for any two points $x, y \in X$, $x \neq y$, there exists $f \in \mathcal{O}(X)$ with $f(x) \neq f(y)$.
(2) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X having no accumulation points, then there exists $f \in \mathcal{O}(X)$ with $\limsup_{n\to\infty} |f(x_n)| = \infty$.

That every non-compact Riemann surface is Stein follows from the next result. Clearly, compact Riemann surfaces are not Stein.

Theorem 26.7. *Let X be a non-compact Riemann surface. Let (a_n) be a sequence of distinct points of X with no accumulation points. Given arbitrary complex numbers $c_n \in \mathbb{C}$, there is a holomorphic function $f \in \mathcal{O}(X)$ such that $f(a_n) = c_n$ for all $n \in \mathbb{N}$.*

Proof. By the Weierstrass theorem 26.5, there is a function $h \in \mathcal{O}(X)$ which vanishes of order 1 at every a_n and has no other zeros. For $i \in \mathbb{N}$ set $U_i := X \setminus \bigcup_{k\neq i}\{a_k\}$ and consider the open cover $\mathfrak{U} = (U_i)_{i\in\mathbb{N}}$ of X. Then $U_i \cap U_j = X \setminus \{a_k : k \in \mathbb{N}\}$ if $i \neq j$. So $1/h$ is holomorphic on $U_i \cap U_j$. It follows that $g_i := c_i/h \in \mathscr{M}(U_i)$ forms a Mittag–Leffler distribution $(g_i) \in C^0(\mathfrak{U}, \mathscr{M})$ on X. By the Mittag–Leffler theorem 26.3, it has a solution $g \in \mathscr{M}(X)$. Define $f = gh$. Then on U_i we have

$$f = gh = g_i h + (g - g_i)h = c_i + (g - g_i)h.$$

Since $g - g_i$ is holomorphic on U_i and $h(a_i) = 0$, we may conclude that f is holomorphic on X and $f(a_i) = c_i$ for all $i \in \mathbb{N}$. □

27. The uniformization theorem

In this section we prove the uniformization theorem: any simply connected Riemann surface is isomorphic to one of the following three normal forms, the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$. Evidently, this is a generalization of the Riemann mapping theorem in the plane.

27.1. The holomorphic deRham group. Let X be a Riemann surface. In analogy to the first deRham group (cf. subsection 11.7)

$$\mathrm{Rh}^1(X) = \frac{\ker(d : \mathscr{E}^1(X) \to \mathscr{E}^2(X))}{\mathrm{im}(d : \mathscr{E}^0(X) \to \mathscr{E}^1(X))}$$

of smooth closed 1-forms modulo exact ones, we also consider the **holomorphic deRham group**

$$\mathrm{Rh}^1_{\mathcal{O}}(X) := \frac{\mathcal{O}^1(X)}{d\mathcal{O}(X)};$$

recall that every holomorphic 1-form is closed, by Proposition 8.2.

If X is simply connected, then $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$, by Corollary 8.7. We will prove the uniformization theorem under the condition $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$, which a posteriori will turn out to be equivalent to X being simply connected.

Lemma 27.1. *Let X be a Riemann surface with $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$. Then:*

(1) *Every non-vanishing $f \in \mathcal{O}(X)$ has a logarithm and a square root, i.e., there exist $g, h \in \mathcal{O}(X)$ with $e^g = f$ and $h^2 = f$.*
(2) *Every real valued harmonic function on X is the real part of a holomorphic function on X.*

Proof. (1) Since $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$, there exists $g \in \mathcal{O}(X)$ such that $dg = df/f$. By adding a constant to g, we may assume that for some $a \in X$ we have $e^{g(a)} = f(a)$. Then

$$d(fe^{-g}) = e^{-g}df - fe^{-g}dg = 0.$$

So $fe^{-g} = 1$ and hence $e^g = f$. Taking $h = e^{g/2}$ gives $h^2 = f$.

(2) Let $u : X \to \mathbb{R}$ be harmonic. By Theorem 18.3, there is $\omega \in \mathcal{O}^1(X)$ with $du = \mathrm{Re}(\omega)$. Since $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$, there is $g \in \mathcal{O}(X)$ such that $du = \mathrm{Re}(\omega) = \mathrm{Re}(dg) = (dg + d\overline{g})/2$. Then $u = \mathrm{Re}(g) + \mathrm{const}$. □

27.2. Towards the uniformization theorem.

Theorem 27.2. *Let X be a non-compact Riemann surface. Let $Y \Subset X$ be relatively compact open, connected, with $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$, and with regular boundary. Then Y is biholomorphic to $\mathbb{D}$.*

Proof. Fix $a \in Y$. By the Weierstrass theorem 26.5, there is a holomorphic function g on X which has a zero of first order at a and is non-zero on $X \setminus \{a\}$. Since the Dirichlet problem has a solution on Y, there is a function $u : \overline{Y} \to \mathbb{R}$ which is continuous on $\overline{Y}$, by Theorem 22.7, harmonic on Y, and such that

$$u(y) = \log |g(y)| \quad \text{for } y \in \partial Y.$$

By Lemma 27.1, $u = \mathrm{Re}(h)$ for some $h \in \mathcal{O}(Y)$. Set $f := e^{-h}g$. We will show that $f : Y \to \mathbb{D}$ is a biholomorphism.

We begin by proving that $f(Y) \subseteq \mathbb{D}$. For $y \in Y \setminus \{a\}$,

$$|f(y)| = e^{-\mathrm{Re}\, h(y)}|g(y)| = e^{-u(y)}e^{\log |g(y)|}.$$

Thus $|f|$ extends to a continuous function $|f| : \overline{Y} \to \mathbb{R}$ which is equal to 1 on ∂Y. By the maximum principle, $|f(y)| < 1$ for all $y \in Y$.

We claim that $f : Y \to \mathbb{D}$ is proper. It suffices to show that $f^{-1}(\overline{D}_r(0))$ is compact in Y for all $r < 1$. But $f^{-1}(\overline{D}_r(0)) = \{y \in \overline{Y} : |f(y)| \le r\}$ and hence it is a closed subset of the compact set $\overline{Y}$. So $f^{-1}(\overline{D}_r(0))$ is compact.

Since $f : Y \to \mathbb{D}$ is proper, each value is attained equally often, by Theorem 3.19. The value 0 is attained exactly once. It follows that $f : Y \to \mathbb{D}$ is bijective and thus biholomorphic. □

Lemma 27.3. *Let Y be proper subregion of $D_R(0)$, for $R \in (0, \infty]$, such that $0 \in Y$ and $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$. Then there exists $r \in (0, R)$ and a holomorphic map $f : Y \to D_r(0)$ with $f(0) = 0$ and $f'(0) = 1$.*

Proof. Suppose that $R < \infty$. Without loss of generality we may assume that $R = 1$. Choose $a \in \mathbb{D} \setminus Y$ and consider

$$\varphi_a(z) := \frac{z - a}{1 - \overline{a}z}.$$

Then 0 is not contained in $\varphi_a(Y)$ and thus there exists $g \in \mathcal{O}(Y)$ with $g^2 = \varphi_a$ on Y, by Lemma 27.1. We have $g(Y) \subseteq \mathbb{D}$. Then $h := \varphi_{g(0)} \circ g : Y \to \mathbb{D}$ satisfies

$h(0) = 0$ and (as $g(0)^2 = -a$)

$$h'(0) = \varphi'_{g(0)}(g(0))g'(0) = \varphi'_{g(0)}(g(0))\frac{\varphi'_a(0)}{2g(0)} = \frac{1}{1-|g(0)|^2}\frac{1-|a|^2}{2g(0)} = \frac{1+|g(0)|^2}{2g(0)}.$$

It follows that $|h'(0)| > 1$. Thus, $f := h/h'(0)$ is a holomorphic map $f : Y \to D_r(0)$, where $r := 1/|h'(0)|$, satisfying $f(0) = 0$ and $f'(0) = 1$.

The case $R = \infty$ is similar. □

Lemma 27.4. *Let X be a non-compact Riemann surface with $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$. If $Y \subseteq X$ is a Runge region, then also $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$.*

Proof. Let $\omega \in \mathcal{O}^1(Y)$. By Corollary 26.6, there is $\omega_0 \in \mathcal{O}^1(X)$ which has no zeros. So $\omega = f\omega_0$ for some $f \in \mathcal{O}(Y)$. By the Runge approximation theorem 25.10, there is a sequence $f_n \in \mathcal{O}(X)$ which converges to f uniformly on compact subsets of Y. Consequently, $\int_\gamma f_n\omega_0 \to \int_\gamma \omega$ for every closed curve γ in Y. The condition $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$ implies that the holomorphic 1-forms $f_n\omega_0$ are exact on X, whence $\int_\gamma f_n\omega_0 = 0$. Thus, $\int_\gamma \omega = 0$. So ω has a primitive, by Corollary 9.4. □

27.3. The uniformization theorem. We recall that the Cauchy integral formula implies that a holomorphic map $f : D_r(0) \to D_{r'}(0)$ satisfies $|f'(0)| \le r'/r$.

Theorem 27.5 (uniformization theorem)**.** *Let X be a Riemann surface with $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$. Then X is isomorphic to the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$.*

Proof. Suppose that X is compact. Then $d\mathcal{O}(X) = 0$, since every holomorphic function on X is constant, by Corollary 1.12. The condition $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$ implies that $\mathcal{O}^1(X) = 0$, i.e., X has genus 0, cf. (14.7). We saw in Corollary 13.7 that X must be isomorphic to $\widehat{\mathbb{C}}$.

Now let X be non-compact. There is an exhaustion $Y_0 \Subset Y_1 \Subset Y_2 \Subset \cdots$ of X by Runge regions with regular boundary, by Proposition 25.2. We have $\mathrm{Rh}^1_{\mathcal{O}}(Y_n) = 0$ for all n, by Lemma 27.4. By Theorem 27.2, every Y_n is isomorphic to $\mathbb{D}$. Fix $a \in Y_0$ and a coordinate neighborhood (U, z) of a. For each n, there is $r_n > 0$ and a biholomorphism $f_n : Y_n \to D_{r_n}(0)$ with $f_n(a) = 0$ and $(df_n/dz)(a) = 1$.

We claim that $r_n \le r_{n+1}$ for all n. In fact, the map $h = f_{n+1} \circ f_n^{-1} : D_{r_n}(0) \to D_{r_{n+1}}(0)$ satisfies $h(0) = 0$ and $h'(0) = 1$, and, by the remark before the theorem, $1 = h'(0) \le r_{n+1}/r_n$. Let $R := \lim_{n\to\infty} r_n \in (0, \infty]$. We will prove that X is mapped biholomorphically onto $D_R(0)$ which completes the proof.

Next we claim that there is a subsequence (f_{n_k}) of (f_n) such that for every m the sequence $(f_{n_k}|_{Y_m})_{k\ge m}$ converges uniformly on compact subsets of Y_m. Consider

$$g_n(z) := \frac{1}{r_0} f_n(f_0^{-1}(r_0 z)), \quad n \ge 0.$$

Then each $g_n : \mathbb{D} \to \mathbb{C}$ is an injective holomorphic function with $g_n(0) = 0$ and $g_n'(0) = 1$, i.e., (g_n) is a sequence of *schlicht functions*. Since the set of schlicht functions is compact in $\mathcal{O}(\mathbb{D})$ (see e.g. [**14**, Exercise 38]), there is a subsequence $(f_{n_{0k}})$ of (f_n) which converges uniformly on compact subsets of Y_0 (for, $z \mapsto f_0^{-1}(r_0 z)$ is a biholomorphism from $\mathbb{D}$ to Y_0). By the same reasoning, there is a subsequence $(f_{n_{1k}})$ of $(f_{n_{0k}})$ which converges uniformly on compact subsets of Y_1. Repeating this process we obtain for each $m \in \mathbb{N}$ a subsequence $(f_{n_{mk}})$ of the previous sequences which converges uniformly on compact subsets of Y_m. Then the sequence $f_{n_k} := f_{n_{kk}}$ has the required properties.

The limit f of the subsequence (f_{n_k}) is a holomorphic function on X which coincides on every Y_m with the limit of $(f_{n_k}|_{Y_m})_{k\geq m}$. Then $f : X \to \mathbb{C}$ is injective and satisfies $f(a) = 0$ and $(df/dz)(a) = 1$.

To finish the proof we show that f maps X biholomorphically onto $D_R(0)$. Clearly, $f(X) \subseteq D_R(0)$, so it is enough to prove that $f : X \to D_R(0)$ is surjective. If not, then, by Lemma 27.3, there is $r \in (0, R)$ and a holomorphic map $g : f(X) \to D_r(0)$ with $g(0) = 0$ and $g'(0) = 1$ (we have $\mathrm{Rh}^1_{\mathcal{O}}(f(X)) = 0$ since $f : X \to f(X)$ is a biholomorphism). Choose n such that $r_n > r$. Then $h := g \circ f \circ f_n^{-1} : D_{r_n}(0) \to D_r(0)$ satisfies $h(0) = 0$ and $h'(0) = 1$, which contradicts $r < r_n$, by the remark before the theorem. The proof is complete. □

Corollary 27.6. *Let X be a Riemann surface with $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$. Then X is simply connected.*

Proof. This follows from the uniformization theorem 27.5, since $\widehat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{D}$ are simply connected. □

27.4. Classification of Riemann surfaces. Let G be a group which acts on a Riemann surface X. We say that G **acts discretely** if every orbit $Gx := \{gx : g \in G\}$, $x \in X$, is a discrete subset of X. We say that G **acts without fixed points** if for all $g \in G \setminus \{\mathrm{id}\}$ and all $x \in X$, we have $gx \neq x$.

Lemma 27.7. *Let G be a group of automorphisms of $\mathbb{C}$ which acts discretely and without fixed points. Then one of the following cases occurs.*

(1) $G = \{\mathrm{id}\}$.
(2) $G = \{z \mapsto z + na : n \in \mathbb{Z}\}$, *where* $a \in \mathbb{C}^*$.
(3) $G = \{z \mapsto z + na + mb : n, m \in \mathbb{Z}\}$, *where* $a, b \in \mathbb{C}^*$ *are linearly independent over* $\mathbb{R}$.

Proof. Recall that $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az+b : a \in \mathbb{C}^*,\ b \in \mathbb{C}\}$. If $a \neq 1$, then $z \mapsto az+b$ has a fixed point. Thus G consists only of translations $z \mapsto z + b$. Let $\Gamma := G0$ be the orbit of 0. Then Γ is a discrete additive subgroup of $\mathbb{C}$ which consists of all translations $z \mapsto z + b$, where $b \in \Gamma$. Let V be the smallest real linear subspace of $\mathbb{C}$ which contains Γ. Depending on whether the (real) dimension of V is 0,1, or 2 the case (1), (2), or (3) occurs; this follows from Proposition 21.1. □

Let X be a Riemann surface and let $\widetilde{X}$ be its universal covering. By the uniformization theorem 27.5, $\widetilde{X}$ is isomorphic to $\widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$. Depending on which case occurs one says that X is **elliptic**, **parabolic**, or **hyperbolic**.

Let $G = \mathrm{Deck}(\widetilde{X} \to X) \cong \pi_1(X)$ be the group of deck transformations of the universal covering of X. The elements of G are automorphisms of $\widetilde{X}$. *The group G acts on $\widetilde{X}$ discretely and without fixed points.*

Indeed, since $p : \widetilde{X} \to X$ is a normal covering, by Theorem 4.4, for all $x \in \widetilde{X}$ we have $Gx = p^{-1}(p(x))$ which is discrete. If $g \in G$ has a fixed point $gx = x$, then $g = \mathrm{id}$, by the uniqueness of liftings 3.6 (cf. subsection 4.2).

The Riemann surface X may be thought of as the orbit space $\widetilde{X}/G$. In particular, a hyperbolic Riemann surface is a quotient of the unit disk $\mathbb{D}$ (respectively, upper half-plane $\mathbb{H}$) modulo a group of automorphisms of $\mathbb{D}$ (respectively, $\mathbb{H}$) acting discretely and without fixed points.

Theorem 27.8 (classification)**.** *We have:*

(1) *The Riemann sphere $\widehat{\mathbb{C}}$ is elliptic.*

(2) *The complex plane* $\mathbb{C}$, *the punctured plane* $\mathbb{C}^*$, *and the complex tori* $\mathbb{C}/\Lambda$ *are parabolic.*
(3) *Every other Riemann surface is hyperbolic.*

Proof. (1) and (2) are clear. To prove (3) we show that every Riemann surface X which is not hyperbolic is isomorphic to a Riemann surface listed in (1) and (2).

Suppose that $\widetilde{X}$ is isomorphic to $\widehat{\mathbb{C}}$. Recall that the automorphism group of $\widehat{\mathbb{C}}$ is the group of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Thus every automorphism of $\widehat{\mathbb{C}}$ has a fixed point. Thus X itself is isomorphic to $\widehat{\mathbb{C}}$.

Now suppose that $\widetilde{X}$ is isomorphic to $\mathbb{C}$. By Lemma 27.7, we may conclude that X is isomorphic to $\mathbb{C}$ if G is trivial, to a complex torus if G is a lattice, and to $\mathbb{C}^*$ if $G = \{z \mapsto z + na : n \in \mathbb{Z}\}$, for some $a \in \mathbb{C}^*$. In the last case the universal covering is isomorphic to $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto \exp(2\pi i z/a)$. □

Corollary 27.9. *A compact Riemann surface is elliptic, parabolic, or hyperbolic depending on whether its genus is zero, one, or greater than one.* □

We remark that compact Riemann surfaces of genus one are often called **elliptic curves**. This should not be confused with the notion of elliptic Riemann surface.

As a nice corollary we present the little Picard theorem.

Corollary 27.10 (little Picard theorem)**.** *A non-constant holomorphic function* $f : \mathbb{C} \to \mathbb{C}$ *attains every value* $c \in \mathbb{C}$ *with at most one exception.*

Proof. Suppose that f does not attain $a \neq b \in \mathbb{C}$. The Riemann surface $X = \mathbb{C} \setminus \{a, b\}$ is hyperbolic, by Theorem 27.8. Then $f : \mathbb{C} \to X$ admits a holomorphic lifting $\widetilde{f} : \mathbb{C} \to \widetilde{X}$. Since $\widetilde{X}$ is isomorphic to $\mathbb{D}$, Liouville's theorem implies that $\widetilde{f}$, and hence f, is constant, a contradiction. □

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Index