CHOOSING ROOTS OF POLYNOMIALS WITH SYMMETRIES SMOOTHLY

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Abstract. The roots of a smooth curve of hyperbolic polynomials may not in general be parameterized smoothly, even not $C^{1,\alpha}$ for any $\alpha > 0$. A sufficient condition for the existence of a smooth parameterization is that no two of the increasingly ordered continuous roots meet of infinite order. We give refined sufficient conditions for smooth solvability if the polynomials have certain symmetries. In general a C^{3n} curve of hyperbolic polynomials of degree n admits twice differentiable parameterizations of its roots. If the polynomials have certain symmetries we are able to weaken the assumptions in that statement.

1. Introduction

Consider a smooth curve of monic hyperbolic (i.e. all roots real) polynomials with fixed degree n:

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Is it possible to find n smooth functions $x_1(t), \ldots, x_n(t)$ which parameterize the roots of $P(t)$ for each t? It has been shown in [28] that real analytic curves $P(t)$ allow real analytic parameterizations of its roots, and in [1] that the roots of smooth curves $P(t)$ may be chosen smoothly if no two of the increasingly ordered continuous roots meet of infinite order. In general, as shown in [15], the roots of a C^{3n} curve $P(t)$ of hyperbolic polynomials can be parameterized twice differentiable. That regularity of the roots is best possible: In general no $C^{1,\alpha}$ parameterizations of the roots for any $\alpha > 0$ exist which is shown by examples in [1], [6], and [11]. Further references related to that topic are [8], [21], and [34].

The space Hyp^n of monic hyperbolic polynomials P of fixed degree n may be identified with a semialgebraic subset in $\mathbb{R}^n$, the coefficients of P being the coordinates. Then $P(t)$ is a smooth curve in $\mathrm{Hyp}^n \subseteq \mathbb{R}^n$. If the curve $P(t)$ lies in some semialgebraic subset of Hyp^n, then it is evident that in general the conditions which guarantee smooth parameterizations of the roots of $P(t)$ are weaker than those mentioned in the previous paragraph. In the present paper we are going to study that phenomenon.

In section 3 we present a class of semialgebraic subsets in spaces of hyperbolic polynomials for which we are able to apply the described strategy. The construction of that class is based on results due to [32].

Our main goal is to investigate the problem of finding smooth roots of P under the assumption that the polynomials $P(t)$ satisfy certain symmetries. More precisely, we shall assume that the roots $x_1(t), \ldots, x_n(t)$ of $P(t)$ fulfill some linear relations, i.e., there is a linear subspace U of $\mathbb{R}^n$ such that $(x_1(t), \ldots, x_n(t)) \in U$ for all

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t. Then the curve $P(t)$ lies in the semialgebraic subset $E(U)$ of the space of hyperbolic polynomials $\mathrm{Hyp}^n = E(\mathbb{R}^n) = \mathbb{R}^n/\mathrm{S}_n$ of degree n, where $E = (E_1, \ldots, E_n)$ and E_i denotes the i-th elementary symmetric function. The symmetries of the roots of $P(t)$ are represented by the action of the group W on U which is inherited from the action of the symmetric group S_n on $\mathbb{R}^n$ by permuting the coordinates:

$$W = W(U) := N(U)/Z(U),$$

where $N(U) := \{\tau \in \mathrm{S}_n : \tau.U = U\}$ and $Z(U) := \{\tau \in \mathrm{S}_n : \tau.x = x \text{ for all } x \in U\}$.

Under the additional assumption that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$ of W-invariant polynomials on U, we will show that the conditions imposed on $P(t)$ in order to guarantee the existence of a smooth parameterization of its roots may be weakened. These conditions will be formulated in terms of the two natural stratifications carried by U and $E(U) = U/W$: the orbit type stratification with respect to W and the restriction of the orbit type stratification with respect to S_n. The latter will be called ambient stratification. See section 4. It will turn out (section 5) that we may find global smooth parameterizations of the roots of $P(t)$, provided that $P(t)$ is normally nonflat with respect to the orbit type stratification of $E(U) = U/W$ at any t. This condition is in general weaker than the condition found in [1], since we prove in section 4 that normal nonflatness with respect to the ambient stratification implies normal nonflatness with respect to the orbit type stratification. For a definition of 'normally nonflat' see 2.5.

These improvements are essentially applications of the lifting problem tackled in [2]. See also [16] and [17]. This generalization of the above problem studies the question whether it is possible to lift smoothly a smooth curve in the orbit space V/G of an orthogonal finite dimensional representation of a compact Lie group G into the representation space V. Here the orbit space V/G is identified with the semialgebraic subset $\sigma(V)$ in $\mathbb{R}^n$ given by the image of the orbit map $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$, where $\sigma_1, \ldots, \sigma_n$ constitute a system of homogeneous generators of the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V. See section 2 for details.

As mentioned before a C^{3n} curve $P(t)$ of hyperbolic polynomials of degree n allows twice differentiable parameterizations of its roots. Using results found for the general lifting problem in [17], we are able to lower the degree of regularity in the assumption of that statement, if the polynomials $P(t)$ satisfy certain symmetries. See section 6.

A class of examples for which the described refinements apply will be constructed in section 7. For illustration we consider the case when W is a finite reflection group in section 8. Moreover, explicit examples will be treated.

The problem of finding regular roots of families of hyperbolic polynomials has relevance in the perturbation theory of selfadjoint operators (e.g. [14], [18], [28]) and in the theory of partial differential equations for the well-posedness of hyperbolic Cauchy problems (e.g. [9], [12]).

2. Preliminaries

2.1. **Representations of compact Lie groups.** Let G be a compact Lie group and let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V with inner product $\langle \quad | \quad \rangle$. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \ldots, d_n$. Consider the *orbit map* $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. The image $\sigma(V)$ is a semialgebraic set in $Z := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$ where I is the ideal of relations between $\sigma_1, \ldots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$, by the following lemma.

Lemma. *Suppose that X and Y are locally compact, Hausdorff spaces and that $f : X \to Y$ is bijective, continuous, and proper. Then f is a homeomorphism.*

Proof. (E.g. [7]) By defining $\tilde{f}(\infty) = \infty$, f extends to a continuous map $\tilde{f} : X \cup \{\infty\} \to Y \cup \{\infty\}$ between the one point compactifications, since it is proper. If $A \subseteq X$ is closed in X, then $A \cup \{\infty\}$ is closed in $X \cup \{\infty\}$ and hence compact. Then, $\tilde{f}(A \cup \{\infty\})$ is compact and hence closed in $Y \cup \{\infty\}$. Consequently, $f(A) = \tilde{f}(A \cup \{\infty\}) \cap Y$ is closed in Y. $\square$

2.2. **Description of $\sigma(V)$.** Let $\langle \quad | \quad \rangle$ denote also the G-invariant dual inner product on V^*. The differentials $d\sigma_i : V \to V^*$ are G-equivariant, and the polynomials $v \mapsto \langle d\sigma_i(v) \mid d\sigma_j(v) \rangle$ are in $\mathbb{R}[V]^G$ and are entries of an $n \times n$ symmetric matrix valued polynomial

$$B(v) := \begin{pmatrix} \langle d\sigma_1(v) \mid d\sigma_1(v) \rangle & \cdots & \langle d\sigma_1(v) \mid d\sigma_n(v) \rangle \\ \vdots & \ddots & \vdots \\ \langle d\sigma_n(v) \mid d\sigma_1(v) \rangle & \cdots & \langle d\sigma_n(v) \mid d\sigma_n(v) \rangle \end{pmatrix}.$$

There is a unique matrix valued polynomial $\tilde{B}$ on Z such that $B = \tilde{B} \circ \sigma$. The following theorem is due to Procesi and Schwarz [27].

Theorem. $\sigma(V) = \{z \in Z : \tilde{B}(z)$ *positive semidefinite*$\}$.

This theorem provides finitely many equations and inequalities describing $\sigma(V)$. Changing the choice of generators may change the equations and inequalities, but not the set they describe.

For each $1 \le i_1 < \cdots < i_s \le n$ and $1 \le j_1 < \cdots < j_s \le n$ $(s \le n)$ consider the matrix with entries $\langle d\sigma_{i_p} \mid d\sigma_{j_q} \rangle$ for $1 \le p, q \le s$. Denote its determinant by $\Delta_{i_1,\dots,i_s}^{j_1,\dots,j_s}$. Then, $\Delta_{i_1,\dots,i_s}^{j_1,\dots,j_s}$ is a G-invariant polynomial on V, and thus there is a unique polynomial $\tilde{\Delta}_{i_1,\dots,i_s}^{j_1,\dots,j_s}$ on Z such that $\Delta_{i_1,\dots,i_s}^{j_1,\dots,j_s} = \tilde{\Delta}_{i_1,\dots,i_s}^{j_1,\dots,j_s} \circ \sigma$.

2.3. **The problem of lifting curves.** Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space; smooth as curve in $\mathbb{R}^n$. A curve $\bar{c} : \mathbb{R} \to V$ is called lift of c to V, if $c = \sigma \circ \bar{c}$ holds. *The problem of lifting smooth curves over invariants is independent of the choice of a system of homogeneous generators of* $\mathbb{R}[V]^G$ in the following sense: Suppose $\sigma_1, \dots, \sigma_n$ and $\tau_1, \dots, \tau_m$ both generate $\mathbb{R}[V]^G$. Then for all i and j we have $\sigma_i = p_i(\tau_1, \dots, \tau_m)$ and $\tau_j = q_j(\sigma_1, \dots, \sigma_n)$ for polynomials p_i and q_j. If $c^\sigma = (c_1, \dots, c_n)$ is a curve in $\sigma(V)$, then $c^\tau = (q_1(c^\sigma), \dots, q_m(c^\sigma))$ defines a curve in $\tau(V)$ of the same regularity. Any lift $\bar{c}$ to V of the curve c^σ, i.e., $c^\sigma = \sigma \circ \bar{c}$, is a lift of c^τ as well (and conversely):

$$c^\tau = (q_1(c^\sigma), \dots, q_m(c^\sigma)) = (q_1(\sigma(\bar{c})), \dots, q_m(\sigma(\bar{c}))) = (\tau_1(\bar{c}), \dots, \tau_m(\bar{c})) = \tau \circ \bar{c}.$$

2.4. **Stratification of the orbit space.** Let $H = G_v$ be the isotropy group of $v \in V$ and (H) the conjugacy class of H in G which is called the type of an orbit $G.v$. The union $V_{(H)}$ of orbits of type (H) is called an orbit type submanifold of the representation ρ and $V_{(H)}/G$ is called an orbit type submanifold of the orbit space V/G. The collection of connected components of the manifolds $\{V_{(H)}/G\}$ forms a stratification of V/G called orbit type stratification, see [26], [30]. The semialgebraic subset $\sigma(V) \subseteq \mathbb{R}^n$ is naturally Whitney stratified ([19]). The homeomorphism of V/G and $\sigma(V)$ induced by σ provides an isomorphism between the orbit type stratification of V/G and the primary Whitney stratification of $\sigma(V)$, see [5]. These facts are essentially consequences of the slice theorem, see e.g. [30].

The inclusion relation on the set of subgroups of G induces a partial ordering on the family of conjugacy classes. There is a unique minimum orbit type, the principal orbit type, corresponding to the open and dense submanifold V_{reg} (respectively V_{reg}/G) consisting of regular points, i.e., points where the isotropy representation is trivial. The points in the complement V_{sing} (respectively V_{sing}/G) are called singular.

Theorem. [27] *Let $\tilde{B}$ be as in 2.2. The k-dimensional primary strata of $\sigma(V)$ are the connected components of the set $\{z \in \sigma(V) : \operatorname{rank} \tilde{B}(z) = k\}$.*

2.5. **Smooth lifts.** Let us recall some results from [2].

Let $s \in \mathbb{N}_0$. Denote by A_s the union of all strata X of the orbit space V/G with $\dim X \le s$, and by I_s the ideal of $\mathbb{R}[Z] = \mathbb{R}[V]^G$ consisting of all polynomials vanishing on A_{s-1}. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve, $t \in \mathbb{R}$, and $s = s(c,t)$ a minimal integer such that, for a neighborhood J of t in $\mathbb{R}$, we have $c(J) \subseteq A_s$. The curve c is called *normally nonflat at t* if there is $f \in I_s$ such that $f \circ c$ is nonflat at t, i.e., the Taylor series of $f \circ c$ at t is not identically zero. A smooth curve $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ is called *generic*, if c is normally nonflat at t for each $t \in \mathbb{R}$.

It is easy to see, that c is normally nonflat at $t \in \mathbb{R}$ if there is some integer $1 \le r \le n$ such that:

(1) The functions $\tilde{\Delta}^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} \circ c$ vanish in a neighborhood of t whenever $k > r$.
(2) There exists a minor $\tilde{\Delta}^{j_1,\ldots,j_r}_{i_1,\ldots,i_r}$ such that $\tilde{\Delta}^{j_1,\ldots,j_r}_{i_1,\ldots,i_r} \circ c$ is nonflat at t.

Theorem. *Let $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve which is normally nonflat at $t \in \mathbb{R}$. Then there exists a smooth lift $\bar{c}$ in V of c, locally near t. If c is generic then there exists a global smooth lift $\bar{c}$ of c.*

2.6. **Smooth roots.** In the special case that the symmetric group S_n is acting on $\mathbb{R}^n$ by permuting the coordinates there is the following interpretation of the described lifting problem. As generators of $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ we may take the elementary symmetric functions

$$E_j(x) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j} \qquad (1 \le j \le n),$$

which constitute the coefficients a_j of a monic polynomial

$$P(x) = x^n - a_1 x^{n-1} + \cdots + (-1) a_n$$

with roots $x_1, \ldots, x_n$ via Vieta's formulas. Then a curve in the orbit space $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n)$ corresponds to a curve $P(t)$ of monic polynomials of degree n with only real roots (such polynomials are called *hyperbolic*), and a lift of $P(t)$ may be interpreted as a parameterization of the roots of $P(t)$.

The first n Newton polynomials

$$N_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_j^i$$

which are related to the elementary symmetric functions by

$$N_k - N_{k-1}E_1 + N_{k-2}E_2 + \cdots + (-1)^{k-1} N_1 E_{k-1} + (-1)^k k E_k = 0 \quad (k \ge 1) \tag{2.1}$$

constitute a different system of generators of $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$. For convenience we shall switch from elementary symmetric functions to Newton polynomials and conversely, if it seems appropriate.

Let us choose $\frac{1}{j}N_j$, $1 \le j \le n$, as generators of $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ and put $\Delta_k := \Delta^{1,\dots,k}_{1,\dots,k}$ and $\tilde{\Delta}_k := \tilde{\Delta}^{1,\dots,k}_{1,\dots,k}$. Then ([1])

$$\Delta_k(x) = \sum_{i_1<\dots<i_k} (x_{i_1} - x_{i_2})^2 \cdots (x_{i_1} - x_{i_k})^2 \cdots (x_{i_{k-1}} - x_{i_k})^2. \tag{2.2}$$

Theorem. [1] *Consider a smooth curve $P(t)$, $t \in \mathbb{R}$, of monic hyperbolic polynomials of fixed degree n. Let one of the following two equivalent conditions be satisfied:*

(1) *If two of the increasingly ordered continuous roots meet of infinite order at t_0 then their germs at t_0 are equal.*
(2) *Let k be maximal with the property that the germ at t_0 of $\tilde{\Delta}_k(P)$ is not 0. Then $\tilde{\Delta}_k(P)$ is not infinitely flat at t_0.*

Then $P(t)$ is smoothly solvable near $t = t_0$. If (1) or (2) are satisfied for any $t_0 \in \mathbb{R}$, then the roots of P may be chosen smoothly globally, and any two choices differ by a permutation.

Lemma. *Condition (1) (and thus condition (2)) in the above theorem is satisfied if and only if P is normally nonflat at t_0 as curve in $E(\mathbb{R}^n) = \mathbb{R}^n/\mathrm{S}_n$.*

Proof. Let P be normally nonflat at t_0. Let s be a minimal integer such that $P(t)$ lies in A_s for t near t_0 and let $f \in I_s$ be such that $f \circ P$ is not infinitely flat at t_0. Denote by $\bar{I}_s$ the ideal in $\mathbb{R}[\mathbb{R}^n]$ defining the closed subset $\pi^{-1}(A_{s-1}) \subseteq \mathbb{R}^n$, where $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathrm{S}_n$ is the quotient projection. It is easy to see that the polynomials

$$f_{i_1\dots i_s} = (x_{i_1} - x_{i_2}) \cdots (x_{i_1} - x_{i_s}) \cdots (x_{i_{s-1}} - x_{i_s}),$$

where $1 \le i_1 < \dots < i_s \le n$, generate $\bar{I}_s$. So there exist polynomials $Q_{i_1\dots i_s} \in \mathbb{R}[\mathbb{R}^n]$ such that

$$f \circ \pi = \sum_{i_1<\dots<i_s} Q_{i_1\dots i_s} f_{i_1\dots i_s}.$$

Denote by $\bar{P}(t)$ the lift of $P(t)$ given by the increasingly ordered continuous roots $x_1(t), \dots, x_n(t)$ of the polynomial $P(t)$. Then we have

$$f \circ P(t) = \sum_{i_1<\dots<i_s} Q_{i_1\dots i_s} \circ \bar{P}(t) \cdot f_{i_1\dots i_s} \circ \bar{P}(t).$$

Since $f \circ P$ is not infinitely flat at t_0, at least one of the summands in this sum is not infinitely flat at t_0 and thus there is a polynomial $f_{i_1\dots i_s}$ such that $f_{i_1\dots i_s} \circ \bar{P}$ is not infinitely flat at t_0. By assumption, among the roots $x_1(t), \dots, x_n(t)$ there are precisely s distinct for t near t_0. Hence the germs at t_0 of the roots $x_{i_1}(t), \dots, x_{i_s}(t)$ are distinct, and no two of them meet of infinite order at t_0. Therefore, condition (1) in the above theorem is satisfied.

The other direction is evident by (2.2). □

3. Lifting smooth curves in spaces of hyperbolic polynomials

3.1. **The problem.** Let us denote by Hyp^n the space of hyperbolic polynomials of degree n

$$P(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j x^{n-j}.$$

We may naturally view Hyp^n as a semialgebraic subset of $\mathbb{R}^n$ by identifying P with $(a_1, \dots, a_n)$. We have $\mathrm{Hyp}^n = E(\mathbb{R}^n) = \mathbb{R}^n/\mathrm{S}_n$, and, by means of 2.2, we may calculate explicitly a set of inequalities defining Hyp^n (no equalities since the ring $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ is polynomial).

Suppose X is a semialgebraic subset of Hyp^n. Let $c : \mathbb{R} \to X$ be a smooth curve in X; smooth as curve in $\mathbb{R}^n$. We may view c as a curve in Hyp^n, i.e., as a smooth curve of monic hyperbolic polynomials of degree n. In 2.6 sufficient conditions for the existence of a smooth lift $\bar{c}$ to $\mathbb{R}^n$, i.e., a smooth parameterization of its roots, are presented. It is evident that a smooth curve c in X in order to be liftable smoothly over E to $E^{-1}(X)$ must in general fulfill weaker genericity conditions. Our purpose is to investigate that phenomenon.

3.2. **Orbit spaces embedded in spaces of hyperbolic polynomials.** We recall a construction due to L. Smith and R.E. Strong [32] (see also [3]) related to E. Noether's [25] proof of Hilbert's finiteness theorem as recounted by H. Weyl [35].

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G in a finite dimensional vector space V. Consider its induced representation in the dual V^*. For an orbit $B \subseteq V^*$ set

$$\phi_B(X) = \prod_{b \in B} (X + b)$$

which we regard as an element of the ring $\mathbb{R}[V][X]$, with X a new variable. The polynomial $\phi_B(X)$ is called the *orbit polynomial* of B. Evidently, $\phi_B \in \mathbb{R}[V]^G[X]$. If $|B|$ denotes the cardinality of the orbit B, we may expand $\phi_B(X)$ to a polynomial of degree $|B|$ in X,

$$\phi_B(X) = \sum_{i+j=|B|} C_i(B) X^j,$$

defining classes $C_i(B) \in \mathbb{R}[V]^G$ called the *orbit Chern classes* of B.

Theorem (L. Smith and R.E. Strong [32])**.** *Let $\rho : G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation of a finite group G. Then there exist orbits $B_1, \ldots, B_l \subseteq V^*$ such that the associated orbit Chern classes $C_i(B_j)$, $1 \le i \le |B_j|$, $1 \le j \le l$, generate $\mathbb{R}[V]^G$.*

The field of real numbers may be replaced by any field of either characteristic zero or characteristic larger than the order of G. For our purpose the reals will suffice.

The Chern classes of the orbit are exactly the elementary symmetric functions in the elements of the orbit. If $B \subseteq V^*$ is an orbit and V_B^* is a vector space with basis identified with the elements of B, then there is a natural map $V_B^* \to V^*$ given by the identification. This map induces a map $\mathbb{R}[V_B]^{\mathrm{S}_{|B|}} \to \mathbb{R}[V]^G$ which sends the k-th elementary symmetric function to the k-th orbit Chern class of B.

In this notation the above theorem says that there exist orbits $B_1, \ldots, B_l \subseteq V^*$ such that the induced map

$$\bigotimes_{i=1}^{l} \mathbb{R}[V_{B_i}]^{\mathrm{S}_{|B_i|}} \longrightarrow \mathbb{R}[V]^G$$

is surjective.

The orbit Chern classes $C_i(B)$ of an orbit B, viewed as invariant polynomials on V, define a G-invariant map

$$C(B) = (C_1(B), \ldots, C_{|B|}(B)) : V \longrightarrow \mathbb{R}^{|B|}$$

whose image $C(B)(V)$ is a semialgebraic subset of the space $\mathrm{Hyp}^{|B|}$ of hyperbolic polynomials of degree $|B|$.

According to 2.1 and the above theorem, for any faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$ of a finite group G there exist orbits $B_1, \ldots, B_l \subseteq V^*$ such that the map

$$C(\rho) = (C(B_1), \ldots, C(B_l)) : V \longrightarrow \mathrm{Hyp}^{|B_1|} \times \cdots \times \mathrm{Hyp}^{|B_l|} \subseteq \mathbb{R}^{|B_1|+\cdots+|B_l|}$$

induces a homeomorphism between the orbit space V/G and the image $C(\rho)(V)$ which is a semialgebraic subset of $\mathrm{Hyp}^{|B_1|} \times \cdots \times \mathrm{Hyp}^{|B_l|}$. By increasing the number of orbits B_i if necessary, we may assume that each irreducible subspace of V contributes at least one orbit B_i. Then, the linear forms $b \in B_1 \cup \cdots \cup B_l$ induce an injective inclusion $V \hookrightarrow \mathbb{R}^{|B_1|+\cdots+|B_l|}$.

Let $c : \mathbb{R} \to C(\rho)(V)$ be a smooth curve. Then $c = (c_1, \ldots, c_l)$ where each $c_i : \mathbb{R} \to C(B_i)(V)$ is smooth. Since $C(B_i)(V) \subseteq \mathrm{Hyp}^{|B_i|}$ we may view c_i as a curve in $\mathrm{Hyp}^{|B_i|}$. If there exist smooth lifts $\bar{c}_i : \mathbb{R} \to \mathbb{R}^{|B_i|}$ with respect to the representations $\mathrm{S}_{|B_i|} : \mathbb{R}^{|B_i|}$, then $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_l) : \mathbb{R} \to \mathbb{R}^{|B_1|+\cdots+|B_l|}$ is a smooth lift with respect to $\mathrm{S}_{|B_1|} \times \cdots \times \mathrm{S}_{|B_l|} : \mathbb{R}^{|B_1|+\cdots+|B_l|}$. Consequently, it suffices to study the case when there is given a smooth curve in a semialgebraic subset of some Hyp^n. That is exactly the problem introduced in 3.1.

Suppose $\tilde{c} : \mathbb{R} \to V$ is a smooth lift of c with respect to ρ. Then, there exists a smooth lift $\bar{c} : \mathbb{R} \to \mathbb{R}^{|B_1|+\cdots+|B_l|}$ of c with respect to the representation of $\mathrm{S}_{|B_1|} \times \cdots \times \mathrm{S}_{|B_l|}$ on $\mathbb{R}^{|B_1|+\cdots+|B_l|}$, namely

$$\begin{tikzcd} & V \arrow[r, hook] \arrow[d] & \mathbb{R}^{|B_1|+\cdots+|B_l|} \arrow[d] \\ \mathbb{R} \arrow[ur, "\tilde{c}"] \arrow[r, "c"'] & C(\rho)(V) \arrow[r, hook] & \mathrm{Hyp}^{|B_1|} \times \cdots \times \mathrm{Hyp}^{|B_l|} \end{tikzcd}$$

It follows, by 2.5, that conditions which guarantee that c is generic as curve in the orbit space V/G suffice to imply the existence of a smooth lift of c with respect to $\mathrm{S}_{|B_1|} \times \cdots \times \mathrm{S}_{|B_l|} : \mathbb{R}^{|B_1|+\cdots+|B_l|}$.

We have seen that the above construction provides a class of semialgebraic subsets of spaces of hyperbolic polynomials, namely orbit spaces of faithful finite group representations, for which we are able to apply the strategy described in 3.1, thanks to the results of 2.5.

In the remaining sections we shall change the point of view. Assume we are given a curve of hyperbolic polynomials with certain symmetries. We will investigate whether we can weaken the conditions in 2.6 which guarantee the existence of smooth parameterizations of the roots. This will be performed in section 5. The following section provides the necessary preparation.

4. Orbit type and ambient stratification

Suppose U is a linear subspace of $\mathbb{R}^n$. Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates and endow U with the induced effective action of

$$W = W(U) := N(U)/Z(U),$$

where $N(U) := \{\tau \in \mathrm{S}_n : \tau.U = U\}$ and $Z(U) := \{\tau \in \mathrm{S}_n : \tau.x = x \text{ for all } x \in U\}$. Then U carries two natural stratifications: the orbit type stratification with respect to the W-action and the restriction to U of the orbit type stratification of $\mathbb{R}^n$ with respect to the S_n-action. It is easily seen that the latter indeed provides a Whitney stratification of U. Let us denote it as the *ambient stratification of U*.

Proposition 4.1. *Let U be a linear subspace in $\mathbb{R}^n$ endowed with the induced action by $W = W(U)$. Then for the ambient and orbit type stratification of U we have:*

1. *Each ambient stratum is contained in a unique orbit type stratum.*
2. *Each orbit type stratum contains at least one ambient stratum of the same dimension and is the union of all contained ambient strata.*

Proof. To (1): Let S be an ambient stratum, i.e., S is a component of $\mathrm{S}_n.\mathbb{R}^n_H \cap U$, where $H = (\mathrm{S}_n)_x$ for a $x \in U$ and $\mathbb{R}^n_H = \{y \in \mathbb{R}^n : (\mathrm{S}_n)_y = H\}$. Since S_n is finite and the manifolds $\tau.\mathbb{R}^n_H$ for $\tau \in \mathrm{S}_n$ either coincide or are pairwise disjoint, the components of $\mathrm{S}_n.\mathbb{R}^n_H$ are open subsets of $\tau.\mathbb{R}^n_H$ for $\tau \in \mathrm{S}_n$. Thus, we may assume that S is a component of $\mathbb{R}^n_H \cap U$.

Denote by π the quotient projection $N(U) \to N(U)/Z(U) = W$. For any $u \in U$ we have $W_u = \pi(N(U) \cap (\mathrm{S}_n)_u)$ and thus $\mathbb{R}^n_H \cap U \subseteq \{u \in U : W_u = W_x\}$. By definition and a similar argument as above, the components of the subset $\{u \in U : W_u = W_x\}$ are orbit type strata of U. So the ambient stratum S is contained in a unique isotropy type stratum R_S.

To (2): Let R be an orbit type stratum and let $\mathfrak{S}$ be the set of all ambient strata S such that $R_S = R$, where R_S is the unique orbit type stratum from (1). Clearly, $R = \bigcup \mathfrak{S}$ and for each $S \in \mathfrak{S}$ we have $\dim S \le \dim R$. Since the set $\mathfrak{S}$ is finite, there is a stratum $S \in \mathfrak{S}$ such that $\dim S = \dim R$. $\square$

Remarks 4.2. (1) It is easy to see that proposition 4.1 is true if one replaces the S_n-module $\mathbb{R}^n$ by any finite dimensional G-module V, where G is a finite group.

(2) Proposition 4.1 implies that the orbit type stratification of U is coarser than its ambient stratification. That means, following [26], that for each ambient stratum S there exists an orbit type stratum R_S such that $S \subseteq R_S$, $\mathrm{id}|_S : S \to R_S$ is smooth, and for all $S \subseteq \overline{S'}$ we have $R_S \subseteq \overline{R_{S'}}$. It remains to check the last condition: Assume that $S \subseteq \overline{S'}$. Since $S \subseteq R_S$ and $S \subseteq \overline{S'} \subseteq \overline{R_{S'}}$, we obtain $R_S \cap \overline{R_{S'}} \ne \emptyset$, and, by the frontier condition, $R_S \subseteq \overline{R_{S'}}$.

Assume that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$. It follows that $E|_U = (E_1|_U, \ldots, E_n|_U)$ induces a homeomorphism between U/W and the semialgebraic subset $E(U)$ of $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$, by 2.1. It is well-known that $U_{(H)} \to U_{(H)}/W$, where $H = W_u$ for some $u \in U$, is a Riemannian submersion. Since W is finite, it is even a local diffeomorphism. By proposition 4.1, this implies that for any ambient stratum S in U the image $E(S)$ is a smooth manifold. The collection $\mathcal{T} = \{E(S) : S \text{ ambient stratum in } U\}$ obviously coincides with the collection obtained by restricting to $E(U)$ the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$. It is easily verified that the frontier condition for the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$ implies the frontier condition for $\mathcal{T}$. Consequently, $\mathcal{T}$ provides a stratification of $E(U)$. Let us denote this stratification as the *ambient stratification of* $E(U)$.

Consider a smooth curve $c : \mathbb{R} \to E(U) = U/W$ in the sense of 2.3. It may then be also viewed as a smooth curve in $\mathbb{R}^n/\mathrm{S}_n = E(\mathbb{R}^n) = \mathrm{Hyp}^n$. Thus it makes sense to speak about the normal nonflatness of c at some point t_0 with respect to the orbit type stratification of U/W on the one hand and with respect to the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n$ on the other hand. To shorten notation we shall say that c is normally nonflat at t_0 with respect to the ambient stratification of U/W iff it is normally nonflat at t_0 with respect to the orbit type stratification of $\mathbb{R}^n/\mathrm{S}_n$.

Proposition 4.3. *Let U be a linear subspace in $\mathbb{R}^n$ endowed with the induced action by $W = W(U)$ and assume that the restrictions $E_i|_U$, $1 \le i \le n$, generate $\mathbb{R}[U]^W$. Consider a smooth curve $c : \mathbb{R} \to E(U) = U/W$. If c is normally nonflat at t_0 with respect to the ambient stratification of U/W, then it is normally nonflat at t_0 with respect to the orbit type stratification of U/W.*

Proof. The set of reflection hyperplanes H of the reflection group S_n is in bijective correspondence with the set of linear functionals ω_H on $\mathbb{R}^n$ of the form $x_j - x_i$ for $1 \le i < j \le n$, namely H is the kernel of ω_H. Let us consider the restrictions $\omega_H|_U$ to U. If c is normally nonflat at t_0 with respect to the ambient stratification, then, by lemma 2.6, any two of the increasingly ordered continuous roots of the polynomial $c(t) \in E(U) \subseteq \mathrm{Hyp}^n$ either coincide identically near t_0 or do not meet at t_0 of infinite order. Then for the continuous lift $\bar{c}$ of c defined by such a choice of roots any function $\omega_H \circ \bar{c}$ either vanishes identically near t_0 or does not vanish at t_0 of infinite order.

Let s be a minimal integer such that $c(t)$ lies in $A_{s,\mathrm{orb}}$ for t near t_0, where $A_{s,\mathrm{orb}}$ is the union of all orbit type strata of U/W of dimension $\le s$.

Denote by π_U the projection $U \to U/W$. Let R be an orbit type stratum contained in $\pi_U^{-1}(A_{s-1,\mathrm{orb}})$ and let $S_1, \dots, S_k$ be the ambient strata of the same dimension as R contained in R (see proposition 4.1). For each $1 \le j \le k$ denote by $\mathcal{H}_j$ the set of reflection hyperplanes for reflections in S_n fixing S_j pointwise. Let Ω_j be the set of linear functionals $\omega_H|_U$ for $H \in \mathcal{H}_j$. Put $f_{R,j} = \sum_{\omega \in \Omega_j} \omega^2$. By definition the equation $f_{R,j} = 0$ defines a linear subspace of U in which S_j is an open subset. Let $f_R = \prod_{j=1}^k f_{R,j}$. Consider the natural action of W on $\mathbb{R}[U]$ and let $W.f_R = \{f_R^1, \dots, f_R^l\}$ be the orbit through f_R with respect to this action. Define $F_R = f_R^1 \cdots f_R^l$. By construction $F_R \in \mathbb{R}[U]^W$ and the set Z_R of zeros of F_R viewed as a function on U/W is contained in $A_{s-1,\mathrm{orb}}$. Moreover, $A_{s-1,\mathrm{orb}}$ is the union of the Z_R, where R ranges over all orbit type strata (of maximal dimension) contained in $\pi_U^{-1}(A_{s-1,\mathrm{orb}})$. Thus $F = \prod_R F_R$, where the product is taken over all orbit type strata (of maximal dimension) R contained in $\pi_U^{-1}(A_{s-1,\mathrm{orb}})$, is a regular function on U/W whose set of zeros equals $A_{s-1,\mathrm{orb}}$. By construction, the function $F \circ c$ is nonflat at t_0.

This proves the statement. □

We define $F_{\mathrm{amb}}(c)$ (resp. $F_{\mathrm{orb}}(c)$) to be the set of all $t \in \mathbb{R}$ such that c is normally flat at t with respect to the ambient (resp. orbit type) stratification of $E(U)$. It follows that in the situation of proposition 4.3 we have $F_{\mathrm{orb}}(c) \subseteq F_{\mathrm{amb}}(c)$.

5. Choosing roots of polynomials with symmetries smoothly

Consider a smooth curve of hyperbolic polynomials

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

We are interested in conditions that guarantee the existence of a smooth parameterization of the roots of P. Such conditions have been found in [1], see 2.6. There no additional assumptions on the polynomials $P(t)$ have been made.

In this section we are going to improve those results if the set of roots $x_1(t), \dots, x_n(t)$ of $P(t)$ has symmetries additional to its invariance under permutations.

Let as assume that the additional symmetries of $P(t)$ are given by linear relations between the roots of $P(t)$. Otherwise put, there is a linear subspace U of $\mathbb{R}^n$ such that $(x_1(t), \dots, x_n(t)) \in U$ for all $t \in \mathbb{R}$. Then, the curve $P(t)$ lies in the semialgebraic subset $E(U)$ of $\mathrm{Hyp}^n = E(\mathbb{R}^n) = \mathbb{R}^n/\mathrm{S}_n$, the space of hyperbolic polynomials of degree n.

The linear subspace $U \subseteq \mathbb{R}^n$ inherits an effective action by the group $W = W(U)$.

Let us suppose that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^W$. Then $E|_U = (E_1|_U, \dots, E_n|_U)$ induces a homeomorphism between U/W and the semialgebraic subset $E(U)$ of Hyp^n, by 2.1.

Lemma 5.1. *Consider a continuous curve of hyperbolic polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Let U be some linear subspace of $\mathbb{R}^n$ and assume that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$. Then the following two conditions are equivalent:

1. *There exists a continuous parameterization $x(t)$ of the roots $x_1(t), \dots, x_n(t)$ of $P(t)$ such that $x(t) \in U$ for all $t \in \mathbb{R}$.*
2. *$P(t) \in E(U)$ for all $t \in \mathbb{R}$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Suppose that $P(t)$ is a continuous curve in $E(U)$. By assumption, we may view $P(t)$ as a curve in the orbit space $U/W(U) \cong E(U)$. It allows a continuous lift $x(t)$ into U, by [16] or [24], which constitutes a parameterization of the roots of $P(t)$. $\square$

The smooth curve of polynomials $P(t)$ which lies in $E(U)$ may be viewed as a smooth curve in the orbit space U/W in the sense of 2.3. A smooth lift of $P(t)$ over the orbit map $E|_U$ to the W-module U provides a smooth parameterization of the roots of the polynomials $P(t)$.

By theorem 2.5, we may conclude: If $P(t)$ is normally nonflat at $t = t_0$ with respect to the orbit type stratification of $E(U)$, then $P(t)$ is smoothly solvable near $t = t_0$.

Consider the closed sets $F_{\text{amb}}(P)$ and $F_{\text{orb}}(P)$, as defined in section 4. By proposition 4.3, the set $F_{\text{orb}}(P)$ is contained in $F_{\text{amb}}(P)$. We have found that that $P(t)$ is smoothly solvable locally near any $t_0 \in \mathbb{R}\backslash F_{\text{orb}}(P)$. Any two smooth parameterizations of the roots of $P(t)$ near such a t_0 differ by a constant permutation, see theorem 2.6. Thus the local solutions may be glued to a smooth solution on $\mathbb{R}\backslash F_{\text{orb}}(P)$.

It follows from a result in [15] (see also [17]) that any smooth curve of monic hyperbolic polynomials of fixed degree allows a global twice differentiable parameterization of its roots. By the methods used in [15], it is easy to combine this with the result above in order to get the following theorem.

Theorem 5.2. *Consider a smooth curve of hyperbolic polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Let U be some linear subspace of $\mathbb{R}^n$ such that:

1. *The restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$.*
2. *$P(t) \in E(U)$ for all $t \in \mathbb{R}$.*

Then: There exists a global twice differentiable parameterization of the roots of $P(t)$ on $\mathbb{R}$ which is smooth on $\mathbb{R}\backslash F_{\text{orb}}(P)$. $\square$

Remark 5.3. The orbit type stratification and the ambient stratification of $E(U)$ do in general not coincide, whence theorem 5.2 provides an actual improvement of the statement of theorem 2.6. In other words, in general we have $F_{\text{orb}}(P) \subsetneq F_{\text{amb}}(P)$. It may, for instance, happen that $P(0)$ is regular in $E(U) = U/W$ but singular in $\text{Hyp}^n = \mathbb{R}^n/\,\mathrm{S}_n$ and $P(t)$ is normally flat at $t = 0$ with respect to the ambient stratification. See examples in section 8.

Let us suppose that a linear subspace U of $\mathbb{R}^n$ is given. It is then a purely computational problem to check whether the assumptions we have made in the forgoing discussion are satisfied. There are algorithms in computational invariant theory (e.g. [10], [33]) which allow to decide whether the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$. If the answer is yes, theorem 2.2 provides an explicit way to describe the semialgebraic subset $E(U) \subseteq \text{Hyp}^n$ by a finite set of polynomial equations and inequalities. So the condition that the curve P lies in $E(U)$ may again be check computationally. The orbit type stratification and the ambient stratification of $E(U)$ can be determined explicitly using theorem 2.4. Then all ingredients are supplied in order to decide whether the curve $P(t)$ is normally nonflat at some $t = t_0$ with respect to the one or the other stratification of $E(U)$.

Note that there are refined approaches and algorithms for computing the orbit space V/G and its orbit type stratification of a G-module V (when identified with the image of its orbit map). In [29] rational parameterizations of the strata are obtained, while [4] provides an algorithm yielding a description of each stratum in terms of a minimal number of polynomial equations and inequalities, if G is finite. We shall carry out that procedure explicitly in example 8.8.

6. Choosing roots of polynomials with symmetries differentiably

Consider a curve of hyperbolic polynomials

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R}).$$

Then the following results are known:

Result 6.1. *We have:*

1. *If all a_i are of class C^n, then there exists a differentiable parameterization of the roots of $P(t)$ with locally bounded derivative,* [8], [34].
2. *If all a_i are of class C^{2n}, then any differentiable parameterization of the roots of $P(t)$ is actually C^1,* [15], [21].
3. *If all a_i are of class C^{3n}, then there exists a twice differentiable parameterization of the roots of $P(t)$,* [15].

In [17] we have proved the following generalizations:

Result 6.2. *Let $\rho : G \to \mathrm{O}(V)$ be a finite dimensional representation of a finite group G. Let $d = d(\rho)$ be the maximum of the degrees of a minimal system of homogeneous generators $\sigma_1, \dots, \sigma_m$ of $\mathbb{R}[V]^G$. Write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i. Define $k_i := \min\{|G.v| : v \in V_i \backslash \{0\}\}$, $1 \le i \le l$, and $k := \max\{d(\rho), k_1, \dots, k_l\}$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^m$ be a curve in the orbit space. Then:*

1. *If c is of class C^k, then there exists a differentiable lift of c to V with locally bounded derivative.*
2. *If c is of class C^{k+d}, then any differentiable lift of c is actually of class C^1.*
3. *If c is of class C^{k+2d}, then there exists a twice differentiable lift of c to V.*

Again we may use these facts in order to improve the results for curves $P(t)$ of hyperbolic polynomials with symmetries.

Let U be some linear subspace of $\mathbb{R}^n$ such that the restrictions $E_i|_U$, $1 \le i \le n$, generate the algebra $\mathbb{R}[U]^{W(U)}$, and $P(t) \in E(U)$ for all $t \in \mathbb{R}$. It follows that we may view $P(t)$ as a curve in the orbit space $U/W(U) = E(U)$, and any lift of $P(t)$ over the orbit map $E|_U$ to U gives a parameterization of the roots of $P(t)$ of the same regularity.

Provided that the integer k, associated to the $W(U)$-module U as above, is less than the degree n of the polynomials in $P(t)$, we are able, using 6.2, to lower the degree of regularity in the assumptions of the statements in 6.1. We shall give examples in section 8.

7. Construction of a class of examples

We will present a class of examples which our considerations apply to.

Let $G \subseteq \mathrm{O}(V)$ be a finite group whose action on the vector space V is irreducible and effective. Choose some non-zero orbit $G.v$. Introducing some numbering we can write $G.v = \{g_1.v, \dots, g_n.v\}$, where $|G.v| = n$ and $g_i \in G$. We define a mapping $F_{G,v} : V \to \mathbb{R}^n$ by

$$F_{G,v}(x) := (\langle g_1.v \mid x\rangle, \dots, \langle g_n.v \mid x\rangle).$$

Since the linear span of $G.v$ spans V, the mapping $F_{G,v}$ is a linear isomorphism onto its image $F_{G,v}(V) =: U_{G,v}$. The linear space $U_{G,v} \subseteq \mathbb{R}^n$ carries the action of $W_{G,v} := W(U_{G,v})$ and a natural G-action given by transformations from $W_{G,v}$. Since the G-action is irreducible, so is the $W_{G,v}$-action. Hence $U_{G,v} \subseteq \{y \in \mathbb{R}^n :$

$y_1+\cdots+y_n=0\}$. Irreducibility and effectiveness of the G-action induce an injection $G \hookrightarrow W_{G,v}$. Thus we may consider G as a subgroup of $W_{G,v}$, and in this picture $F_{G,v}$ is G-equivariant.

Remark 7.1. The linear space $U_{G,v}$ always intersects the submanifold of regular points in the S_n-module $\mathbb{R}^n$. Namely: For $1 \le i < j \le n$ we define $U_{i,j} = \{F_{G,v}(x) : \langle g_i.v \mid x\rangle = \langle g_j.v \mid x\rangle, x \in V\}$. By definition, $U_{i,j}$ is a linear subspace of $U_{G,v}$ and $\bigcup_{i<j} U_{i,j}$ is the set of singular points of the S_n-module $\mathbb{R}^n$ contained in $U_{G,v}$. Since, by definition, $g_i.v \neq g_j.v$ for any $i<j$, we have $\dim U_{i,j} = n-1$. Thus, $\bigcup_{i<j} U_{i,j} \neq U_{G,v}$, which gives the assertion.

Put $P_{G,v} := E \circ F_{G,v}$. Then $P_{G,v}$ is proper, since E and $F_{G,v}$ are proper.

Lemma 7.2. *Suppose that $P_{G,v}$ separates G-orbits. Then we have $G = W_{G,v}$.*

Proof. The groups G and $W_{G,v}$ have the same orbits in $U_{G,v}$. For: Suppose that $\tau \in W_{G,v}$ and $x, y \in V$ such that $F_{G,v}(y) = \tau.F_{G,v}(x)$. Since $P_{G,v}$ separates orbits, it follows that there exists some $g \in G$ such that $y = g.x$, whence $g.F_{G,v}(x) = \tau.F_{G,v}(x)$.

Now choose $x \in V$ such that $F_{G,v}(x)$ is a regular point of the $W_{G,v}$-module $U_{G,v}$. The regular points of any effective linear finite group representation are precisely those with trivial isotropy groups. We may conclude that x is a regular point of the G-module V. So $|W_{G,v}| = |W_{G,v}.F_{G,v}(x)| = |G.x| = |G|$, and thus $G = W_{G,v}$. □

If $P_{G,v}$ separates G-orbits, then, by lemma 7.2, the $G = W_{G,v}$-modules V and $U_{G,v}$ are equivalent. In particular, it follows that the restriction $E|_{U_{G,v}}$ separates $W_{G,v}$-orbits, $F_{G,v}$ induces a homeomorphism between V/G and $U_{G,v}/W_{\rho,v}$, and $F_{G,v}^* : \mathbb{R}[U_{G,v}]^{W_{G,v}} \to \mathbb{R}[V]^G$ is an algebra isomorphism.

Proposition 7.3. *The following conditions are equivalent:*

1. *$P_{G,v}$ separates G-orbits.*
2. *For all $x \in V$ we have $F_{G,v}(G.x) = \mathrm{S}_n.F_{G,v}(x) \cap U_{G,v}$.*
3. *$P_{G,v}$ induces a homeomorphism between V/G and $P_{G,v}(V)$.*

Proof. Since E separates S_n-orbits, for each $x \in V$ there exists a $z \in \mathbb{R}^n$ such that $E^{-1}(z) = \mathrm{S}_n.F_{G,v}(x)$. Then the equivalence of (1) and (2) follows from

$$P_{G,v}^{-1}(z) = F_{G,v}^{-1}(\mathrm{S}_n.F_{G,v}(x)) = F_{G,v}^{-1}(\mathrm{S}_n.F_{G,v}(x) \cap U_{G,v}).$$

The equivalence of (1) and (3) follows easily from lemma 2.1. □

Note that the introduced construction of $F_{G,v}$ and $P_{G,v}$ essentially coincides with the construction of orbit Chern classes as described in 3.2.

Let us discuss uniqueness of the above construction. Suppose $G \subseteq \mathrm{O}(V)$ is a finite group. Denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. Let S be the set of all reflections belonging to G. Denote by $\mathrm{Aut}(G,S)$ the group of automorphisms of G preserving the set S. Let $a \in \mathrm{Aut}(G,S)$. A diffeomorphism $T : V \to V$ is called a-equivariant, if $T \circ g = a(g) \circ T$ for any $g \in G$ (cf. [20]).

Lemma 7.4. *Suppose $G \subseteq \mathrm{O}(V)$ is a finite group. Let $a \in \mathrm{Aut}(G,S)$ and let $T : V \to V$ be an a-equivariant diffeomorphism. Then the isotropy groups of x and $T(x)$ are isomorphic, for all $x \in V$, T maps orbits onto orbits, and T induces an automorphism of the orbit type stratification of V.*

Proof. It is easily seen that $G_{T(x)} = a(G_x)$ and $T(G.x) = G.T(x)$ for all $x \in V$. Further, it is evident that $G_x = gHg^{-1}$ if and only if $G_{T(x)} = a(g)a(H)a(g)^{-1}$. The statement follows. □

Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve and $\bar{c} : \mathbb{R} \to V$ a smooth lift of c. The orbit space V/G has a smooth structure given by the sheaf $C^\infty(V/G) = C^\infty(V)^G$ of smooth G-invariant functions on V. Then c induces a continuous algebra morphism $c^* : C^\infty(V/G) \to C^\infty(\mathbb{R})$ and $\bar{c}$ induces a continuous algebra morphism $\bar{c}^* : C^\infty(V) \to C^\infty(\mathbb{R})$ such that $c^* = \bar{c}^* \circ \sigma^*$. This algebraic lifting problem is equivalent to the geometrical one. It is evident that to determine $\bar{c}^*$ it suffices to know the images under $\bar{c}^*$ of some system of global coordinate functions $x_1, \ldots, x_m$, where $m = \dim V$. The same is true for c^*, and in this case we may take the basic invariants $\sigma_1, \ldots, \sigma_n$ as global coordinates functions, by Schwarz's theorem [31]. If $f : V/G \to V/G$ is a smooth diffeomorphism one can take instead of the σ_i the functions $f^*(\sigma_i)$ with the same result. Thus, the problem of smooth lifting is invariant with respect to the group of diffeomorphisms of V/G. Each such diffeomorphism has a smooth lift to V which is an a-equivariant diffeomorphism, for some $a \in \mathrm{Aut}(G, S)$, see [20]. Conversely, any smooth a-equivariant diffeomorphism of V induces a smooth diffeomorphism of V/G, by lemma 7.4.

Therefore, we may regard two constructions as described above, carried out for distinct points v and w in V, as equivalent with respect to our lifting problem, if there exists a smooth a-equivariant diffeomorphism $T : V \to V$ with $v = T(w)$, for some $a \in \mathrm{Aut}(G, S)$.

If T is of a particular form, we can even say more.

Proposition 7.5. *Suppose $G \subseteq \mathrm{O}(V)$ is a finite group. Let $v, w \in V \backslash \{0\}$. If there exists a homothety or an a-equivariant linear orthogonal map $T : V \to V$, for some $a \in \mathrm{Aut}(G, S)$, such that $v = T(w)$, then $P_{G,v}(V)$ and $P_{G,w}(V)$ are homeomorphic, and $\mathbb{R}[E_1 \circ F_{G,v}, \ldots, E_n \circ F_{G,v}]$ and $\mathbb{R}[E_1 \circ F_{G,w}, \ldots, E_n \circ F_{G,w}]$ are isomorphic.*

Moreover, in both cases, the ambient stratifications of $U_{G,v}$ and $U_{G,w}$ are isomorphic, i.e., there exists a linear isomorphism $U_{G,v} \to U_{G,w}$ mapping strata onto strata.

Proof. If T is a homothety, then it is equivariant ($a = \mathrm{id}$) and $U_{G,v} = U_{G,w}$. If T is a-equivariant linear orthogonal, then, by lemma 7.4, the linear subspaces $U_{G,v}$ and $U_{G,w}$ of $\mathbb{R}^n$ differ only by a permutation from S_n. In both cases $P_{G,v}(V)$ and $P_{G,w}(V)$ are homeomorphic, and $T^* : \mathbb{R}[E_1 \circ F_{G,v}, \ldots, E_n \circ F_{G,v}] \to \mathbb{R}[E_1 \circ F_{G,w}, \ldots, E_n \circ F_{G,w}]$ is an algebra isomorphism.

The supplement in the lemma follows immediately from the fact that $U_{G,v}$ and $U_{G,w}$ differ only by a permutation of S_n. □

If $P(t)$ is a smooth curve of hyperbolic polynomials lying in $P_{G,v}(V)$ and provided that the polynomials $E_i \circ F_{G,v}$, $1 \leq i \leq n$, generate $\mathbb{R}[V]^G$, we may apply the results of sections 5 and 6.

We will investigate the case of finite reflection groups in the next section.

8. Finite reflection groups

Suppose U is a linear subspace of $\mathbb{R}^n$. Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates and endow U with the induced action of $W = W(U)$. We shall assume in this section that W is a finite reflection group.

Remark 8.1. If W is a finite reflection group, proposition 4.1 reduces to the following statement: *Any reflection hyperplane of W in U is the intersection with U of some reflection hyperplane of S_n in $\mathbb{R}^n$.* For: Let H be a reflection hyperplane of W in U. By proposition 4.1, there exists a ambient stratum S of U such that $S \subseteq H$ and $\dim S = \dim H$. Obviously, $S \subseteq (\mathbb{R}^n)_{\mathrm{sing}} \cap U$, and so there are reflection hyperplanes $P_1, \ldots, P_l$ of S_n in $\mathbb{R}^n$ which contain S. Since $\dim S = \dim U - 1$, there is a $1 \leq i \leq n$ such that $P_i \cap U$ is a hyperplane in U. Since S is contained in both H and $P_i \cap U$, we have $H = P_i \cap U$.

For any finite reflection group $W \subseteq \mathrm{O}(U)$ we may write U as the orthogonal direct sum of W-invariant subspaces $U_0 = U^W, U_1, \dots, U_l$ such that W is isomorphic to $W_0 \times W_1 \times \cdots \times W_l$, where $W_i = \{\tau|_{U_i} : \tau \in W\}$. Each W_i $(i \geq 1)$ is one of the groups (e.g. [13])

$$\mathrm{A}_m, m \geq 1; \mathrm{B}_m, m \geq 2; \mathrm{D}_m, m \geq 4; \mathrm{I}_2^m, m \geq 5, m \neq 6;$$
$$\mathrm{G}_2; \mathrm{H}_3; \mathrm{H}_4; \mathrm{F}_4; \mathrm{E}_6; \mathrm{E}_7; \mathrm{E}_8 \,.$$

It follows that $\mathbb{R}[U]^W \cong \mathbb{R}[U_1]^{W_1} \otimes \cdots \otimes \mathbb{R}[U_l]^{W_l}$ and $U/W \cong U_1/W_1 \times \cdots \times U_l/W_l$. A smooth curve $c = (c_1, \dots, c_l)$ in the orbit space U/W is then smoothly liftable to U if and only if, for all $1 \leq i \leq l$, c_i is smoothly liftable to U_i. Note that the orbit type stratification of U/W coincides with the product stratification of the orbit type stratifications $\mathcal{Z}_i$ of the factors U_i/W_i, i.e., the strata of U/W are $S_1 \times \cdots \times S_l$, where $S_i \in \mathcal{Z}_i$. Consequently, in order to apply the results of section 5 and section 6 we may consider each factor U_i/W_i separately. So let us assume that U is an irreducible W-module.

To this end we have to check whether the restrictions $E_i|_U$, $1 \leq i \leq n$, generate the algebra $\mathbb{R}[U]^W$. In practice this is easily accomplishable: The unique degrees $d_1, \dots, d_m$, where $m = \dim U$, of the elements in a minimal system of homogeneous generators of $\mathbb{R}[U]^W$ are well known. It suffices to compute the Jacobian J of the polynomials $E_{d_i}|_U$, $1 \leq i \leq m$. If $J \neq 0 \in \mathbb{R}[U]$ then they generate $\mathbb{R}[U]^W$. Note that a necessary condition for the $E_i|_U$, $1 \leq i \leq n$, to generate $\mathbb{R}[U]^W$ is that the degrees $d_1, \dots, d_m$ must be pairwise distinct, see remark 8.4.

Let us carry out the construction presented in section 7 for finite irreducible reflection groups $G \subseteq \mathrm{O}(V)$. Let $v \in V \backslash \{0\}$. If the polynomials $E_i \circ F_{G,v}$ generate the algebra $\mathbb{R}[V]^G$, then $W_{G,v}$ is a finite irreducible reflection group as well, by lemma 7.2.

Fix a system Π of simple roots of G. For any v in the fundamental domain $C = \{x \in V : \langle x \mid r \rangle \geq 0 \text{ for all } r \in \Pi\}$, the isotropy group G_v is generated by the simple reflections it contains (e.g. [13]).

Lemma 8.2. *Let $G \subseteq \mathrm{O}(V)$ be a finite reflection group. Each automorphism of the corresponding Coxeter diagram $\Gamma(G)$ induces an a-equivariant orthogonal automorphism of V for some $a \in \mathrm{Aut}(G, S)$.*

Proof. ([20]) Since the vertices in the Coxeter diagram $\Gamma(G)$ represent the simple roots of G, an automorphism φ of $\Gamma(G)$, defines uniquely an automorphism $a_\varphi \in \mathrm{Aut}(G, S)$. Suppose the simple roots have unit length. Since they form a basis for V the automorphism φ defines naturally an orthogonal automorphism T_φ of V. It is easily checked that T_φ is a_φ-equivariant. □

Theorem 8.3. *Suppose $G \subseteq \mathrm{O}(V)$ is a finite irreducible reflection group. Let $v \in V \backslash \{0\}$ such that the cardinality of G_v is maximal. Then: The polynomials $E_i \circ F_{G,v}$, $1 \leq i \leq n$, generate $\mathbb{R}[V]^G$ and $P_{G,v}$ induces a homeomorphism between V/G and $P_{G,v}(V)$ if and only if $G \neq \mathrm{D}_m$, $m \geq 4$.*

Proof. By proposition 7.5 and lemma 8.2 it suffices to check the statement for one single $v \neq 0$ with maximal G_v. Choosing $e_1 + \cdots + e_m - me_{m+1}$, e_1, and e_1 for A_m, B_m, and I_2^m, respectively, one obtains the usual systems of basic invariants. The choice e_1 for D_m yields $F_{\mathrm{D}_m,e_1} = F_{\mathrm{B}_m,e_1}$, whence the polynomials $E_i \circ F_{\mathrm{D}_m,e_1}$, $1 \leq i \leq n = 2m$, cannot separate D_m-orbits. For the remaining irreducible reflection groups the necessary computations have been carried out by Mehta [23]. □

Remark 8.4. If for D_m with m odd one chooses $v = e_1 + \cdots + e_m$, then the polynomials $E_i \circ F_{\mathrm{D}_m,v}$, $1 \leq i \leq n = 2^{m-1}$, generate $\mathbb{R}[\mathbb{R}^m]^{\mathrm{D}_m}$, since the Jacobian of the polynomials $N_i \circ F_{\mathrm{D}_m,w}$, $i = 2, 4, \cdots, 2n-2, n$, is up to a constant factor given by $\prod_{i<j}(x_i^2 - x_j^2)$. If $m(\geq 4)$ is even, this cannot be true since there have to be two basic invariants of degree $m/2$.

The following theorem is a corollary of theorem 8.3 and theorem 5.2.

Theorem 8.5. *Suppose $G \subseteq \mathrm{O}(V)$ is a finite irreducible reflection group and $G \neq \mathrm{D}_m$, $m \geq 4$. Let $v \in V \backslash \{0\}$ such that the cardinality of G_v is maximal. Let*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R})$$

be a smooth curve of hyperbolic polynomials of degree $n = |G.v|$ lying in $P_{G,v}(V)$ for all $t \in \mathbb{R}$. Then there exists a global twice differentiable parameterization of the roots of $P(t)$ on $\mathbb{R}$ which is smooth on $\mathbb{R} \backslash F_{\mathrm{orb}}$. □

Remark 8.6. It is easy to see that, under the assumption that the cardinality of G_v is maximal, the orbit type stratification and the ambient stratification of $U_{G,v}$ coincide only for $G = \mathrm{A}_m, \mathrm{B}_m, \mathrm{I}_2^m$. In general, if $|G_v|$ is not maximal, the orbit type stratification of $U_{G,v}$ will be strictly coarser than its ambient stratification.

It is easy to compute the integer k, associated to orthogonal representations of finite groups G in 6.2, if G is a finite irreducible reflection group. See figure 1.

G	A_m	B_m	D_m	I_2^m	G_2	H_3	H_4	F_4	E_6	E_7	E_8
k	$m+1$	$2m$	$2m$	m	6	12	120	24	27	56	240

FIGURE 1. Irreducible Coxeter groups with associated integer k.

In the situation of theorem 8.5 the strategy discussed in section 6 will lead to no improvement, since $k = n$ by definition. But, if we choose $v \in V \backslash \{0\}$ such that $|G_v|$ is not maximal, then $k < n$ and the methods of section 6 will yield refinements.

In many cases the following theorem provides an improvement of 6.1.

Theorem 8.7. *Suppose $G \subseteq \mathrm{O}(V)$ is a finite irreducible reflection group. Choose some $v \in V \backslash \{0\}$. Put $n = |G.v|$ and let k be as in figure 1. Suppose that the restrictions $E_i|_{U_{G,v}}$, $1 \leq i \leq n$, generate $\mathbb{R}[U_{G,v}]^{W_{G,v}}$. Let*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + a_2(t)x^{n-2} - \cdots + (-1)^n a_n(t) \qquad (t \in \mathbb{R})$$

be a curve of hyperbolic polynomials lying in $P_{G,v}(V)$ for all $t \in \mathbb{R}$. Then:

1. *If all a_i are of class C^k, then there exists a differentiable parameterization of the roots of $P(t)$ with locally bounded derivative.*
2. *If all a_i are of class C^{k+d}, then any differentiable parameterization of the roots of $P(t)$ is actually C^1.*
3. *If all a_i are of class C^{k+2d}, then there exists a twice differentiable parameterization of the roots of $P(t)$.* □

Example 8.8. Consider the Coxeter group B_3 and choose $v = e_1 + e_2 + e_3$. We find

$$\begin{aligned} F_{\mathrm{B}_3,v}(x) = (&x_1 + x_2 + x_3, -x_1 + x_2 + x_3, x_1 - x_2 + x_3, x_1 + x_2 - x_3, \\ &- x_1 - x_2 + x_3, -x_1 + x_2 - x_3, x_1 - x_2 - x_3, -x_1 - x_2 - x_3) \end{aligned}$$

and $U_{\mathrm{B}_3,v} = \{y \in \mathbb{R}^8 : y_i + y_j = 0 \text{ for } i + j = 9, y_1 = y_2 + y_3 + y_4\}$. It is easy to check that $N_{2i} \circ F_{\mathrm{B}_3,v}$, $1 \leq i \leq 3$, generate $\mathbb{R}[\mathbb{R}^3]^{\mathrm{B}_3}$, by computing their Jacobian. It is readily verified that the set of all reflection hyperplanes of $W_{\mathrm{B}_3,v}$ is given by intersecting the following hyperplanes in $\mathbb{R}^8$ with $U_{\mathrm{B}_3,v}$ (compare with remark 8.1):

$$\{y_1 = y_2, y_1 = y_3, y_1 = y_4, y_1 = y_5, y_1 = y_6, y_1 = y_7, y_2 = y_3, y_2 = y_4, y_3 = y_4\}.$$

Furthermore, the intersections with $U_{\mathrm{B}_3,v}$ of the following hyperplanes in $\mathbb{R}^8$,

$$\{y_1 = y_8, y_2 = y_7, y_3 = y_6, y_4 = y_5\},$$

are not among the set of reflection hyperplanes of $W_{\mathrm{B}_3,v}$. Therefore, the orbit type stratification of $U_{\mathrm{B}_3,v}$ is strictly coarser than its ambient stratification.

We follow the recipe for computing orbit type and ambient stratification of $E(U_{\mathrm{B}_3,v}) = N(U_{\mathrm{B}_3,v})$ given at the end of section 5. We will present only the outcome of the calculations. Using $N_{2i} \circ F_{\mathrm{B}_3,v}$, $1 \le i \le 3$, as basic invariants of $\mathbb{R}[\mathbb{R}^3]^{\mathrm{B}_3}$, we find that the symmetric matrix $\tilde{B} = (\tilde{b}_{ij})$ from 2.2 has entries

$$\tilde{b}_{11} = 32z_2,\ \tilde{b}_{12} = 64z_4,\ \tilde{b}_{13} = 96z_6,\ \tilde{b}_{22} = -3z_2^3 + 36z_2z_4 + 32z_6,$$
$$\tilde{b}_{23} = \frac{1}{8}(5z_2^4 - 108z_2^2z_4 + 192z_4^2 + 544z_2z_6),$$
$$\tilde{b}_{33} = \frac{1}{64}(27z_2^5 - 300z_2^3z_4 - 1140z_2z_4^2 + 1140z_2^2z_6 + 7680z_4z_6).$$

Put $\tilde{\Delta}_{ij} = \det \begin{pmatrix} \tilde{b}_{ii} & \tilde{b}_{ij} \\ \tilde{b}_{ji} & \tilde{b}_{jj} \end{pmatrix}$ where $i < j$. Then $N(U_{\mathrm{B}_3,v})$ is the subset in $\mathbb{R}^8$ defined by the following relations

$$z_2 \ge 0, \tilde{\Delta}_{12} \ge 0, \det \tilde{B} \ge 0$$
$$z_1 = z_3 = z_5 = z_7 = 0,$$
$$384z_8 = 5z_2^4 - 72z_2^2z_4 + 48z_4^2 + 256z_2z_6.$$

The 3-dimensional principal orbit type stratum is given by

$$R^{(3)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} > 0, \det \tilde{B} > 0\}.$$

Put

$$\tilde{f}_1 = 53z_2^6 - 840z_2^4z_4 + 1680z_2^2z_4^2 + 6144z_4^3 + 2752z_2^3z_6 - 16128z_2z_4z_6 + 9216z_6^2,$$
$$\tilde{f}_2 = z_2^3 - 12z_2z_4 + 32z_6.$$

There are three 2-dimensional orbit type strata

$$R_1^{(2)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} > 0, \tilde{f}_1 = 0\}$$
$$R_2^{(2)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} = 0, \tilde{\Delta}_{23} > 0, \tilde{f}_1 = 0\}$$
$$R_3^{(2)} = N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{13} > 0, \tilde{f}_2 = 0\},$$

the three 1-dimensional orbit type strata $R_1^{(1)}$, $R_2^{(1)}$, $R_3^{(1)}$ are the connected components of

$$N(U_{\mathrm{B}_3,v}) \cap \{z_2 > 0, \tilde{\Delta}_{12} = \tilde{\Delta}_{13} = \tilde{\Delta}_{23} = 0\},$$

and $R^{(0)} = \{0\}$ is the only 0-dimensional stratum.

The ambient stratification of $N(U_{\mathrm{B}_3,v})$ is obtained by cutting with the surface $\{z_2^2 - 4z_4 = 0\}$. There are two 3-dimensional ambient strata

$$S_1^{(3)} = R^{(3)} \cap \{z_2^2 - 4z_4 > 0\} \quad \text{and} \quad S_2^{(3)} = R^{(3)} \cap \{z_2^2 - 4z_4 < 0\},$$

five 2-dimensional ambient strata

$$S_1^{(2)} = R^{(3)} \cap \{z_2^2 - 4z_4 = 0\},\ S_2^{(2)} = R_1^{(2)} \cap \{z_2^2 - 4z_4 > 0\},$$
$$S_3^{(2)} = R_1^{(2)} \cap \{z_2^2 - 4z_4 < 0\},\ S_4^{(2)} = R_2^{(2)},\ S_5^{(2)} = R_3^{(2)},$$

four 1-dimensional ambient strata $S_1^{(1)} = R_1^{(1)}$, $S_2^{(1)} = R_2^{(1)}$, $S_3^{(1)} = R_3^{(1)}$, $S_4^{(1)} = R_1^{(2)} \cap \{z_2^2 - 4z_4 = 0\}$, and $S^{(0)} = R^{(0)} = \{0\}$ is the only 0-dimensional ambient stratum. See figure 2.

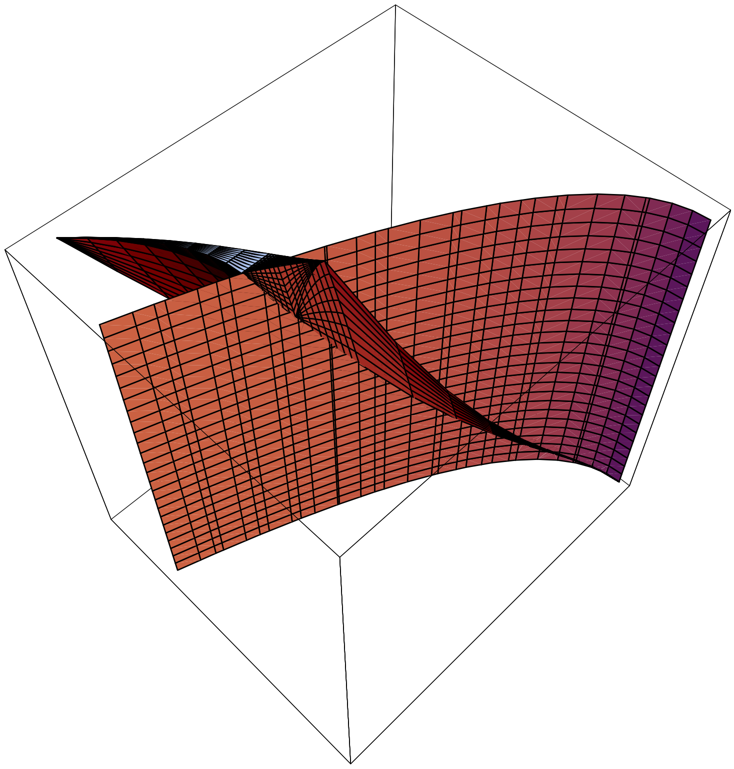

FIGURE 2. The projection of $N(U_{\mathrm{B}_3,v})$ to the $\{z_2, z_4, z_6\}$-subspace and intersection with the surface $\{z_2^2 - 4z_4 = 0\}$.

Let f, g, h be functions defined in some neighborhood of $0 \in \mathbb{R}$. Suppose that f and g are infinitely flat at 0 and $h(0) = 0$. For t near 0 consider the curve of polynomials $P(t)(x) = x^8 + \sum_{j=1}^{8}(-1)^j a_j(t) x^{8-j}$ where

$$a_1 = a_3 = a_5 = a_7 = 0,$$
$$a_2 = -56 + f,\ a_4 = 784 + g,\ a_6 = -2304 + h,$$
$$1024 a_8 = 16 a_2^4 - 128 a_2^2 a_4 + 256 a_4^2.$$

Then, for t near 0, $P(t)$ is a curve in $N(U_{\mathrm{B}_3,v})$ with $P(0) \in S_1^{(2)}$. At $t = 0$ it is normally flat with respect to the ambient stratification but normally nonflat with respect to the orbit type stratification.

If f, g and h are smooth, then $P(t)$ is smoothly solvable near $t = 0$, by theorem 5.2. Note that in this example we have $d = k = 6 < 8 = n$ and thus theorem 8.7 provides an actual improvement, too.

The following example shows that $W(U)$ must not necessarily be a finite reflection group, even though the $E_i|_U$ generate $\mathbb{R}[U]^{W(U)}$.

Example 8.9. Let U be the subspace of $\mathbb{R}^6$ defined by the following equations

$$x_1 + x_2 + x_3 = 0, \quad x_4 + x_5 + x_6 = 0.$$

The subgroup $N(U)$ of S_6 is generated by all permutations of x_1, x_2, x_3, all permutations of x_4, x_5, x_6, and the simultaneous transpositions of x_1 and x_4, x_2 and x_5, x_3 and x_6. The subgroup $Z(U)$ is trivial. Thus $W(U)$ is isomorphic to the semidirect product of $\mathrm{S}_3 \times \mathrm{S}_3$ and S_2.

One can get the subspace U above as follows. Consider the point $v = (x, x, x, y, y, y) \in \mathbb{R}^6$, where $x, y \neq 0$ and $x \neq y$. The isotropy group $H = (\mathrm{S}_6)_v$ of v is evidently isomorphic to $\mathrm{S}_3 \times \mathrm{S}_3$. Then $U = ((\mathbb{R}^6)^H)^\perp$. The group H is the normal subgroup of $W(U)$ generated by reflections.

First consider the action of H on U. It is clear that the algebra $\mathbb{R}[U]^H$ is a polynomial algebra generated by the basic generators

$$y_1 = x_1^2 + x_2^2 + x_1x_2, \; z_1 = x_1x_2(x_1 + x_2),$$
$$y_2 = x_4^2 + x_5^2 + x_4x_5, \; z_2 = x_4x_5(x_4 + x_5).$$

Consider the space $\mathbb{R}^4$ with the coordinates y_1, z_1, y_2, z_2 and the action of the group S_2 on it induced by the action of $\mathrm{S}_2 = W(U)/(\mathrm{S}_3 \times \mathrm{S}_3)$ on the above basic generators. It is easy to check that this action coincides with the diagonal action of S_2 on $(\mathbb{R}^2)^2$ for the standard action of S_2 on $\mathbb{R}^2$. Since the algebra of S_2-invariant polynomials on $(\mathbb{R}^2)^2$ is generated by the polarizations of basic invariants for the standard action of S_2 ob $\mathbb{R}^2$ we get the following system of generators of $\mathbb{R}[U]^{W(U)}$:

$$f_1 = y_1 + y_2, \; f_2 = z_1 + z_2, \; f_3 = y_1^2 + y_2^2, \; f_4 = y_1z_1 + y_2z_2, \; f_5 = z_1^2 + z_2^2.$$

Simple calculations for the restrictions of the Newton polynomials N_i on $\mathbb{R}^6$ to U gives the following result:

$$N_1|_U = 0, \; N_2|_U = 2f_1, \; N_3|_U = -3f_2,$$
$$N_4|_U = 2f_3, \; N_5|_U = -5f_4, \; N_6|_U = 3f_5 + 3f_1f_3 - f_1^3.$$

This proves that the morphism $\mathbb{R}[\mathbb{R}^6]^{\mathrm{S}_6} \to \mathbb{R}[U]^{W(U)}$ defined by restriction is surjective.

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