

QUANTITATIVE TAME PROPERTIES OF DIFFERENTIABLE FUNCTIONS WITH CONTROLLED DERIVATIVES

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ABSTRACT. We show that differentiable functions, defined on a convex body $K \subseteq \mathbb{R}^d$, whose derivatives are controlled by a suitable given sequence of positive real numbers share many properties with polynomials. The role of the degree of a polynomial is hereby played by an integer associated with the given sequence of reals, the diameter of K , and a real parameter linked to the C^0 -norm of the function. We give quantitative information on the size of the zero set, show that it admits a local parameterization by Sobolev functions, and prove an inequality of Remez-type. From the latter, we deduce several consequences, for instance, a bound on the volume of sublevel sets and a comparison of L^p -norms reversing Hölder's inequality. The validity of many of the results only depends on the derivatives up to some finite order; the order can be specified in terms of the given data.

1. INTRODUCTION

Yomdin showed in [31] that the zero set Z_f of a non-zero C^∞ -function $f : B \rightarrow \mathbb{R}$ defined on the unit ball $B \subseteq \mathbb{R}^d$ behaves in many respects as the zero set of a polynomial, if the partial derivatives of some order $j \geq 2$ are sufficiently small on B . To be precise, let $\|f\|_{j,B} := \sum_{|\alpha|=j} \frac{j!}{\alpha!} \|f^{(\alpha)}\|_B$ and $\|f\|_B := \|f\|_{0,B} = \sup_{x \in B} |f(x)|$. If, for some $j \geq 2$,

$$(1) \quad \|f\|_{j,B} \leq \frac{1}{2^{j+1}} \|f\|_{0,B},$$

then Z_f has the following properties:

- (a) There is some ball $B' \subseteq B$ such that for each affine line ℓ in \mathbb{R}^d that meets B' the restriction $f|_\ell$ has at most $j - 1$ zeros (counted with multiplicities).
- (b) Z_f is contained in a countable union of compact C^∞ -hypersurfaces.
- (c) The $(d - 1)$ -dimensional Hausdorff measure of Z_f satisfies

$$\mathcal{H}^{d-1}(Z_f) \leq C(d, j),$$

where $C(d, j)$ is a positive constant depending only on d and j .

An important fact (which is needed to get (c)) is that the radius of B' depends only on j .

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This result is in drastic contrast to the classical result that any closed subset of \mathbb{R}^d is the zero set of some \mathcal{C}^∞ -function. In fact, by (a), a non-zero function satisfying (1) cannot have points of infinite flatness.

In this paper, we will show that the zero sets of smooth functions with a different type of constraints have similar quantitative tame properties. Instead of assuming that the Fréchet derivative of *some* order is “small”, as in (1), we ask that the growth of the sequence of derivatives of *all* orders is controlled.

In a spirit similar to [31], Yomdin proved in [32] a Remez-type inequality for smooth functions involving a “remainder term” expressible through bounds on the derivatives. We will prove a Remez-type inequality (without “remainder term”) and deduce several consequences for smooth functions with controlled derivatives.

1.1. Functions with controlled derivatives. Let a positive increasing sequence $(\mu_j)_{j \geq 1}$ and a positive real number $M_0 > 0$ be given and set $M_j := M_0 \mu_1 \mu_2 \cdots \mu_j$ for all $j \geq 1$. Let $K \subseteq \mathbb{R}^d$ be a d -dimensional convex body (i.e., compact with non-empty interior K°). We will study \mathcal{C}^∞ -functions $f : K \rightarrow \mathbb{R}$ such that

$$(2) \quad \|f\|_{j,K} \leq M_j, \quad j \in \mathbb{N}.$$

Now it is well-known that the qualitative behavior of functions satisfying (2) — let us call them $(M_j)_j$ -smooth functions — fundamentally depends on the convergence or divergence of the series $\sum_j \frac{1}{\mu_j}$ (the *quasianalyticity threshold*). The series $\sum_j \frac{1}{\mu_j}$ and its partial sums play a central role in our quantitative analysis. In fact, this analysis is based on a quantity $\mathfrak{d}_{(\delta_K \mu_j)_j}(b)$ which depends on the sequence $(\delta_K \mu_j)_j$ and an additional parameter $b > 0$. Here $\delta_K := \text{diam}(K)$ is the diameter of K . Roughly speaking, $\mathfrak{d}_{(\delta_K \mu_j)_j}(b)$ is the greatest integer n (possibly infinite) such that

$$\sum_{j=j_0(b)+1}^n \frac{1}{\mu_j} < \delta_K e,$$

where the integer $j_0(b)$ is explicitly computed from b ; see Section 2 for precise definitions. We call $\mathfrak{d}_{(\delta_K \mu_j)_j}(b)$ the $(\delta_K \mu_j)_j$ -degree, since it has similar properties as the degree of a polynomial. Note that it is closely related to the *Bang degree* introduced by Nazarov, Sodin, and Volberg [20].

The $(\delta_K \mu_j)_j$ -degree $\mathfrak{d}_{(\delta_K \mu_j)_j}(b)$ is finite if $\sum_{j \geq j_0(b)+1} \frac{1}{\mu_j}$ exceeds $\delta_K e$ which is always the case provided that the series $\sum_j \frac{1}{\mu_j}$ diverges. But also if $\sum_j \frac{1}{\mu_j}$ converges to a large enough sum, $\mathfrak{d}_{(\delta_K \mu_j)_j}(b)$ is finite and contains useful information on functions satisfying (2). In fact, this may occur if δ_K is relatively small.

1.2. The zero set of functions with controlled derivatives. We will see that the zero set Z_f of $(M_j)_j$ -smooth functions $f : K \rightarrow \mathbb{R}$ has properties similar to (a), (b), and (c) whenever the $(\delta_K \mu_j)_j$ -degree $\mathfrak{d}_{(\delta_K \mu_j)_j}(\frac{b}{2M_0})$ is finite. The parameter $b > 0$ hereby acts as a lower bound on the \mathcal{C}^0 -norm of f , that is, we require that $\|f\|_K \geq b$.

Let us summarize our results in the following statements (A)–(D):

- (A) There is a ball B contained in the interior K° of K whose radius depends only on K and the ratio $\frac{b}{M_1}$ such that for each affine line ℓ in \mathbb{R}^d that meets B the restriction $f|_\ell$ has at most $2\mathfrak{d}_{(\delta_K \mu_j)_j}(\frac{b}{2M_0})$ zeros (counted with multiplicities). (Theorem 4.2)

The key to this result is a bound for the number of zeros of $(M_j)_j$ -smooth univariate functions which goes back to Bang [2]; see also [20] for an exposition of Bang's ideas. We revisit Bang's result in some detail in Section 3.

As a consequence of Malgrange's preparation theorem and Yomdin's observation [31, Lemma 6] we obtain:

- (B) Z_f is contained in a countable union of compact C^∞ -hypersurfaces. (Theorem 4.6)

Then, using a Crofton-type argument, (A) and (B) allow to conclude:

- (C) The $(d - 1)$ -dimensional Hausdorff measure of Z_f satisfies

$$\mathcal{H}^{d-1}(Z_f) \leq C \mathfrak{d}_{(\delta_K \mu_j)_j} \left(\frac{b}{2M_0} \right) \delta_K^{d-1},$$

where the constant $C > 0$ depends only on d and the ratio $\frac{\delta_B}{\delta_K}$; B is the ball from (A). (Theorem 4.6)

Combining Malgrange's preparation theorem with results of Parusiński and Rainer [22], we find that Z_f locally admits a Sobolev parameterization:

- (D) There is a finite cover of K by rectangular boxes U with corresponding orthogonal coordinates $(x_1, x_2, \dots, x_d) = (x', x_d)$ such that $Z_f \cap U$ is contained in the graphs $\{x_d = \xi_i(x')\}$ of at most $2\mathfrak{d}_{(\delta_K \mu_j)_j} \left(\frac{b}{2M_0} \right)$ continuous functions ξ_i of Sobolev class $W^{1,p}$, for all $1 \leq p < p_0$, where

$$p_0 := \frac{2\mathfrak{d}_{(\delta_K \mu_j)_j} \left(\frac{b}{2M_0} \right)}{2\mathfrak{d}_{(\delta_K \mu_j)_j} \left(\frac{b}{2M_0} \right) - 1}.$$

(Theorem 4.8)

That means that the partial derivatives $\partial_k \xi_i$ of first order of ξ_i exist almost everywhere, agree with the weak partial derivatives, and ξ_i as well as $\partial_k \xi_i$ are in the Lebesgue space L^p .

As a by-product, our reasoning shows that (D) remains true for C^∞ -functions satisfying (1) for some $j \geq 2$ with the role of $2\mathfrak{d}_{(\delta_K \mu_j)_j} \left(\frac{b}{2M_0} \right)$ replaced by $j - 1$.

We discuss uniformity of these results at the end of Section 4.

1.3. A Remez-type inequality for functions with controlled derivatives.

The classical Remez inequality [27] states that for a Lebesgue measurable subset E of $[0, 1]$ with Lebesgue measure $|E| > 0$ we have

$$\|p\|_{[0,1]} \leq T_n \left(\frac{2 - |E|}{|E|} \right) \|p\|_E$$

for all polynomials $p \in \mathbb{R}[x]$ with degree $\deg p \leq n$. Here $T_n(x) = \cos(n \arccos x)$ is the n -th Chebyshev polynomial. In higher dimensions, there is the generalization due to Yu. Brudnyi and Ganzburg [11]: for a convex body $K \subseteq \mathbb{R}^d$ and a Lebesgue measurable subset $E \subseteq K$ with $|E| > 0$ we have the sharp inequality

$$\|p\|_K \leq T_n \left(\frac{1 + (1 - \frac{|E|}{|K|})^{1/d}}{1 - (1 - \frac{|E|}{|K|})^{1/d}} \right) \|p\|_E$$

for all polynomials $p \in \mathbb{R}[x_1, \dots, x_d]$ with degree $\deg p \leq n$. It implies

$$\|p\|_K \leq \left(\frac{4d|K|}{|E|} \right)^n \|p\|_E.$$

A version of the latter inequality for analytic functions was obtained by A. Brudnyi [10], where the role of n is played by the *analytic degree*; cf. Section 6.2. As already mentioned there is a version for smooth function due to [32] which however contains a “remainder term”.

In this paper, we are interested in Remez-type inequalities for $(M_j)_j$ -smooth functions $f : K \rightarrow \mathbb{R}$ (without “remainder term”). On the unit interval $K = [0, 1]$, such an inequality was obtained in [20, Theorem B]. We recall and slightly adjust this result in Theorem 5.1. The crucial difference is that, instead of the Bang degree, we work with the $(\delta_K \mu_j)_j$ -degree $\mathfrak{d}_{(\delta_K \mu_j)_j}(\frac{b}{M_0})$ which in a sense allows more “leeway”: it works for all $(M_j)_j$ -smooth functions $f : K \rightarrow \mathbb{R}$ with $\|f\|_K \geq b$. (In contrast to above, there is no factor 2 in the denominator.)

In this way, we are able to apply the strategy of restriction to 1-dimensional sections of Brudnyi and Ganzburg and obtain a Remez-type inequality for $(M_j)_j$ -smooth functions in several variables (Theorem 5.4). Actually, the Remez-type inequality is invariant under the \mathbb{R}^* -action on f by multiplication with non-zero constants. This observation leads to a useful version (Theorem 5.5) for $(M_j)_j$ -smooth functions f , where $M_j = \|f\|_K \cdot \mu_1 \cdots \mu_j$, for $j \geq 1$, and $M_0 = \|f\|_K > 0$, and with $(\delta_K \mu_j)_j$ -degree $\mathfrak{d}_{(\delta_K \mu_j)_j}(1)$.

From this version, we deduce (in a standard way, see for instance [9, 10] and [11]) a number of consequences for functions with controlled derivatives:

- (1) A bound for the volume of sublevel sets. (Corollary 5.6)
- (2) A comparison of L^p -norms (reversing Hölder’s inequality). (Corollary 5.8)
- (3) A bound for the mean oscillation of $\log |f|$. (Corollary 5.10 and Corollary 5.11)

Generally, we deduce the main multivariate results from respective univariate ones by restriction to affine lines. See Remark 4.13 for how this relates to results of Bochnak–Siciak type for controlled functions.

1.4. Complementary results. In Section 6.1 and Section 6.2, we attempt to clarify the relation between the $(\mu_j)_j$ -degree on the one hand and polynomial and analytic degree on the other hand. The $(\mu_j)_j$ -degree is a more general object, but in a polynomial or analytic setting it is generally larger than the polynomial or analytic degree, respectively. This is related to the general assumption that $(\mu_j)_j$ is increasing, which is crucial for many of the results.

As already mentioned, the $(\mu_j)_j$ -degree can be finite also in the *non-quasianalytic* setting (i.e., when $\sum_j \frac{1}{\mu_j}$ converges). In fact, we obtain in Corollary 6.3 a quantitative necessary condition for the existence of \mathcal{C}^∞ -functions with uniform bounds, like (2), and compactly supported in a given ball, in terms of the size of the ball and the \mathcal{C}^0 -norm of the function. For completeness, we sketch a proof of the fact that, if $\sum_j \frac{1}{\mu_j} < \infty$, each non-empty closed subset of \mathbb{R}^d is the zero set of a \mathcal{C}^∞ -function f such that $\|f\|_{j, \mathbb{R}^d} \leq A^{j+1} M_j$ for all $j \in \mathbb{N}$ and some $A > 0$ (Remark 6.4).

Bang’s univariate result, i.e. Proposition 3.2, not only contains useful information on the numbers of zeros but also on the locus of points, where derivatives of higher order vanish. For instance, we get a uniform bound for the number of critical points on certain affine lines (Proposition 6.5).

1.5. Finite determinacy. So far we only considered \mathcal{C}^∞ -functions with controlled derivatives of all orders. Actually, many of our results depend only on the derivatives up to some finite order and remain true even for functions that are only

differentiable up to said finite order. We will make this precise below, in particular, we will specify the finite order. To account for this fact, we introduce a customized terminology in Section 2, Section 4.1, and Section 5.4.

1.6. Notation. We often write $|E|$ for the Lebesgue measure $\mathcal{L}^d(E)$ of a measurable set $E \subseteq \mathbb{R}^d$. If not stated otherwise, “measurable” always means “Lebesgue measurable”. The k -dimensional Hausdorff measure of E is denoted by $\mathcal{H}^k(E)$.

Let $B_r(a) := \{x \in \mathbb{R}^d : |x - a| < r\}$ and $\bar{B}_r(a) := \{x \in \mathbb{R}^d : |x - a| \leq r\}$ denote the open and closed Euclidean ball in \mathbb{R}^d centered at a with radius $r > 0$, respectively. If a is the origin, we simply write $B_r := B_r(0)$ and $\bar{B}_r := \bar{B}_r(0)$.

Throughout the paper, we write $\delta_K := \text{diam}(K)$ for the diameter of a set $K \subseteq \mathbb{R}^d$. We denote by \bar{K} , K° , and ∂K the closure, the interior, and the boundary of K in \mathbb{R}^d , respectively.

The integral part of a real number x is denoted by $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$. Similarly, $\lceil x \rceil := \min\{n \in \mathbb{Z} : n \geq x\}$, but most of the time we will use $\lceil x \rceil_{\mathbb{N}} := \min\{n \in \mathbb{N} : n \geq x\}$, where $\mathbb{N} := \mathbb{Z}_{\geq 0} := \{n \in \mathbb{Z} : n \geq 0\}$. Similarly, we use $\mathbb{N}_{\geq m} := \{n \in \mathbb{N} : n \geq m\}$, $\mathbb{R}_{>a} := \{x \in \mathbb{R} : x > a\}$, and variations thereof.

2. $M^{|\mathbb{N}|}$ -SMOOTH FUNCTIONS AND $\mu^{|\mathbb{N}|}$ -DEGREE

2.1. Admissible weights. Let $(\mu_j)_{j \geq 1}$ be an infinite increasing sequence of elements in $\mathbb{R}_{>0} \cup \{\infty\}$, i.e., $0 < \mu_1 \leq \mu_2 \leq \dots$. We allow that sequence elements attain the value ∞ . By the requirement that the sequence is increasing, if $\mu_{j_0} = \infty$, then $\mu_j = \infty$ for all $j \geq j_0$. We say that such a sequence $(\mu_k)_{k \geq 1}$ is an *admissible weight*. It will be convenient to keep track of the index (if any), where the sequence ceases to be finite. So we write

$$\begin{aligned} \mu^{|\mathbb{N}|} &:= (\mu_j)_{j \geq 1} = (\mu_1, \dots, \mu_N, \infty, \infty, \dots) \quad \text{if } \mu_N < \infty = \mu_{N+1} \text{ for } N \geq 1, \\ \mu^{|\mathbb{N}|} &:= (\mu_j)_{j \geq 1} = (\infty, \infty, \dots) \quad \text{if } \mu_j = \infty \text{ for all } j \geq 1, \\ \mu^{|\mathbb{N}|} &:= (\mu_j)_{j \geq 1} \quad \text{if } \mu_j < \infty \text{ for all } j \geq 1. \end{aligned}$$

From now on, we let N be an element of $\mathbb{N}_{\geq 1} \cup \{\infty\}$ so that the notation $\mu^{|\mathbb{N}|}$ also includes the case $\mu^{|\mathbb{N}|}$. (We exclude the case $N = 0$, where the weight is worthless.) Let us adopt the usual conventions for the arithmetic with ∞ : $\infty \pm 1 = \infty$, $\frac{1}{\infty} = 0$, and $r \cdot \infty = \infty$ if $r \in \mathbb{R}_{>0} \cup \{\infty\}$.

If $\mu^{|\mathbb{N}|}$ is an admissible weight and $M_0 > 0$ is any positive real number, then we call the pair $(\mu^{|\mathbb{N}|}, M_0)$ a *full admissible weight*. With the full admissible weight $(\mu^{|\mathbb{N}|}, M_0)$ we associate a sequence $M^{|\mathbb{N}|} = (M_j)_{j \geq 0}$ by setting

$$M_j := M_0 \mu_1 \mu_2 \cdots \mu_j, \quad j \geq 1.$$

Then $M_j = \infty$ if $N < \infty$ and $j > N$. That $\mu^{|\mathbb{N}|}$ is increasing amounts to the property $M_j^2 \leq M_{j-1} M_{j+1}$ for $j \geq 1$. Given a sequence $M^{|\mathbb{N}|} = (M_j)_{j \geq 0}$ with this property, we may recover the pair $(\mu^{|\mathbb{N}|}, M_0)$ in a unique way by

$$\mu_j := \frac{M_j}{M_{j-1}} \quad \text{for } 1 \leq j \leq N \quad \text{and} \quad \mu_j := \infty \quad \text{for } j > N,$$

if $N < \infty$, and simply by $\mu_j := \frac{M_j}{M_{j-1}}$ for all $j \geq 1$ otherwise. Thus there is a one-to-one correspondence between $(\mu^{|\mathbb{N}|}, M_0)$ and $M^{|\mathbb{N}|}$. We also call $M^{|\mathbb{N}|}$ a *full admissible weight*.

Notice that, given a full admissible weight (μ^{lN}, M_0) and positive constants $r, C > 0$, the pair $(r\mu^{lN}, CM_0) = ((r\mu_j)_{j \geq 1}, CM_0)$ is again a full admissible weight; it corresponds to $(Cr^j M_j)_{j \geq 0}$.

2.2. M^{lN} -smooth functions. Let M^{lN} be a full admissible weight. Let $K \subseteq \mathbb{R}^d$ be a convex body and $f : K \rightarrow \mathbb{R}$ a function. We say that f is M^{lN} -smooth (or (μ^{lN}, M_0) -smooth) if f is of class \mathcal{C}^N (in an open neighborhood of K) and

$$(2) \quad \|f\|_{j,K} \leq M_j, \quad 0 \leq j < N + 1,$$

where we set

$$\|f\|_{j,K} := \sum_{|\alpha|=j} \frac{j!}{\alpha!} \|f^{(\alpha)}\|_K$$

and $\|f\|_K = \|f\|_{0,K} = \sup_{x \in K} |f(x)|$ denotes the sup-norm. If $N = \infty$, this means that f is of class \mathcal{C}^∞ and $\|f\|_{j,K} \leq M_j$ for all $j \in \mathbb{N}$.

2.3. The μ^{lN} -degree. Let μ^{lN} be an admissible weight. We consider the function $\Sigma_{\mu^{lN}} : \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1} \rightarrow [0, \infty)$ defined by

$$\Sigma_{\mu^{lN}}(m, n) := \sum_{j=m}^n \frac{1}{\mu_j}$$

and extend the definition to $(\mathbb{N}_{\geq 1} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$ by setting

$$\Sigma_{\mu^{lN}}(m, n) := \begin{cases} 0 & \text{if } 1 \leq m \leq \infty, n = 0, \\ \sum_{j=m}^{\infty} \frac{1}{\mu_j} & \text{if } 1 \leq m < \infty, n = \infty, \\ 0 & \text{if } m = \infty, 1 \leq n \leq \infty. \end{cases}$$

Note that, if $m > n$, then $\Sigma_{\mu^{lN}}(m, n)$ is an empty sum and hence has the value 0. For $n \geq N$, we have $\Sigma_{\mu^{lN}}(m, n) = \Sigma_{\mu^{lN}}(m, N)$ (since $\frac{1}{\infty} = 0$). The map $(m, n) \mapsto \Sigma_{\mu^{lN}}(m, n)$ is increasing in n and decreasing in m (even strictly in the range, where μ_j is finite).

For $b > 0$ let $j_0(b)$ be the smallest integer $j \in \mathbb{N}$ with $j \geq \log b^{-1}$, i.e.,

$$j_0(b) := \lceil \log b^{-1} \rceil_{\mathbb{N}}.$$

We define the μ^{lN} -degree $\mathfrak{d}_{\mu^{lN}}(b) \in \mathbb{N} \cup \{\infty\}$ by setting

$$\mathfrak{d}_{\mu^{lN}}(b) := \sup \{n \in \mathbb{N} : \Sigma_{\mu^{lN}}(j_0(b) + 1, n) < e\}.$$

We remark that $\mathfrak{d}_{\mu^{lN}}(b) \geq j_0(b)$ and $\mathfrak{d}_{\mu^{lN}}(b) = j_0(b)$ occurs precisely if $\mu_{j_0(b)+1} \leq 1/e$. Clearly, $\mathfrak{d}_{\mu^{lN}}(b) = \infty$ if $\Sigma_{\mu^{lN}}(j_0(b) + 1, n) < e$ for all n . The map $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \ni (a, b) \mapsto \mathfrak{d}_{a\mu^{lN}}(b)$ is increasing in a and decreasing in b .

We could assign a μ^{lN} -degree to a (μ^{lN}, M_0) -smooth function $f : [0, 1] \rightarrow \mathbb{R}$ by putting

$$\mu^{lN}\text{-degree of } f := \mathfrak{d}_{\mu^{lN}}\left(\frac{\|f\|_{[0,1]}}{M_0}\right);$$

like the *Bang degree* in [20]. But the unspecified argument b allows for more flexibility; it will however always be related to the ratio of the \mathcal{C}^0 -norm of f by M_0 .

If the domain K of f is a d -dimensional convex body, then, in our results, the diameter δ_K of the domain K (not the dimension!) will affect the μ^{lN} -degree as a multiplicative factor of the sequence μ^{lN} , i.e.,

$$\mathfrak{d}_{\delta_K \mu^{lN}}(b).$$

See, for instance, the proof of Theorem 4.2, where the function defined on K is restricted to affine lines and a rescaling allows to consider it on the interval $[0, 1]$.

3. ZEROS OF UNIVARIATE FUNCTIONS

In this section, we revisit some results of Bang [2] (see also [20]). Since we have to rephrase them in our terminology, we give detailed proofs.

3.1. Bang's metric theory revisited. Let $M^{|\mathbb{N}} = (M_j)_{j \geq 0}$ be a full admissible weight. Let $I \subseteq \mathbb{R}$ be a non-trivial compact interval and $f : I \rightarrow \mathbb{R}$ an $M^{|\mathbb{N}}$ -smooth function.

Following Bang [2], we associate with f and $M^{|\mathbb{N}}$ a sequence $(b_n)_{n \geq 0}$ of functions defined by

$$(4) \quad b_n(t) = b_{f, M^{|\mathbb{N}}, n}(t) := \sup_{j \geq n} \frac{|f^{(j)}(t)|}{e^j M_j}, \quad t \in I,$$

with the interpretation that $\frac{|f^{(j)}(t)|}{e^j M_j} = 0$ if $M_j = \infty$ (even if $f^{(j)}(t)$ is not defined). In other words, if N is finite, then $b_n(t) = \sup_{n \leq j \leq N} \frac{|f^{(j)}(t)|}{e^j M_j}$ for $n \leq N$ and $b_n \equiv 0$ for $n > N$.

Lemma 3.1. *The sequence $(b_n)_{n \geq 0}$ has the following properties.*

- (i) $e^{-n} \geq b_n$ for all $n \geq 0$.
- (ii) $b_{n-1} \geq b_n$ and $f^{(n-1)}(t_0) = 0$ implies $b_{n-1}(t_0) = b_n(t_0)$ for all $n \geq 1$.
- (iii) For all $k > n$ and all distinct $t, s \in I$,

$$(5) \quad b_n(s) < \max\{b_n(t), e^{-k}\} e^{e|t-s|\mu_k},$$

where the right-hand side is interpreted as ∞ if $\mu_k = \infty$. In particular, all b_n are continuous.

Proof. (i) and (ii) are obvious (by (3)).

Let us prove (iii). Now (5) is trivial if N is finite and $k > N$ which means that $\mu_k = \infty$. So we may assume that $k < N + 1$ and that μ_k and M_k are finite. Let $n \leq j < k$ and $t, s \in I$. Then, by Taylor's formula, for some ξ between t and s ,

$$\begin{aligned} \frac{|f^{(j)}(s)|}{e^j M_j} &\leq \sum_{i=0}^{k-j-1} \frac{|f^{(j+i)}(t)| |t-s|^i}{e^j M_j i!} + \frac{|f^{(k)}(\xi)| |t-s|^{k-j}}{e^j M_j (k-j)!} \\ &= \sum_{i=0}^{k-j-1} \frac{M_{j+i}}{M_j} \frac{|f^{(j+i)}(t)|}{e^{j+i} M_{j+i}} \frac{(e|t-s|)^i}{i!} + e^{-k} \frac{M_k}{M_j} \frac{|f^{(k)}(\xi)|}{M_k} \frac{(e|t-s|)^{k-j}}{(k-j)!} \\ &\leq b_n(t) \sum_{i=0}^{k-j-1} \mu_k^i \frac{(e|t-s|)^i}{i!} + e^{-k} \mu_k^{k-j} \frac{(e|t-s|)^{k-j}}{(k-j)!} \\ &< \max\{b_n(t), e^{-k}\} e^{e|t-s|\mu_k}, \end{aligned}$$

where we used that $(\mu_j)_j$ is increasing. If $j \geq k$, then trivially

$$\frac{|f^{(j)}(s)|}{e^j M_j} \leq e^{-j} < \max\{b_n(t), e^{-k}\} e^{e|t-s|\mu_k}.$$

This implies (5).

To see the continuity of the b_n , we treat separately the cases $N < \infty$ and $N = \infty$. First, if N is finite, then b_n , for $n \leq N$, is a maximum of finitely many continuous functions and $b_n \equiv 0$, for $n > N$; thus continuity is clear. Second, if $N = \infty$, then continuity of b_n , for all $n \geq 0$, follows from (5): fix $t \in I$ and a sequence $t_\nu \rightarrow t$ (with $t_\nu \neq t$) in I . By (5), for each $k > n$ and all ν ,

$$b_n(t_\nu) < \max\{b_n(t), e^{-k}\} e^{e|t-t_\nu|\mu_k},$$

whence

$$\limsup_{\nu \rightarrow \infty} b_n(t_\nu) \leq \max\{b_n(t), e^{-k}\}.$$

Since this holds for all $k > n$, we have

$$\limsup_{\nu \rightarrow \infty} b_n(t_\nu) \leq b_n(t).$$

Again by (5), for each $k > n$ and all ν ,

$$b_n(t) < \max\{b_n(t_\nu), e^{-k}\} e^{e|t-t_\nu|\mu_k},$$

and so we find

$$b_n(t) \leq \max\{\liminf_{\nu \rightarrow \infty} b_n(t_\nu), e^{-k}\}$$

for all $k > n$, and therefore

$$b_n(t) \leq \liminf_{\nu \rightarrow \infty} b_n(t_\nu).$$

It follows that $\lim_{\nu \rightarrow \infty} b_n(t_\nu) = b_n(t)$. \square

The following proposition is due to [2]; we implement several modifications.

Proposition 3.2. *Let $M^{|N|} = (M_j)_{j \geq 0}$ be a full admissible weight. Let $I \subseteq \mathbb{R}$ be a non-trivial compact interval and $f : I \rightarrow \mathbb{R}$ an $M^{|N|}$ -smooth function. Let $m \in \mathbb{N}$ be such that $m + 1 \leq N$. Assume that for all $0 \leq j \leq m$ there is $x_j \in I$ such that $f^{(j)}(x_j) = 0$. Let x_{-1} be an arbitrary point in I . Then,*

$$(6) \quad \sum_{j=0}^m |x_{j-1} - x_j| \geq \frac{1}{e} \Sigma_{\mu^{|N|}}(j_0 + 1, m + 1),$$

where

$$(7) \quad j_0 = j_0(b_{f, M^{|N|}, 0}(x_{-1})) = \lceil \log b_{f, M^{|N|}, 0}(x_{-1})^{-1} \rceil_{\mathbb{N}}.$$

If $j_0 \leq m$, then the inequality (6) is strict.

Remark 3.3. A few remarks are in order.

(1) Note that $j_0 < \infty$ if and only if there exists $0 \leq j \leq N$ with $f^{(j)}(x_{-1}) \neq 0$.

(2) The right-hand side of (6) is zero if $j_0 > m$. In that case, the statement is trivial. For instance, if $N = \infty$, we do here not exclude the case that f has points of infinite flatness. If x_{-1} is such a point, then $j_0 = \infty$ and hence (6) is trivially true. In that case, the assumptions of the proposition are satisfied for the choice $x_j := x_{-1}$, $j = 0, \dots, m$, entailing that also the left-hand side of (6) is zero.

Proof of Proposition 3.2. Let $(b_n)_{n \geq 0}$ be the decreasing sequence of continuous functions (4); cf. Lemma 3.1. We will construct a new continuous function β by tracing through the graphs of the b_n (for $0 \leq n \leq m$) and switching from b_n to b_{n+1}

at x_n . For $0 \leq k \leq m$, set $\tau_k := \sum_{j=0}^k |x_{j-1} - x_j|$ and $\tau_{-1} := 0$. For $t \in [\tau_{n-1}, \tau_n]$, where $0 \leq n \leq m$, define

$$\beta_n(t) := \begin{cases} b_n(x_{n-1} + \tau_{n-1} - t) & \text{if } x_n < x_{n-1}, \\ b_n(x_{n-1} - \tau_{n-1} + t) & \text{if } x_n \geq x_{n-1}. \end{cases}$$

Then each β_n is continuous and $\beta_n(\tau_n) = b_n(x_n) = b_{n+1}(x_n) = \beta_{n+1}(\tau_n)$; by Lemma 3.1. Thus,

$$\beta(t) := \beta_n(t) \quad \text{if } t \in [\tau_{n-1}, \tau_n], \quad 0 \leq n \leq m,$$

defines a continuous function on $[0, \tau_m]$. By Lemma 3.1, we have $\beta(t) \leq e^{-n}$ for all $t \geq \tau_{n-1}$ as well as

$$(8) \quad \beta(\tau_m) = \beta_m(\tau_m) = b_m(x_m) = b_{m+1}(x_m) \leq e^{-m-1}.$$

On the other hand, with j_0 as defined in (7),

$$(9) \quad \beta(0) = \beta_0(\tau_{-1}) = b_0(x_{-1}) \geq e^{-j_0}.$$

It might be that $e^{-j_0} \leq e^{-m-1}$ (including the case $j_0 = \infty$). Then the right-hand side of (6) is zero so that (6) is trivially true. Thus we may assume that $j_0 \leq m$. In view of (8) and (9), the range of β then contains all numbers e^{-j} for $j_0 \leq j \leq m+1$. So we find a strictly increasing sequence t_j , for $j_0 \leq j \leq m+1$, such that $\beta(t_j) = e^{-j}$ and $\beta(t) > e^{-j}$ if $t < t_j$ (starting in the point $(0, \beta(0))$ let t_j be the first time that the graph of β meets the horizontal line with ordinate e^{-j}). Then

$$(10) \quad \beta(t_{j-1}) < \beta(t_j) e^{e(t_j - t_{j-1})\mu_j}, \quad j_0 + 1 \leq j \leq m+1.$$

To see this, we apply (iii) of Lemma 3.1 to each interval in the subdivision of (t_{j-1}, t_j) induced by the points τ_n between t_{j-1} and t_j , and notice that, since $t_j \leq \tau_{j-1}$ (as $\beta(t) \leq e^{-j}$ if $t \geq \tau_{j-1}$), we have $n < j$ for all such n and $\max\{b_n(t), e^{-j}\} = b_n(t)$ for all $t \in (t_{j-1}, t_j)$.

In view of $\beta(t_j) = e^{-j}$, (10) amounts to

$$t_j - t_{j-1} > \frac{1}{e^{\mu_j}}, \quad j_0 + 1 \leq j \leq m+1.$$

Summing over j , we find

$$t_{m+1} \geq t_{m+1} - t_{j_0} > \frac{1}{e} \sum_{k=j_0+1}^{m+1} \frac{1}{\mu_k}.$$

Since $\tau_m \geq t_{m+1}$, this yields (6). It also shows that the inequality is strict provided that $j_0 \leq m$. \square

3.2. Bounds for the number of zeros. Now it is easy to deduce a lower bound for the length of the interval and an upper bound for the number of zeros.

Corollary 3.4. *Let $M^{|\mathbb{N}} = (M_j)_{j \geq 0}$ be a full admissible weight. Let $I \subseteq \mathbb{R}$ be a non-trivial compact interval and $f : I \rightarrow \mathbb{R}$ an $M^{|\mathbb{N}}$ -smooth function. Let $z_1 \leq z_2 \leq \dots \leq z_m$ be an increasing enumeration of some of the zeros of f , where $m \leq N$, and let $x_{-1} \in I \setminus (z_1, z_m)$ be arbitrary. Then we have a lower bound for the length $|I|$ of I ,*

$$(11) \quad |I| > \frac{1}{e^{\sum_{\mu^{|\mathbb{N}}} (j_0 + 1, m)}}, \quad (j_0 = \lceil \log b_{f, M^{|\mathbb{N}}, 0}(x_{-1})^{-1} \rceil_{\mathbb{N}}),$$

and an upper bound for the number of zeros,

$$(12) \quad m \leq \mathfrak{d}_{|I|\mu^{|N|}}(b_{f,M^{|N|},0}(x_{-1})).$$

Proof. Suppose that $x_{-1} \leq z_1$; if $x_{-1} \geq z_m$ the proof is similar. By Rolle's theorem, there is a sequence of points $x_{-1} \leq x_0 = z_1 \leq x_1 \leq \dots \leq x_{m-1}$ in I such that $f^{(j)}(x_j) = 0$ for all $0 \leq j \leq m-1$. By (6),

$$|I| \geq \sum_{j=0}^{m-1} (x_j - x_{j-1}) \geq \frac{1}{e} \Sigma_{\mu^{|N|}}(j_0 + 1, m),$$

with strict inequality if $j_0 \leq m-1$. If $j_0 \geq m$, then $\Sigma_{\mu^{|N|}}(j_0 + 1, m) = 0$ and also in that case the inequality is strict, since I is assumed to be non-trivial. So we proved (11). By the definition of $\mathfrak{d}_{\mu^{|N|}}$, also (12) follows. \square

Remark 3.5. Variations of the argument yield further useful information. For instance, if $f(x_{-1}) \neq 0$ and x_0 is an m -fold zero of f , then

$$|x_{-1} - x_0| \geq \frac{1}{e} \Sigma_{\mu^{|N|}}(j_0 + 1, m) \geq \frac{1}{e} \Sigma_{\mu^{|N|}}(\lceil \log \frac{M_0}{|f(x_{-1})|} \rceil_{\mathbb{N}} + 1, m),$$

since in this case $\sum_{j=0}^{m-1} |x_j - x_{j-1}| = |x_{-1} - x_0|$ and $b_{f,M^{|N|},0}(x_{-1}) \geq \frac{|f(x_{-1})|}{M_0}$.

In Corollary 3.4, the list of zeros z_j may not comprise all zeros of f . If z_j is in the list and z_j is a multiple zero of f , we do not even require that all multiplicities of z_j appear in the list.

Under an additional assumption, we find that the total number of zeros is finite and get an upper bound for it.

Corollary 3.6. *Let $M^\infty = (M_j)_{j \geq 0}$ be a full admissible weight. Let $I \subseteq \mathbb{R}$ be a non-trivial compact interval and $f : I \rightarrow \mathbb{R}$ an M^∞ -smooth function. Let $x_{-1} \in I$. If*

$$(13) \quad \Sigma_{\mu^\infty}(j_0 + 1, \infty) > |I|e, \quad (j_0 = \lceil \log b_{f,M^\infty,0}(x_{-1})^{-1} \rceil_{\mathbb{N}}),$$

then the total number m of zeros of f in I counted with multiplicities is finite. In that case, $\mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1})) < \infty$ and

$$(14) \quad m \leq 2\mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1})).$$

For each $x_{-1} \in I$ satisfying (13) and not lying strictly between the smallest and the largest zero of f , we even have

$$(15) \quad m \leq \mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1})).$$

If $\mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1})) = 0$, then f has no zeros in I .

Proof. By (13), it is obvious that $\mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1})) < \infty$. Suppose for contradiction that there are infinitely many zeros of f . Then we find an increasing infinite sequence of zeros on the right of x_{-1} or a decreasing infinite sequence of zeros on the left of x_{-1} (or on both sides). But that contradicts Corollary 3.4. The upper bounds for positive m follow again from Corollary 3.4; if there are m_l zeros left and m_r zeros right of x_{-1} , then $m = m_l + m_r \leq 2\mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1}))$. In particular, if f has zeros in I , then $\mathfrak{d}_{|I|\mu^\infty}(b_{f,M^\infty,0}(x_{-1})) \neq 0$. \square

Remark 3.7. If in Corollary 3.6 we assume that f is an $M^{|N|}$ -smooth function with finite N , then in general f could have more zeros than N so that Corollary 3.4 is not applicable.

On the other hand, in the setting of Corollary 3.6 we conclude that the number of zeros of f left and right of x_{-1} is bounded by

$$(16) \quad N := \mathfrak{d}_{|I|\mu^{|\infty}}(b_{f,M^{|\infty}},0(x_{-1})).$$

So *a posteriori* we obtain that the total number of zeros m of f in I satisfies

$$(17) \quad m \leq 2\mathfrak{d}_{|I|\mu^{|N}}(b_{f,M^{|N}},0(x_{-1})),$$

where N is given by (16) and $\mu^{|N|}$, $M^{|N|}$ are obtained from $\mu^{|\infty}$, $M^{|\infty}$ simply by setting all elements with index $\geq N + 1$ equal to ∞ . In particular, the bound on the number of zeros in (17) depends only on the derivatives up order N of f .

3.3. Quasianalyticity. Let $\mu^{|\infty} = (\mu_j)_{j \geq 0}$ be an admissible weight. We see that, if

$$(18) \quad \sum_j \frac{1}{\mu_j} = \infty$$

and f is not identically zero, then condition (13) is always satisfied (provided that j_0 is finite) and Corollary 3.6 applies. An admissible weight $\mu^{|\infty}$ satisfying (18) is said to be *quasianalytic*; if (18) is not fulfilled we say that $\mu^{|\infty}$ is *non-quasianalytic* (analogously for full admissible weights $M^{|\infty}$). If we speak of (non-)quasianalytic admissible weights, we always presuppose that $N = \infty$.

In fact, let $I \subseteq \mathbb{R}$ be a non-trivial compact interval and let $\mathcal{C}^{M^{|\infty}}(I)$ denote the set of all functions $f : I \rightarrow \mathbb{R}$ such that there exists a constant $\rho > 0$ such that $\|f^{(j)}\|_I \leq \rho^{j+1}M_j$ for all $j \in \mathbb{N}$. Then $\mathcal{C}^{M^{|\infty}}(I)$ is called *quasianalytic* if, for all $f \in \mathcal{C}^{M^{|\infty}}(I)$ and $x_0 \in I$, triviality of the Taylor series \widehat{f}_{x_0} of f at x_0 , i.e., $\widehat{f}_{x_0} = 0$, implies that f is identically zero. By the Denjoy–Carleman theorem (see e.g. [25]), $\mathcal{C}^{M^{|\infty}}(I)$ is quasianalytic if and only if $M^{|\infty}$ is quasianalytic (that is, (18) holds).

Note that Corollary 3.6 implies one direction of this equivalence. For, assume that $f \in \mathcal{C}^{M^{|\infty}}(I)$, $x_0 \in I$, and $\widehat{f}_{x_0} = 0$. Then there exists $\rho > 0$ such that f is $(\rho\mu^{|\infty}, \rho M_0)$ -smooth. If f is not identically zero, we find $x_{-1} \in I$ with $f(x_{-1}) \neq 0$ and thus $j_0 = \lceil \log b_{f,(\rho^{j+1}M_j)_j,0}(x_{-1})^{-1} \rceil_{\mathbb{N}}$ is finite. Since $\widehat{f}_{x_0} = 0$ and thus the number of zeros of f is infinite, Corollary 3.6 implies that (13) must be violated. Since j_0 is finite, we may infer that (18) is violated, that is, $M^{|\infty}$ is non-quasianalytic.

4. THE ZERO SET OF MULTIVARIATE FUNCTIONS

Let $K \subseteq \mathbb{R}^d$ be a convex body. We will derive quantitative information on the zero set of $M^{|\infty}$ -smooth functions $f : K \rightarrow \mathbb{R}$ in terms of the $\delta_K \mu^{|\infty}$ -degree; recall that δ_K is the diameter of K . It turns out that these results actually depend only on a finite number of derivatives. To account for this we introduce the following bit of notation.

4.1. Finite determinacy. Let $M^{|\infty} = (M_j)_{j \geq 0}$ be a full admissible weight, $K \subseteq \mathbb{R}^d$ a convex body, and $b > 0$. Assume that

$$(19) \quad \Sigma_{\mu^{|\infty}}(j_0 + 1, \infty) > \delta_K e, \quad (j_0 = \lceil \log(\frac{b}{2M_0})^{-1} \rceil_{\mathbb{N}}).$$

(Note that (19) is in any case satisfied if $M^{|\infty}$ is a quasianalytic full admissible weight. The factor 2 in the definition of j_0 could be replaced by any real number > 1 without changing the validity of the results.) Then

$$(20) \quad \mathbf{N} := \mathfrak{d}_{\delta_K \mu^{|\infty}}\left(\frac{b}{2M_0}\right)$$

is a nonnegative (finite) integer. Let $\mu^{|\mathbf{N}+1}$ (resp. $M^{|\mathbf{N}+1}$) be the (resp. full) admissible weight obtained from $\mu^{|\infty}$ (resp. $M^{|\infty}$) by setting all elements with index $\geq \mathbf{N} + 2$ equal to ∞ . Then $\Sigma_{\mu^{|\infty}}(j_0 + 1, n) = \Sigma_{\mu^{|\mathbf{N}+1}}(j_0 + 1, n)$ for $n \leq \mathbf{N} + 1$ so that

$$(21) \quad \mathfrak{d}_{\delta_K \mu^{|\infty}}\left(\frac{b}{2M_0}\right) = \mathfrak{d}_{\delta_K \mu^{|\mathbf{N}+1}}\left(\frac{b}{2M_0}\right) = \mathbf{N}.$$

Furthermore,

$$(22) \quad \Sigma_{\mu^{|\infty}}(j_0 + 1, \infty) > \Sigma_{\mu^{|\mathbf{N}+1}}(j_0 + 1, \infty) = \Sigma_{\mu^{|\mathbf{N}+1}}(j_0 + 1, \mathbf{N} + 1) \geq \delta_K e$$

which follows easily from the definitions.

Definition 4.1 (Admissible data). We call the triple $(M^{|\infty}, K, b)$, where $M^{|\infty} = (M_j)_{j \geq 0}$ is a full admissible weight, $K \subseteq \mathbb{R}^d$ a convex body, and $b > 0$, *admissible data* if (19) holds. In that case, \mathbf{N} defined by (20) is called the *integer associated with the data* $(M^{|\infty}, K, b)$. In this setting, $\mu^{|\mathbf{N}+1}$ (resp. $M^{|\mathbf{N}+1}$) will always denote the (resp. full) admissible weight resulting from $\mu^{|\infty}$ (resp. $M^{|\infty}$) by setting all elements with index $\geq \mathbf{N} + 2$ equal to ∞ .

4.2. Number of zeros on affine lines. Recall that K° denotes the interior of K .

Theorem 4.2. *Let $(M^{|\infty}, K, b)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $M^{|\mathbf{N}+1}$ -smooth function such that $\|f\|_K \geq b$. Then there is a ball $B \subseteq K^\circ$, whose radius only depends on K and the ratio $\frac{b}{M_1}$, such that for each affine line ℓ in \mathbb{R}^d that meets B the restriction $f|_\ell$ has at most $2\mathbf{N}$ zeros.*

Proof. Consider $U := \{x \in K : |f(x)| > \frac{b}{2}\}$. Since $\|f\|_K \geq b$, there is $a \in U$ with $|f(a)| \geq b$. For all $x \in B_s(a) \cap K$ with $s := \frac{b}{3M_1}$, we have

$$|f(x) - f(a)| \leq \|f\|_{1,K} |x - a| \leq M_1 s = \frac{b}{3},$$

so that

$$|f(x)| \geq |f(a)| - |f(x) - f(a)| \geq \frac{2b}{3}.$$

It follows that $a \in B_s(a) \cap K \subseteq U$. But the intersection $B_s(a) \cap K^\circ$ contains an open ball B , whose radius depends only on s and on the ‘‘thickness’’ of K near a .

Let $\ell \subseteq \mathbb{R}^d$ be any affine line that meets the ball B . Let x_0 and x_1 be the intersection points of ℓ with the boundary of K . Then $g : [0, 1] \rightarrow \mathbb{R}$ given by $g(t) := f(x_0 + t(x_1 - x_0))$ defines a $\mathcal{C}^{\mathbf{N}+1}$ -function satisfying

$$\|g^{(j)}\|_{[0,1]} \leq |x_1 - x_0|^j \|f\|_{j,K} \leq \delta_K^j M_j, \quad 0 \leq j \leq \mathbf{N} + 1.$$

Thus, $g : [0, 1] \rightarrow \mathbb{R}$ is $(\delta_K \mu^{|\mathbf{N}+1}, M_0)$ -smooth. For all $t \in [0, 1]$ with $x_0 + t(x_1 - x_0) \in B$, we find

$$b_{g, (\delta_K^j M_j)_{j,0}}(t) = \sup_{j \geq 0} \frac{|g^{(j)}(t)|}{(\delta_K e)^j M_j} \geq \frac{|g(t)|}{M_0} > \frac{b}{2M_0},$$

whence $\mathfrak{d}_{\delta_K \mu^{|\mathbf{N}+1}}(b_{g,(\delta_K^j M_j)_j,0}(t)) \leq \mathfrak{d}_{\delta_K \mu^{|\mathbf{N}+1}}(\frac{b}{2M_0}) = \mathbf{N}$ (see (21)). Fix such a t . Suppose that g has (at least) $\mathbf{N} + 1$ zeros left or right of t . Then Corollary 3.4 implies

$$\mathbf{N} + 1 \leq \mathfrak{d}_{\delta_K \mu^{|\mathbf{N}+1}}(b_{g,(\delta_K^j M_j)_j,0}(t)) \leq \mathbf{N},$$

a contradiction. Thus the total number of zeros of g , and hence of $f|_{\ell}$, is finite and bounded by $2\mathbf{N}$. \square

Corollary 4.3. *In the setting of Theorem 4.2, if $K = \overline{B}_r$, then $\delta_K = 2r$ and the radius of B is at least $\frac{1}{2} \min\{\frac{b}{3M_1}, r\}$.*

Proof. The intersection $B_s(a) \cap B_r$ contains an open ball B of radius at least $\frac{1}{2} \min\{s, r\}$. \square

Remark 4.4. In analogy to Remark 3.5, we can give at each point in K a lower bound for the distance to the closest zero of f of a specific multiplicity; often the bound is trivial, but not always:

Let $M^{|\mathbf{N}+1}$ be a full admissible weight and $K \subseteq \mathbb{R}^d$ a convex body. Let $f : K \rightarrow \mathbb{R}$ be an $M^{|\mathbf{N}+1}$ -smooth function and $x_{-1} \in K \setminus Z_f$. Then f has no zeros of multiplicity at least m , where $m \leq \mathbf{N} + 1$, in the ball $B_{\epsilon_m}(x_{-1})$ with center x_{-1} and radius

$$\epsilon_m := \frac{1}{e} \Sigma_{\delta_K \mu^{|\mathbf{N}+1}}(j_0 + 1, m), \quad (j_0 = \lceil \log \frac{M_0}{|f(x_{-1})|} \rceil_{\mathbb{N}}).$$

This follows from Remark 3.5 applied to all affine lines through x_{-1} .

In particular, f has no zeros in $B_{\epsilon_1}(x_{-1})$. But $\epsilon_1 \neq 0$ only if $j_0 = 0$ which is equivalent to $|f(x_{-1})| = M_0$; in that case $\epsilon_1 = \frac{1}{\delta_K e \mu_1}$. Furthermore, f has no zeros that are also critical points in $B_{\epsilon_2}(x_{-1})$, where

$$\epsilon_2 = \begin{cases} \frac{1}{\delta_K e} (\frac{1}{\mu_1} + \frac{1}{\mu_2}), & \text{if } |f(x_{-1})| = M_0, \\ \frac{1}{\delta_K e \mu_2}, & \text{if } \frac{M_0}{e} \leq |f(x_{-1})| < M_0, \\ 0, & \text{otherwise.} \end{cases}$$

For later use, we remark that the zero set of a function f as in Theorem 4.2 has zero Lebesgue measure. This will be a trivial consequence of Theorem 4.6, if f is additionally assumed to be of class \mathcal{C}^∞ , but we will need it without this assumption.

Corollary 4.5. *In the setting of Theorem 4.2, $\mathcal{L}^d(Z_f) = 0$.*

Proof. First of all, f is continuous so that Z_f is closed, hence measurable. The union of all affine lines of a fixed direction meeting B form an open cylinder U . By compactness, K is covered by finitely many such cylinders U . Now $Z_f \cap U$ is measurable, whence we can apply Fubini's theorem to its characteristic function. By Theorem 4.2, this gives $\mathcal{L}^d(Z_f \cap U) = 0$. \square

4.3. Hausdorff measure of the zero set.

Theorem 4.6. *Let $(M^{|\infty}, K, b)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $M^{|\mathbf{N}+1}$ -smooth \mathcal{C}^∞ -function such that $\|f\|_K \geq b$. Then the zero set Z_f is contained in a countable union of compact \mathcal{C}^∞ -hypersurfaces and its $(d-1)$ -dimensional Hausdorff measure satisfies*

$$(23) \quad \mathcal{H}^{d-1}(Z_f) \leq C \mathbf{N} \delta_K^{d-1},$$

where the constant $C > 0$ depends only on d and on the ratio $\frac{\delta_B}{\delta_K}$ and where B is the ball from Theorem 4.2. In the case $K = \overline{B}_r$, the constant C depends only on d and $\min\{\frac{b}{6rM_1}, \frac{1}{2}\}$; it blows up as $b \rightarrow 0$ or $r \rightarrow \infty$.

Proof. The first statement follows from Malgrange's preparation theorem and [31, Lemma 6]. Thus, we may apply the Crofton-type result [31, Lemma 7] to conclude (23) from Theorem 4.2.

If $K = \overline{B}_r$, then we may take $\delta_B = \min\{\frac{b}{3M_1}, r\}$, by Corollary 4.3, so that $\frac{\delta_B}{\delta_K} = \min\{\frac{b}{6rM_1}, \frac{1}{2}\}$. This also yields the asymptotics of the constant C . \square

Remark 4.7. In Theorem 4.6 (and in the next result), we assume that f is of class C^∞ in order to apply Malgrange's preparation theorem. That $f : K \rightarrow \mathbb{R}$ is an $M^{\mathbb{N}+1}$ -smooth C^∞ -function means that f is of class C^∞ and satisfies $\|f\|_{j,K} \leq M_j$ for all $0 \leq j \leq \mathbb{N} + 1$.

4.4. Sobolev parameterization of the zero set. Let $(M^{\mathbb{N}}, K, b)$ be admissible data and \mathbb{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $M^{\mathbb{N}+1}$ -smooth C^∞ -function such that $\|f\|_K \geq b$.

The collection \mathcal{L} of all affine lines in \mathbb{R}^d that meet the open ball B from Theorem 4.2 covers \mathbb{R}^d , i.e., $\mathbb{R}^d = \bigcup_{\ell \in \mathcal{L}} \ell$. Fix $z_0 \in Z_f$ and any line $\ell \in \mathcal{L}$ with $z_0 \in \ell$. By Theorem 4.2, the multiplicity n of z_0 as a zero of $f|_\ell$ is at most $2\mathbb{N}$. Let x_1, \dots, x_d be an orthogonal coordinate system such that ℓ coincides with the x_d -axis. By Malgrange's preparation theorem, there is a rectangular neighborhood U of z_0 of the form $U := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \alpha_j < x_j < \beta_j \text{ for all } 1 \leq j \leq d\}$ such that $f(x) = p(x)u(x)$ for $x \in U$, where u does not vanish on U and

$$(24) \quad p(x', x_d) = x_d^n + a_1(x')x_d^{n-1} + \dots + a_d(x')$$

is a polynomial in x_d with C^∞ -coefficients a_j which are defined on the projection U' of U onto the first $d-1$ coordinates $x' := (x_1, \dots, x_{d-1})$. Thus,

$$Z_f \cap U = Z_p \cap U.$$

Let $[\zeta_1(x'), \dots, \zeta_n(x')]$ be the unordered n -tuple of complex roots (with multiplicities) of the polynomial $p(x', x_d)$ in x_d for any $x' \in U'$. Note that the number of real roots among the $\zeta_j(x')$ may change with varying x' . Let $(\xi_1(x'), \dots, \xi_n(x'))$ be the n -tuple consisting of the real parts of the $\zeta_j(x')$, ordered increasingly such that $\xi_1(x') \leq \xi_2(x') \leq \dots \leq \xi_n(x')$ for all x' . Since all the real roots among the $\zeta_j(x')$ clearly are also among the $\xi_i(x')$ (with correct multiplicities), it follows that

$$\begin{aligned} Z_f \cap U &= Z_p \cap U \\ &= \{(x', x_d) \in U : \text{there is } 1 \leq j \leq n \text{ with } x_d = \zeta_j(x')\} \\ &\subseteq \{(x', x_d) \in U : \text{there is } 1 \leq i \leq n \text{ with } x_d = \xi_i(x')\} \subseteq \bigcup_{i=1}^n \Gamma(\xi_i), \end{aligned}$$

where $\Gamma(\xi_i) := \{(x', \xi_i(x')) : x' \in U'\}$ denotes the graph of $\xi_i : U' \rightarrow \mathbb{R}$. The increasing order of the ξ_i implies that each ξ_i is a continuous function $U' \rightarrow \mathbb{R}$ and so, applying [22, Remark 9] and [23, Theorem A.1] (see also [21]), we may conclude that each ξ_i is of Sobolev class $\xi_i \in W^{1,p}(U')$, for all $1 \leq p < \frac{n}{n-1}$, and $\|\xi_i\|_{W^{1,p}(U')}$ depends uniformly on f ; see Remark 4.10. In view of [16, Theorem 1.2], we obtain

that each graph $\Gamma(\xi_i)$ is countably \mathcal{H}^{d-1} -rectifiable and

$$(25) \quad \mathcal{H}^{d-1}(\{(x', \xi_i(x')) : x' \in E\}) = \int_E \sqrt{1 + |\nabla \xi_i(x')|^2} dx',$$

for all measurable subsets $E \subseteq U'$. The right-hand side of (25) depends uniformly on f (see Remark 4.10).

Now it is easy to conclude

Theorem 4.8. *Let $(M^{|\infty}, K, b)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $M^{|\mathbf{N}+1}$ -smooth \mathcal{C}^∞ -function such that $\|f\|_K \geq b$. There is a finite cover of K by rectangular boxes U with corresponding orthogonal coordinates (x', x_d) such that $Z_f \cap U$ is contained in the graphs $\{x_d = \xi_i(x')\}$ of at most $2\mathbf{N}$ continuous functions ξ_i of Sobolev class $W^{1,p}$, for all $1 \leq p < \frac{2\mathbf{N}}{2\mathbf{N}-1}$, whose $W^{1,p}$ -norm depends uniformly on f (see Remark 4.10). Furthermore, (25) holds.*

Note that the Sobolev regularity in the results of [22, 23] used above is optimal.

Remark 4.9. It is evident from the above arguments that the statement of Theorem 4.8 remains true for \mathcal{C}^∞ -functions $f : K \rightarrow \mathbb{R}$ that satisfy

$$\|f\|_{j,K} \leq \frac{1}{2\delta_K^j} \|f\|_{0,K}, \quad \text{for some } j \geq 2,$$

where $2\mathbf{N}$ is replaced by $j - 1$. Use [31, Theorem 3(ii)].

Remark 4.10. In general, the coefficients a_j of the polynomial (24) will no longer be $M^{|\mathbf{N}+1}$ -smooth (cf. [1]). But the choice of the functions a_j (and u) can be made to depend linearly and continuously (with respect to the Whitney \mathcal{C}^∞ topology) on f (cf. [19]). Furthermore, we have uniform bounds for the $W^{1,p}$ -norm of the functions ξ_i in terms of the $\mathcal{C}^{2\mathbf{N}}$ -norm of the a_j (cf. [22] and [23]).

4.5. Further remarks. Let $(M^{|\infty}, K, b)$ be admissible data and \mathbf{N} the associated integer.

Remark 4.11. By definition, the number \mathbf{N} depends only on $(M^{|\infty}, K, b)$. The above results on Z_f remain unchanged as long as f satisfies all the respective assumptions. E.g., the bound on $\mathcal{H}^{d-1}(Z_f)$ in (23) is the same as long as f fulfills the assumptions of Theorem 4.6.

Let $f : K \rightarrow \mathbb{R}$ be an $M^{|\mathbf{N}+1}$ -smooth function with $\|f\|_K \geq b$. If we set $V(f) := \sup_{x,y \in K} |f(x) - f(y)|$, then $V(f) \leq 2\|f\|_K$ and hence

$$\|f - c\|_K \geq \frac{V(f - c)}{2} = \frac{V(f)}{2} \quad \text{for all } c \in \mathbb{R}.$$

On the other hand,

$$\|f - c\|_K \leq M_0 \quad \text{if } |c| \leq M_0 - \|f\|_K.$$

So, if $2b \leq V(f)$, we get the same uniform results for the level sets $Z_{f-c} = f^{-1}(c)$ for all $|c| \leq M_0 - \|f\|_K$.

Remark 4.12. If we assume that $f : K \rightarrow \mathbb{R}$ is $(\delta_K^{-1} \mu^{|\infty}, M_0)$ -smooth with $\|f\|_K \geq b$ and accordingly

$$\Sigma_{\mu^{|\infty}}(j_0 + 1, \infty) > e, \quad (j_0 = \lceil \log(\frac{b}{2M_0})^{-1} \rceil_{\mathbb{N}}),$$

instead of (19), then we get all the above results with both the $\mu^{|\infty}$ -degree $\mathfrak{d}_{\mu^{|\infty}}(\frac{b}{2M_0})$ and the constant C in (23) independent of δ_K .

Remark 4.13. A closer inspection of the proof of Theorem 4.2 reveals that it would suffice to assume that each restriction $f|_\ell : K \cap \ell \rightarrow \mathbb{R}$ of f is $M^{|\mathbf{N}+1}$ -smooth, where ℓ is any affine line intersecting K . But this is not far from the assumption that $f : K \rightarrow \mathbb{R}$ is $M^{|\mathbf{N}+1}$ -smooth: for simplicity assume that $f : K \rightarrow \mathbb{R}$ is of class \mathcal{C}^∞ (which is needed in Theorem 4.6 and Theorem 4.8 anyway). Then the uniformity of the bounds and the polarization inequality ([15, 7.13.1]),

$$\sup_{v \in \overline{B}_1} |d_v^j f(x)| \leq \|d^j f(x)\|_{L^j(\mathbb{R}^d, \mathbb{R})} \leq (2e)^j \sup_{v \in \overline{B}_1} |d_v^j f(x)|,$$

where $d_v^j f(x) := \partial_t^j|_{t=0} f(x + tv)$, imply that $f : K \rightarrow \mathbb{R}$ is $M^{|\mathbf{N}+1}$ -smooth, after slight modification of $M^{|\mathbf{N}+1}$. (Note that, by results of Boman [7], the assumption that the function is \mathcal{C}^∞ is actually not necessary, at least if the domain of the function is open.)

This is somewhat reminiscent of a result of Bochnak and Siciak [5, 6, 29] that a \mathcal{C}^∞ function is real analytic if its restrictions to all affine lines are real analytic. For general (even quasianalytic) weights, this is however not true if the bounds are not uniform (i.e. they depend on the affine lines); see [14] and [24].

5. REMEZ INEQUALITY FOR FUNCTIONS WITH CONTROLLED DERIVATIVES

In this section, we prove a Remez-type inequality for $M^{|\mathbf{N}}$ -smooth (and $\mu^{|\mathbf{N}}$ -smooth) functions in several variables and derive several consequences. Our results are based on a univariate version due to [20] which we recall in slightly modified form in Theorem 5.1.

5.1. Definitions and conventions. Let $\mu^{|\infty} = (\mu_j)_{j \geq 1}$ be an admissible weight. Let us assume that there is an increasing continuous function $\tilde{\mu} : [1, \infty) \rightarrow (0, \infty)$ that is (piecewise) \mathcal{C}^1 such that

$$\mu_j = \tilde{\mu}(j), \quad j \geq 1.$$

This is no real restriction, since we may always take $\tilde{\mu}$ piecewise affine and work consistently with the left derivative at points, where $\tilde{\mu}$ is not differentiable.

Once we have $\tilde{\mu}$, we define (following [20])

$$\gamma_{\tilde{\mu}}(n) := \sup_{1 \leq s \leq n} \frac{s \tilde{\mu}'(s)}{\tilde{\mu}(s)} \quad \text{and} \quad \Gamma_{\tilde{\mu}}(n) := 4e^{4 + \gamma_{\tilde{\mu}}(n)},$$

for all positive integers n . Note that $\gamma_{\tilde{\mu}}$ and $\Gamma_{\tilde{\mu}}$ depend on the choice of $\tilde{\mu}$ which is not unique.

In this section, we will again make use of the terminology introduced in Section 4.1 (see Definition 4.1). But here it is better to replace b by $2b$. So let $(M^{|\infty}, K, 2b)$ be admissible data and \mathbf{N} the associated integer. Recall that this means the following: $M^{|\infty}$ is a full admissible weight, $K \subseteq \mathbb{R}^d$ a convex body, and $b > 0$ such that

$$(26) \quad \Sigma_{\mu^{|\infty}}(j_0 + 1, \infty) > \delta_K e, \quad (j_0 = \lceil \log(\frac{b}{M_0})^{-1} \rceil_{\mathbf{N}}).$$

The associated nonnegative integer \mathbf{N} is given by

$$\mathbf{N} := \mathfrak{d}_{\delta_K \mu^{|\infty}}(\frac{b}{M_0}).$$

It satisfies (21) and (22) with b replaced by $2b$, in particular,

$$(27) \quad \mathfrak{d}_{\delta_K \mu^{|\infty}}(\frac{b}{M_0}) = \mathfrak{d}_{\delta_K \mu^{|\mathbf{N}+1}}(\frac{b}{M_0}) = \mathbf{N}.$$

In all occurrences of the Remez inequality for functions with controlled derivatives, the constant $\Gamma_{\bar{\mu}}(2\mathbf{N})^{2\mathbf{N}}$ will appear. If $\mathfrak{d}_{\delta_K \mu^{|\infty}}(\frac{b}{M_0}) = \mathbf{N} = 0$ it is undefined. For ease of notation, we set

$$(28) \quad C_{\mathbf{N}} := \Gamma_{\bar{\mu}}(2\mathbf{N})$$

with the interpretation $C_{\mathbf{N}}^{2\mathbf{N}} =: \infty$ if $\mathbf{N} = 0$; see Remark 5.2.

5.2. Remez inequality for univariate functions. We recall and slightly modify [20, Theorem B]. The statement in [20] involves the so-called Bang degree \mathbf{n}_f (which depends on f) and it is formulated for a quasianalytic full admissible weight $M^{|\infty}$ with $M_0 = 1$. Here we work with $\mathfrak{d}_{\mu^{|\infty}}(\frac{b}{M_0}) = \mathfrak{d}_{\mu^{|\mathbf{N}+1}}(\frac{b}{M_0})$ (which is independent of f) instead of \mathbf{n}_f .

Theorem 5.1. *Let $(M^{|\infty}, [0, 1], 2b)$ be admissible data and \mathbf{N} the associated integer. Let $f : [0, 1] \rightarrow \mathbb{R}$ be any $M^{|\mathbf{N}}$ -smooth function such that $\|f\|_{[0,1]} \geq b$. Then for any interval $I \subseteq [0, 1]$ and any Lebesgue measurable set $E \subseteq I$ with $|E| > 0$ we have*

$$(29) \quad \|f\|_I \leq \left(\frac{C_{\mathbf{N}} |I|}{|E|} \right)^{2\mathbf{N}} \|f\|_E.$$

Remark 5.2. If $\mathfrak{d}_{\mu^{|\infty}}(\frac{b}{M_0}) = \mathbf{N} = 0$, then, by the interpretation $C_{\mathbf{N}}^{2\mathbf{N}} = \infty$, (29) is trivially true.

Proof of Theorem 5.1. First of all, we may assume that $M_0 = 1$, by dividing f by M_0 . In fact, $\bar{f} := \frac{1}{M_0} f$ is $(\mu^{|\mathbf{N}}, 1)$ -smooth and $\|\bar{f}\|_{[0,1]} \geq \frac{b}{M_0} =: \bar{b} > 0$ so that (26), (29), and the integer \mathbf{N} remain unchanged.

By Remark 5.2, we may assume that $\mathbf{N} > 0$ and follow the proof of [20]; it involves only derivatives up to order $2\mathfrak{d}_{\mu^{|\infty}}(b) = 2\mathfrak{d}_{\mu^{|\mathbf{N}+1}}(b) = 2\mathbf{N}$ (cf. (27) with $K = [0, 1]$ and $M_0 = 1$).

In the only place, where the definition of \mathbf{n}_f actually plays a role in the argument, we use the following estimate (cf. [20, p. 72]):

$$(30) \quad \min_{t \in [0,1]} b_0(t) = \min_{t \in [0,1]} \sup_{j \geq 0} \frac{|f^{(j)}(t)|}{e^j M_j} > e^{-\mathbf{N}-1}.$$

Let us justify (30). Suppose it is not true, i.e., $\min_{t \in [0,1]} b_0(t) \leq e^{-\mathbf{N}-1}$. On the other hand, $\max_{t \in [0,1]} b_0(t) \geq b \geq e^{-j_0}$, by the definition of j_0 (recall that $M_0 = 1$). We have $j_0 < \mathfrak{d}_{\mu^{|\mathbf{N}+1}}(b) + 1 = \mathbf{N} + 1$ (cf. Section 2.3), that is $e^{-\mathbf{N}-1} < e^{-j_0}$. Since b_0 is continuous (cf. Lemma 3.1), there is a monotonic sequence $x_j \in [0, 1]$ such that $b_0(x_j) = e^{-j}$ for all $j_0 \leq j \leq \mathbf{N} + 1$ (cf. the proof of Proposition 3.2). By Lemma 3.1, we have $|x_j - x_{j-1}| > \frac{1}{e\mu_j}$ for $j_0 + 1 \leq j \leq \mathbf{N} + 1$ so that

$$1 \geq \sum_{j=j_0+1}^{\mathbf{N}+1} |x_j - x_{j-1}| > \frac{1}{e} \sum_{j=j_0+1}^{\mathbf{N}+1} \frac{1}{\mu_j},$$

and thus $\mathbf{N} = \mathfrak{d}_{\mu^{|\infty}}(\frac{b}{M_0}) \geq \mathbf{N} + 1$, a contradiction. Hence (30) is proved. \square

Remark 5.3. As pointed out in [20], if $\bar{\gamma}_{\bar{\mu}} := \sup_{s \geq 1} \frac{s\bar{\mu}'(s)}{\bar{\mu}(s)} < \infty$ (as is the case for $\bar{\mu}(s) = s$ or $\bar{\mu}(s) = s(\log(s+e))^\delta$ if $0 < \delta \leq 1$), then in (29) the constant $C_{\mathbf{N}}$ can be replaced by $\bar{\Gamma}_{\bar{\mu}} = 4e^{4+\bar{\gamma}_{\bar{\mu}}}$, provided that $\mathbf{N} \neq 0$ (see Remark 5.2).

5.3. Remez inequality for multivariate functions. Next we prove a Remez-type inequality for multivariate functions with controlled derivatives. The proof is inspired by the technique of [11].

Theorem 5.4. *Let $(M^{|\infty}, K, 2b)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $M^{|\infty}$ -smooth function and $L \subseteq K$ any convex body such that $\|f\|_L \geq b$. If $E \subseteq L$ is a Lebesgue measurable subset with $|E| > 0$, then*

$$\|f\|_L \leq \left(\frac{C_{\mathbf{N}} |L|^{1/d}}{|L|^{1/d} - (|L| - |E|)^{1/d}} \right)^{2\mathbf{N}} \|f\|_E.$$

Proof. Let $\mathcal{F}(E, L) = \mathcal{F}_{M^{|\infty}}(E, L, b)$ denote the set of all $M^{|\infty}$ -smooth functions $f : K \rightarrow \mathbb{R}$ satisfying $\|f\|_L \geq b$ and $\|f\|_E \leq 1$. For $\lambda > 0$ consider

$$\mathcal{R}(\lambda, L) = \mathcal{R}_{M^{|\infty}}(\lambda, L, b) := \sup_{\substack{E \subseteq L \\ |E| \geq \lambda}} \sup_{f \in \mathcal{F}(E, L)} \|f\|_L.$$

We claim that

$$(31) \quad \mathcal{R}(\lambda, L) \leq \left(\frac{C_{\mathbf{N}} |L|^{1/d}}{|L|^{1/d} - (|L| - \lambda)^{1/d}} \right)^{2\mathbf{N}}.$$

Fix a measurable subset $E \subseteq L$ with $|E| \geq \lambda$ and $f \in \mathcal{F}(E, L)$. There is $x_0 \in L$ with $\|f\|_L = |f(x_0)|$. Let ℓ be any half-line emanating from x_0 and let x_1 be the (other) intersection point of ℓ with ∂L . Define $g : [0, 1] \rightarrow \mathbb{R}$ by $g(t) := f(x_0 + t(x_1 - x_0))$. As seen in the proof of Theorem 4.2, $g : [0, 1] \rightarrow \mathbb{R}$ is $(\delta_K \mu^{|\infty}, M_0)$ -smooth and $\|g\|_{[0,1]} \geq |g(0)| = |f(x_0)| \geq b$. Thus (in view of (26) and since $\|f\|_E \leq 1$) Theorem 5.1 implies

$$\|f\|_L = |f(x_0)| = |g(0)| \leq \left(\frac{C_{\mathbf{N}} \mathcal{L}^1(L \cap \ell)}{\mathcal{L}^1(E \cap \ell)} \right)^{2\mathbf{N}},$$

because of the scaling properties of the 1-dimensional Lebesgue measure \mathcal{L}^1 (induced on ℓ); note that $\gamma_{\delta_K \bar{\mu}} = \gamma_{\bar{\mu}}$ and hence $\Gamma_{\delta_K \bar{\mu}} = \Gamma_{\bar{\mu}}$. Taking the essential infimum over all half-lines emanating from x_0 , the supremum over all $f \in \mathcal{F}(E, L)$, and the supremum over all measurable $E \subseteq L$ with $|E| \geq \lambda$, we find

$$\mathcal{R}(\lambda, L) \leq \left(C_{\mathbf{N}} \sup_{\substack{E \subseteq L \\ |E| \geq \lambda}} \operatorname{ess\,inf}_{\ell} \frac{\mathcal{L}^1(L \cap \ell)}{\mathcal{L}^1(E \cap \ell)} \right)^{2\mathbf{N}},$$

and, using [11, Lemma 3 and Remark 2], we conclude (31).

Now we may prove the statement of the theorem. Let f , L , and E be as in the assumptions of the theorem. Since $|E| > 0$, we have $\|f\|_E > 0$, by Corollary 4.5. Then $F := \frac{f}{\|f\|_E}$ is $\frac{1}{\|f\|_E} M^{|\infty}$ -smooth and satisfies $\|F\|_L \geq \frac{b}{\|f\|_E}$ and $\|F\|_E = 1$. Thus $F \in \mathcal{F}_{\frac{1}{\|f\|_E} M^{|\infty}}(E, L, \frac{b}{\|f\|_E})$; note that (26), \mathbf{N} , and $C_{\mathbf{N}}$ remain unchanged. Then (31) implies

$$\|F\|_L \leq \left(\frac{C_{\mathbf{N}} |L|^{1/d}}{|L|^{1/d} - (|L| - |E|)^{1/d}} \right)^{2\mathbf{N}}.$$

Since $\|F\|_L = \frac{\|f\|_L}{\|f\|_E}$, this completes the proof. \square

5.4. An important consequence. The next result is a simple but important consequence of Theorem 5.4.

For its formulation, it is convenient to adapt our terminology. We say that the pair $(\mu^{|\infty}, K)$ is *admissible data with associated integer* \mathbf{N} if $\mu^{|\infty} = (\mu_j)_{j \geq 1}$ is an admissible weight and $K \subseteq \mathbb{R}^d$ a convex body such that

$$(32) \quad \Sigma_{\mu^{|\infty}}(1, \infty) > \delta_K e$$

and

$$\mathbf{N} := \mathfrak{d}_{\delta_K \mu^{|\infty}}(1).$$

As before, $C_{\mathbf{N}} := \Gamma_{\bar{\mu}}(2\mathbf{N})$ and $C_{\mathbf{N}}^{2\mathbf{N}} =: \infty$ if $\mathbf{N} = 0$.

We will specialize Theorem 5.4 to the case $b = M_0 = \|f\|_K$, that is to $(\mu^{|\infty}, \|f\|_K)$ -smooth functions $f : K \rightarrow \mathbb{R}$. Recall that, by the definition in Section 2.2, $f : K \rightarrow \mathbb{R}$ is called $(\mu^{|\infty}, \|f\|_K)$ -smooth if $\|f\|_K > 0$ and

$$\|f\|_{j,K} \leq \|f\|_K \cdot \mu_1 \mu_2 \cdots \mu_j, \quad 1 \leq j \leq 2\mathbf{N}.$$

Theorem 5.5. *Let $(\mu^{|\infty}, K)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $(\mu^{|\infty}, \|f\|_K)$ -smooth function. If $E \subseteq K$ is a Lebesgue measurable subset with $|E| > 0$, then*

$$(33) \quad \|f\|_K \leq \left(\frac{C_{\mathbf{N}} |K|^{1/d}}{|K|^{1/d} - (|K| - |E|)^{1/d}} \right)^{2\mathbf{N}} \|f\|_E.$$

Proof. Set $L = K$ and $b = M_0 := \|f\|_K$ in Theorem 5.4. □

There is no *a priori* condition on $\|f\|_K$, except $\|f\|_K > 0$, in Theorem 5.5. Visibly, (33) is invariant under the action of \mathbb{R}^* on f .

It can happen that $\mathfrak{d}_{\delta_K \mu^{|\infty}}(1) = \mathbf{N} = 0$, in which case Theorem 5.5 contains no information (cf. Remark 5.2). In fact, this occurs precisely if $\delta_K \mu_1 \leq 1/e$.

5.5. Volume of sublevel sets. Theorem 5.5 has several interesting corollaries. We begin with a bound on the growth of the volume of sublevel sets.

Corollary 5.6. *Let $(\mu^{|\infty}, K)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $(\mu^{|\infty}, \|f\|_K)$ -smooth function. Then the sublevel set $S_t := \{x \in K : |f(x)| \leq t\}$ satisfies*

$$(34) \quad |S_t| \leq C_{\mathbf{N}} d |K| \left(\frac{t}{\|f\|_K} \right)^{\frac{1}{2\mathbf{N}}}, \quad t > 0,$$

with the understanding that the right-hand side is ∞ if $\mathbf{N} = 0$.

Proof. We may assume that $\mathbf{N} \neq 0$ and that $|S_t| > 0$ (the inequality being trivial otherwise). Apply Theorem 5.5 to $E = S_t$. Then, putting $\theta := \frac{|S_t|}{|K|}$,

$$\|f\|_K \leq \left(\frac{C_{\mathbf{N}}}{1 - (1 - \theta)^{1/d}} \right)^{2\mathbf{N}} t,$$

and consequently,

$$\frac{\theta}{d} \leq 1 - (1 - \theta)^{1/d} \leq C_{\mathbf{N}} \left(\frac{t}{\|f\|_K} \right)^{\frac{1}{2\mathbf{N}}}.$$

The statement follows. □

Corollary 5.6 implies useful estimates for the distribution function and the decreasing rearrangement of f (more generally, of $f|_E$ for a measurable subset $E \subseteq K$). Recall that the distribution function of $f : K \rightarrow \mathbb{R}$ is defined by

$$d_f(t) := |\{x \in K : |f(x)| > t\}| = |K| - |S_t|$$

and the decreasing rearrangement of f by

$$f^*(y) := \inf\{t > 0 : d_f(t) \leq y\}.$$

Corollary 5.7. *Let $(\mu^{|\infty}, K)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $(\mu^{2\mathbf{N}}, \|f\|_K)$ -smooth function. Let $E \subseteq K$ be a Lebesgue measurable subset with $|E| > 0$. Then*

$$(35) \quad (f|_E)^*(|E|\lambda) \geq \|f\|_K \left(\frac{|E|}{|K|} \cdot \frac{1-\lambda}{C_{\mathbf{N}}d} \right)^{2\mathbf{N}}, \quad \lambda \in (0, 1),$$

where the right-hand side is identically zero if $\mathbf{N} = 0$.

Proof. Assume that $\mathbf{N} \neq 0$. We have

$$d_{f|_E}(t) := |\{x \in E : |f(x)| > t\}| = |E| - |S_t \cap E| = |E|\lambda_t,$$

where $\lambda_t := 1 - |S_t \cap E|/|E|$. Let s_λ denote the right-hand side of (35). By Corollary 5.6,

$$(36) \quad d_{f|_E}(t) = |E|\lambda_t > |E|\lambda, \quad \text{if } t \in (0, s_\lambda).$$

Indeed, by (34),

$$\begin{aligned} |E|(1-\lambda_t) = |S_t \cap E| &\leq |S_t| \leq C_{\mathbf{N}}d|K| \left(\frac{t}{\|f\|_K} \right)^{\frac{1}{2\mathbf{N}}} \\ &< C_{\mathbf{N}}d|K| \left(\frac{s_\lambda}{\|f\|_K} \right)^{\frac{1}{2\mathbf{N}}} = |E|(1-\lambda). \end{aligned}$$

Now (36) implies $(f|_E)^*(|E|\lambda) \geq s_\lambda$, and (35) is proved. \square

5.6. Comparison of L^p -norms. Let us write

$$\begin{aligned} \|f\|_{L^p(E)}^\sharp &:= \left(\frac{1}{|E|} \int_E |f(x)|^p dx \right)^{1/p}, \quad 0 < p < \infty, \\ \|f\|_{L^\infty(E)}^\sharp &:= \operatorname{ess\,sup}_E |f|, \end{aligned}$$

for the normalized L^p -norms (respectively, quasinorms if $0 < p < 1$) of f on a measurable set E with $0 < |E| < \infty$. Then, as a consequence of Hölder's inequality,

$$\|f\|_{L^q(E)}^\sharp \leq \|f\|_{L^p(E)}^\sharp, \quad \text{if } 0 < q \leq p \leq \infty.$$

Indeed, since $p/q \geq 1$,

$$\int_E |f|^q dx \leq |E|^{1-q/p} \left(\int_E |f|^p dx \right)^{q/p}.$$

For functions with controlled derivatives also suitable opposite inequalities hold.

Corollary 5.8. *Let $(\mu^{|\infty}, K)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $(\mu^{2\mathbf{N}}, \|f\|_K)$ -smooth function. Let $E \subseteq K$ be a Lebesgue measurable subset with $|E| > 0$. Then, for all $0 < q < p \leq \infty$,*

$$(37) \quad \|f\|_{L^p(K)}^\sharp \leq \left(\frac{C_{\mathbf{N}}d|K|}{|E|} \right)^{2\mathbf{N}(1-\frac{q}{p})} (2q\mathbf{N} + 1)^{\frac{1}{q}-\frac{1}{p}} (\|f\|_{L^q(K)}^\sharp)^{\frac{q}{p}} (\|f\|_{L^q(E)}^\sharp)^{1-\frac{q}{p}}.$$

In particular,

$$(38) \quad \|f\|_{L^p(K)}^\sharp \leq (C_{\mathbf{N}d})^{2\mathbf{N}(1-\frac{q}{p})} (2q\mathbf{N} + 1)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^q(K)}^\sharp$$

and

$$(39) \quad \|f\|_{L^p(K)}^\sharp \leq \left(\frac{C_{\mathbf{N}d}|K|}{|E|} \right)^{2\mathbf{N}} (2q\mathbf{N} + 1)^{\frac{1}{q}} \|f\|_{L^q(E)}^\sharp.$$

Proof. We may assume that $\mathbf{N} \neq 0$; otherwise the statements are trivially true. Note that both (38) and (39) follow from (37): it suffices to specialize to $E = K$ and to use $\|f\|_{L^q(K)}^\sharp \leq \|f\|_{L^p(K)}^\sharp$, respectively.

Let us prove (37). We begin with the case $p = \infty$. Using (35), we get

$$\begin{aligned} \frac{1}{|E|} \int_E |f(x)|^q dx &= \frac{1}{|E|} \int_0^{|E|} ((f|_E)^*(y))^q dy = \int_0^1 ((f|_E)^*(|E|\lambda))^q d\lambda \\ &\geq \left(\frac{|E|}{C_{\mathbf{N}d}|K|} \right)^{2q\mathbf{N}} \int_0^1 (1-\lambda)^{2q\mathbf{N}} d\lambda \|f\|_K^q \\ &= \left(\frac{|E|}{C_{\mathbf{N}d}|K|} \right)^{2q\mathbf{N}} \frac{1}{2q\mathbf{N} + 1} \|f\|_K^q, \end{aligned}$$

whence

$$(40) \quad \|f\|_K \leq \left(\frac{C_{\mathbf{N}d}|K|}{|E|} \right)^{2\mathbf{N}} (2q\mathbf{N} + 1)^{\frac{1}{q}} \|f\|_{L^q(E)}^\sharp$$

which is (37) for $p = \infty$.

If $0 < q < p < \infty$, then (40) implies

$$\begin{aligned} \frac{1}{|K|} \int_K |f(x)|^p dx &\leq \|f\|_K^{p-q} \frac{1}{|K|} \int_K |f(x)|^q dx = \|f\|_K^{p-q} (\|f\|_{L^q(K)}^\sharp)^q \\ &\leq \left(\left(\frac{C_{\mathbf{N}d}|K|}{|E|} \right)^{2\mathbf{N}} (2q\mathbf{N} + 1)^{\frac{1}{q}} \right)^{p-q} (\|f\|_{L^q(K)}^\sharp)^q (\|f\|_{L^q(E)}^\sharp)^{p-q} \end{aligned}$$

from which (37) follows easily. The proof is complete. \square

Remark 5.9. Bourgain proved in [8] the following inequality for polynomials: Let $K \subseteq \mathbb{R}^d$ be a convex body of volume 1 and $p \in \mathbb{R}[x_1, \dots, x_d]$ a polynomial of degree n . Then, for each $q > 0$,

$$\|p\|_{L^q(K)} \leq C(n, q) \|p\|_{L^1(K)}.$$

More precisely,

$$\|p\|_{L^\psi(K)} \leq C_1 \|p\|_{L^1(K)},$$

where L^ψ is the Orlicz space with Orlicz function $\psi(t) = \exp(t^{C_2/n}) - 1$, where the constants C_1 and C_2 are absolute.

These inequalities have been generalized by Brudnyi [10] to analytic functions, where the role of n is played by the *analytic degree*; see Section 6.2.

In contrast to (37), (38), and (39), here the constants do not depend on the dimension d . Following Bourgain's approach, it should be possible to obtain versions of (37), (38), and (39) that are independent of the dimension: use Theorem 5.5 for $d = 1$ and $K = [0, 1]$ to get a replacement for [8, Lemma 3.1].

5.7. A bound for the mean oscillation of $\log |f|$. Let $K \subseteq \mathbb{R}^d$ be a convex body. Recall that the *mean oscillation* of a locally integrable function $g : K \rightarrow \mathbb{R}$ over a ball $B \subseteq K$ is

$$\text{mo}_B(g) := \frac{1}{|B|} \int_B |g(x) - g_B| dx,$$

where

$$g_B := \frac{1}{|B|} \int_B g(x) dx.$$

We have the following bound for the mean oscillation of $\log |f|$ if f has suitably controlled derivatives.

Corollary 5.10. *Let $(\mu^{|\infty}, K)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $(\mu^{2\mathbf{N}}, \|f\|_K)$ -smooth function. Then, for each ball $B \subseteq K$,*

$$(41) \quad \text{mo}_B(\log |f|) \leq 4\mathbf{N} \left(\log \left(\frac{C_{\mathbf{N}d}|K|}{|B|} \right) + 1 \right)$$

where the right-hand side is interpreted as ∞ if $\mathbf{N} = 0$.

Proof. Observe that

$$\text{mo}_B(\log |f|) \leq \frac{2}{|B|} \int_B \left| \log \frac{|f(x)|}{\|f\|_K} \right| dx.$$

Indeed,

$$\begin{aligned} \left| \log |f| - \frac{1}{|B|} \int_B \log |f| dx \right| &\leq \left| \log |f| - \log \|f\|_K \right| + \left| \log \|f\|_K - \frac{1}{|B|} \int_B \log |f| dx \right| \\ &= \left| \log \frac{|f|}{\|f\|_K} \right| + \left| \frac{1}{|B|} \int_B \log \|f\|_K - \log |f| dx \right| \\ &\leq \left| \log \frac{|f|}{\|f\|_K} \right| + \frac{1}{|B|} \int_B \left| \log \frac{|f|}{\|f\|_K} \right| dx. \end{aligned}$$

We may assume that $\mathbf{N} \neq 0$. Now, by (35),

$$\begin{aligned} \frac{1}{|B|} \int_B \left| \log \frac{|f(x)|}{\|f\|_K} \right| dx &= \frac{1}{|B|} \int_0^{|B|} \left| \log \frac{(f|_B)^*(y)}{\|f\|_K} \right| dy \\ &= \int_0^1 \left| \log \frac{(f|_B)^*(|B|\lambda)}{\|f\|_K} \right| d\lambda \\ &\leq 2\mathbf{N} \int_0^1 -\log \left(\frac{|B|}{|K|} \cdot \frac{1-\lambda}{C_{\mathbf{N}d}} \right) d\lambda \\ &= 2\mathbf{N} \left(\log \left(\frac{C_{\mathbf{N}d}|K|}{|B|} \right) + 1 \right), \end{aligned}$$

and (41) follows. \square

Corollary 5.10 has similarity with the log-BMO property of analytic functions [10, Corollary 1.10]. But from (41) it seems not possible to deduce that $\log |f|$ has bounded mean oscillation, since the right-hand side tends to infinity if the radius of B tends to zero. Also using (41) in the case $K = B$ does not help, because then \mathbf{N} becomes eventually zero if B gets small enough.

This is related to the fact that we do not have good enough control away from points in K , where $\|f\|_K$ is attained. Indeed, if $x \in K$ is such that $|f(x)| = \|f\|_K$, then

$$\sup_{\substack{\text{balls } B \subseteq K \\ x \in B}} \text{mo}_B(\log |f|) \leq 4\mathbf{N} (\log(C_{\mathbf{N}}d) + 1)$$

as follows from the next corollary.

Corollary 5.11. *Let $(\mu^{|\infty}, K)$ be admissible data and \mathbf{N} the associated integer. Let $f : K \rightarrow \mathbb{R}$ be any $(\mu^{2\mathbf{N}}, \|f\|_K)$ -smooth function. Then, for each convex body $L \subseteq K$ such that $\|f\|_L = \|f\|_K$,*

$$(42) \quad \text{mo}_L(\log |f|) \leq 4\mathbf{N} (\log(C_{\mathbf{N}}d) + 1)$$

where the right-hand side is interpreted as ∞ if $\mathbf{N} = 0$.

Proof. If $L \subseteq K$ is a convex body such that $\|f\|_L = \|f\|_K$ and E is a measurable subset of L with $|E| > 0$, then we get

$$\|f\|_L \leq \left(\frac{C_{\mathbf{N}} |L|^{1/d}}{|L|^{1/d} - (|L| - |E|)^{1/d}} \right)^{2\mathbf{N}} \|f\|_E$$

from Theorem 5.4 (in analogy to Theorem 5.5). For $E := |\{x \in L : |f(x)| \leq t\}|$ we may conclude

$$|E| \leq C_{\mathbf{N}}d |L| \left(\frac{t}{\|f\|_L} \right)^{\frac{1}{2\mathbf{N}}}, \quad t > 0,$$

(in analogy to Corollary 5.6) and thus

$$(f|_L)^*(|L|\lambda) \geq \|f\|_L \left(\frac{1-\lambda}{C_{\mathbf{N}}d} \right)^{2\mathbf{N}}, \quad \lambda \in (0, 1),$$

(in analogy to Corollary 5.7). Using this estimate in the computations of the proof of Corollary 5.10, yields (42). \square

6. COMPLEMENTARY RESULTS

In this section, we compare the $\mu^{|\infty}$ -degree to the polynomial and the analytic degree, respectively. Moreover, we discuss the existence of non-quasianalytic bump functions and show how the technique from Section 3 can be used to extract information on critical points.

6.1. The $\mu^{|\infty}$ -degree vs. the polynomial degree. What is the relation between the $\mu^{|\infty}$ -degree and the usual degree of a polynomial?

Since most of our results are based on restriction to one-dimensional sections, let us assume that $d = 1$. Let $p \in \mathbb{R}[t]$ be a univariate polynomial of degree n . We want to find upper and lower bounds in n for the $2\mu^{|\infty}$ -degree $\mathfrak{d}_{2\mu^{|\infty}}(1)$ for any suitable full admissible weight $(\mu^{|\infty}, M_0)$ such that $p : [-1, 1] \rightarrow \mathbb{R}$ is $(\mu^{|\infty}, M_0)$ -smooth. (It is natural to take $N = \infty$, since all derivatives of high order of p are identically zero anyway. The factor 2 appears, since $\delta_{[-1,1]} = 2$.)

By Markov's inequality ([17], [18]),

$$(43) \quad \|p^{(k)}\|_{[-1,1]} \leq \frac{n^2(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdot 5 \cdots (2k-1)} \|p\|_{[-1,1]}, \quad 1 \leq k \leq n.$$

Thus a most natural full admissible weight $(\mu^{|\infty}, M_0)$ such that $p : [-1, 1] \rightarrow \mathbb{R}$ is $(\mu^{|\infty}, M_0)$ -smooth is

$$(44) \quad M_0 := \|p\|_{[-1,1]}, \quad \mu_j := n^2, \quad j \geq 1.$$

In fact, $(\mu_j)_j$ has to be increasing so that any admissible choice must satisfy $\mu_j \geq n^2$ for all $j \geq 1$. Making $(\mu_j)_j$ bigger increases also $\mathfrak{d}_{2\mu^{|\infty}}(1)$ and in that way we could make $\mathfrak{d}_{2\mu^{|\infty}}(1)$ as large as we please. It is also natural to take the second parameter equal to 1, since $\frac{b}{M_0} = 1$ for the choice $b = \|p\|_{[-1,1]}$ and $b \mapsto \mathfrak{d}_{2\mu^{|\infty}}(b)$ is decreasing.

For the choice (44) we have

$$\Sigma_{\mu^{|\infty}}(1, m) = \sum_{j=1}^m \frac{1}{\mu_j} = \frac{m}{n^2}$$

and, consequently,

$$\mathfrak{d}_{\mu^{|\infty}}(1) = \lfloor en^2 \rfloor.$$

Note that in this case

$$(45) \quad 2\mathfrak{d}_{\mu^{|\infty}}(1) \leq \mathfrak{d}_{2\mu^{|\infty}}(1) \leq 2\mathfrak{d}_{\mu^{|\infty}}(1) + 1;$$

the first inequality is always true, because $(\mu_j)_j$ is increasing.

We get a better result for complex polynomials. Let $p \in \mathbb{C}[z]$ be a univariate polynomial of degree n . By Bernstein's inequality ([3], [28]),

$$\|p^{(k)}\|_{\overline{D}_1} \leq \frac{n!}{(n-k)!} \|p\|_{\overline{D}_1}, \quad k \geq 1,$$

where $\overline{D}_1 := \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disk. Thus $p|_{\overline{D}_1}$ is $(\mu^{|\infty}, M_0)$ -smooth for

$$M_0 := \|p\|_{\overline{D}_1}, \quad \mu_j = Cn, \quad j \geq 1,$$

where the constant $C > 0$ accounts for the conversion of the bounds for complex derivatives to real partial derivatives. In this case, we find

$$\mathfrak{d}_{\mu^{|\infty}}(1) = \lfloor Cen \rfloor,$$

and (45) is still valid.

Remark 6.1. Strictly speaking, a $(\mu^{|\infty}, M_0)$ -smooth function is real valued by definition (see Section 2.2). Its definition clearly makes sense for complex valued functions as well, but crucial results of the paper are based on Rolle's theorem.

6.2. The $\mu^{|\infty}$ -degree vs. the analytic degree. The *analytic degree* $d_f(2\epsilon)$ of a holomorphic function f on the disk $D_{1+2\epsilon} := \{z \in \mathbb{C} : |z| < 1 + 2\epsilon\}$, for some $\epsilon > 0$, is defined in [10] as the best constant d in the inequality

$$\|f\|_I \leq \left(\frac{4|I|}{|E|} \right)^d \|f\|_E,$$

where I is any interval in the intersection of a real affine line in \mathbb{C} with the unit disk D_1 and $E \subseteq I$ is any measurable subset with $|E| > 0$.

By the Cauchy estimates,

$$\|f^{(k)}\|_{\overline{D}_1} \leq \frac{k! \|f\|_{D_{1+\epsilon}}}{\epsilon^{k+1}}, \quad k \geq 1.$$

So f on \overline{D}_1 is $(\mu^{|\infty}, M_0)$ -smooth for

$$M_0 := \frac{\|f\|_{D_{1+\epsilon}}}{\epsilon}, \quad \mu_j := \frac{Cj}{\epsilon}, \quad j \geq 1,$$

where the constant $C > 0$ accounts for the conversion of the bounds for complex derivatives to real partial derivatives. For $\tilde{\mu}(s) = s$ we have $\bar{\gamma}_{\tilde{\mu}} = 1$ and $\bar{\Gamma}_{\tilde{\mu}} = 4e^5$; see Remark 5.3. By Theorem 5.1, we find (as in the proof of Theorem 5.4)

$$\|f\|_I \leq \left(\frac{4e^5 |I|}{|E|} \right)^{2\mathfrak{d}_{2\mu^{|\infty}}(\epsilon)} \|f\|_E.$$

Since $(4e^5)^{2\mathfrak{d}_{2\mu^{|\infty}}(\epsilon)} \leq 4^{10\mathfrak{d}_{2\mu^{|\infty}}(\epsilon)}$, we conclude that

$$d_f(2\epsilon) \leq 10\mathfrak{d}_{2\mu^{|\infty}}(\epsilon).$$

We do not know if there is a lower bound for $d_f(2\epsilon)$ in terms of $\mathfrak{d}_{2\mu^{|\infty}}(\epsilon)$.

Remark 6.2. For $\mu^{|\infty} = (j)_{j \geq 1}$, $\Sigma_{\mu^{|\infty}}(1, n)$ is the partial harmonic sum $\sum_{j=1}^n \frac{1}{j} =: H_n$, and, more generally, $\Sigma_{\mu^{|\infty}}(m, n) = H_n - H_{m-1}$ if $1 < m \leq n$. Let $x \geq 1$ and define the positive integer $n(x)$ by

$$(46) \quad H_{n(x)} \leq x < H_{n(x)+1}.$$

Comtet [12] (see also Boas and Wrench [4]) showed that, for $x \geq 2$,

$$\left\lfloor e^{x-\gamma} - \frac{1}{2} - \frac{3}{2} \frac{1}{e^{x-1} - 1} \right\rfloor \leq n(x) \leq \left\lfloor e^{x-\gamma} - \frac{1}{2} + \frac{1}{12} \frac{1}{e^{x-1} - 1} \right\rfloor,$$

where γ is the Euler–Mascheroni constant, which determines one or two possible values of $n(x)$ for any $x \geq 2$. With this formula it is not difficult to find explicit estimates of $\mathfrak{d}_{a\mu^{|\infty}}(b)$, for $a, b > 0$. Note that the possible equality $H_{n(x)} = x$ in (46), in contrast to the strict inequality in the definition of $\mathfrak{d}_{a\mu^{|\infty}}(1)$, might effect a deviation of at most 1 between $n(ae)$ and $\mathfrak{d}_{a\mu^{|\infty}}(1)$.

6.3. Conditions for non-quasianalytic bump functions. Let $M^{|\infty}$ be a non-quasianalytic full admissible weight. We may infer from Theorem 4.2 a quantitative necessary condition for the existence of $M^{|\infty}$ -smooth functions f with compact support contained in the interior K° of the convex body K . Of course, this is most informative if K has equal width in all directions, e.g., if K is a ball.

Corollary 6.3. *Let $M^{|\infty}$ be a non-quasianalytic full admissible weight, $K \subseteq \mathbb{R}^d$ a convex body, and $b > 0$. Suppose that $f : K \rightarrow \mathbb{R}$ is an $M^{|\infty}$ -smooth function compactly supported in K° and $\|f\|_K \geq b$. Then*

$$\Sigma_{\mu^{|\infty}}(j_0 + 1, \infty) \leq \delta_K e, \quad (j_0 = \lceil \log(\frac{b}{2M_0})^{-1} \rceil_{\mathbb{N}}).$$

Proof. Theorem 4.2 shows that (19) must be violated. \square

Remark 6.4. For completeness, we sketch a proof of the following fact: *Let Z be any non-empty closed subset of \mathbb{R}^d and $M^{|\infty} = (M_j)_{j \geq 0}$ a non-quasianalytic full admissible weight. There exist $f \in C^\infty(\mathbb{R}^d)$ and $A > 0$ such that $\|f^{(\alpha)}\|_{\mathbb{R}^d} \leq A^{|\alpha|+1} M_{|\alpha|}$, for all $\alpha \in \mathbb{N}^d$, and $Z_f = Z \subseteq Z_{f^{(\alpha)}}$ for all $\alpha \neq 0$.*

By [25, Lemma 2.3 and Corollary 3.5], there exists a non-quasianalytic full admissible weight $L^{|\infty} = (L_k)_{k \geq 0}$ such that

$$(47) \quad \left(\frac{M_j}{L_j} \right)^{1/j} \rightarrow \infty.$$

By [13, Theorem 1.4.2] or [25, Proposition 3.11], there is a C^∞ -function $0 \leq \varphi \leq 1$ with $\varphi(0) = 1$ and support contained in the unit ball B_1 and $A > 0$ such that $\|\varphi^{(\alpha)}\|_{\mathbb{R}^d} \leq A^{|\alpha|+1} L_{|\alpha|}$ for all α . For every $x \in \mathbb{R}^d \setminus Z$, let $d(x) := \frac{1}{2} \inf_{z \in Z} |x - z|$ and $\varphi_x(y) := \varphi(\frac{y-x}{d(x)})$. Then $\varphi_x(x) = 1$, $U_x := \{y : \varphi_x(y) \neq 0\} \subseteq \mathbb{R}^d \setminus Z$, and

$$\|\varphi_x^{(\alpha)}\|_{\mathbb{R}^d} = \frac{1}{d(x)^{|\alpha|}} \|\varphi^{(\alpha)}\|_{\mathbb{R}^d} \leq \frac{A^{|\alpha|+1}}{d(x)^{|\alpha|}} L_{|\alpha|}.$$

The family $\{U_x\}$ forms an open cover of $\mathbb{R}^d \setminus Z$ which admits a countable subcover $\{U_n := U_{x_n}\}$. Let $\varphi_n := \varphi_{x_n}$, $d_n := 1/d(x_n)$, and choose constants $s_n > 0$ such that

$$d_n^j s_n \leq \frac{M_j}{L_j} \quad \text{for all } n, j \geq 1.$$

This is possible by (47), since for each d_n we find j_n such that $d_n \leq (\frac{M_j}{L_j})^{1/j}$ for all $j > j_n$; so we may take

$$s_n := \min \left\{ \min_{j \leq j_n} \frac{M_j}{d_n^j L_j}, 1 \right\}.$$

Then $f := \sum_{n \geq 1} \frac{s_n}{2^n} \varphi_n$ converges uniformly in all derivatives,

$$\left| \sum_{n \geq 1} \frac{s_n}{2^n} \varphi_n^{(\alpha)}(x) \right| \leq \sum_{n \geq 1} \frac{s_n}{2^n} \cdot A^{|\alpha|+1} d_n^{|\alpha|} L_{|\alpha|} \leq A^{|\alpha|+1} M_{|\alpha|},$$

so that f defines a C^∞ -function on \mathbb{R}^d with $f^{(\alpha)} = \sum_{n \geq 1} \frac{s_n}{2^n} \varphi_n^{(\alpha)}$ and $\|f^{(\alpha)}\|_{\mathbb{R}^d} \leq A^{|\alpha|+1} M_{|\alpha|}$ for all α . Since the U_n cover $\mathbb{R}^d \setminus Z$, f is strictly positive on $\mathbb{R}^d \setminus Z$ and vanishes, together with all partial derivatives $f^{(\alpha)}$, on Z .

6.4. Critical points. Let $M^{|\infty}$ be a full admissible weight and $f : K \rightarrow \mathbb{R}$ an $M^{|\infty}$ -smooth function, where $K \subseteq \mathbb{R}^d$ is a convex body. In order to get quantitative information on the critical points of f , one can consider the function $g := |\nabla f|^2 = \sum_{j=1}^d (\partial_j f)^2$ and apply the results of Section 4 to g . Note that g is $\hat{M}^{|\infty}$ -smooth for a full admissible weight $\hat{M}^{|\infty}$ which can be computed from $M^{|\infty}$ in view of the Faà di Bruno formula.

But the one-dimensional analysis from Section 3 allows us to extract information in a more direct way.

Proposition 6.5. *Let $M^{|\infty} = (M_j)_{j \geq 0}$ be a full admissible weight, $K \subseteq \mathbb{R}^d$ a convex body, and $b > 0$ such that*

$$(48) \quad \Sigma_{\mu^{|\infty}}(j_0 + 1, \infty) > 2\delta_K e, \quad (j_0 = \lceil \log(\frac{b}{2M_0})^{-1} \rceil_{\mathbb{N}}).$$

Then $\mathbf{N} := \mathfrak{d}_{2\delta_K \mu^{|\infty}}(\frac{b}{2M_0})$ is a nonnegative integer. Let $f : K \rightarrow \mathbb{R}$ be any $M^{|\mathbf{N}+1}$ -smooth function such that $\|f\|_K \geq b$. Let ℓ be any affine line that meets B (i.e. the ball contained in $\{x \in K : |f(x)| > \frac{b}{2}\}$ from Theorem 4.2) such that either $\ell \cap Z_f \neq \emptyset$ or $V(f|_\ell) \geq 2b$. Then f has at most $2\mathbf{N}$ critical points on ℓ . Thus at all but possibly $2\mathbf{N}$ points of ℓ the level sets of f are $C^{\mathbf{N}+1}$ -submanifolds.

Notice the factor 2 in (48) and in the definition of \mathbf{N} .

Proof. Let ℓ be an affine line that meets B . If no zero of f lies on ℓ , we may assume without loss of generality that f is positive on ℓ and that $c := \min_{\ell \cap K} f > 0$. The

assumption $V(f|_\ell) \geq 2b$ guarantees that we can replace f by $f - c$; clearly the set of critical points of f and $f - c$ is the same. Indeed, cf. Remark 4.11,

$$b \leq \frac{V(f|_\ell)}{2} \leq \|f - c\|_{\ell \cap K} = \max_{\ell \cap K} f - c \leq M_0 - c \leq M_0,$$

and $\|f - c\|_{j,K} = \|f\|_{j,K}$ for all $j \geq 1$. Since $f - c$ has a zero on ℓ , we may assume from now on that f has a zero on ℓ .

Now we restrict f to ℓ and work with the $(\delta_K \mu^{|\mathbf{N}+1}, M_0)$ -function $g : [0, 1] \rightarrow \mathbb{R}$ as in the proof of Theorem 4.2. Since ℓ meets B , we have $\|g\|_{[0,1]} > \frac{b}{2}$. And g has at least one zero s_0 in $[0, 1]$, because f vanishes on ℓ . As in Theorem 4.2, we see that the number of critical points of g left and right of s_0 is finite and bounded by \mathbf{N} . Since each critical point of f on ℓ is a critical point of g , the proof is complete.

Let us explain in more detail why we need the factor 2 in the definition of \mathbf{N} . To this end, assume that $t_1 \leq t_2 \leq \dots \leq t_{m_r}$ is an increasing enumeration of the critical points of g that lie to the right of s_0 . By Rolle's theorem, we find $t_1 = s_1 \leq s_2 \leq \dots \leq s_{m_r-1}$ such that $g^{(j)}(s_j) = 0$ for $0 \leq j \leq m_r - 1$. Choose $s_{-1} \in [0, 1]$ such that $|g(s_{-1})| > \frac{b}{2}$. If $s_{-1} < s_0$, we find, as in the proof of Corollary 3.4,

$$1 \geq \sum_{j=0}^{m_r-1} |s_j - s_{j-1}| > \frac{1}{e} \Sigma_{\delta_K \mu^{|\infty}}(j_0 + 1, m_r),$$

whence $m_r \leq \mathfrak{d}_{\delta_K \mu^{|\infty}}(\frac{b}{2M_0})$. If $s_{-1} > s_0$, then

$$\sum_{j=0}^{m_r-1} |s_j - s_{j-1}| = (s_{-1} - s_0) + \sum_{j=1}^{m_r-1} (s_j - s_{j-1}) \leq 2$$

so that we may conclude that $m_r \leq \mathfrak{d}_{2\delta_K \mu^{|\infty}}(\frac{b}{2M_0})$. In any case $m_r \leq \mathbf{N}$. \square

Remark 6.6. If M has additional regularity properties so that the implicit function theorem holds in the associated Denjoy–Carleman class, then the submanifolds are of the respective Denjoy–Carleman class; see e.g. [26].

A comprehensive quantitative study of the critical and near-critical values of differentiable mappings can be found in [30].

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