INVARIANT FUNCTIONS IN DENJOY–CARLEMAN CLASSES

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Abstract. Let V be a real finite dimensional representation of a compact Lie group G. It is well-known that the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is finitely generated, say by $\sigma_1, \ldots, \sigma_p$. Schwarz [38] proved that each G-invariant C^∞-function f on V has the form $f = F(\sigma_1, \ldots, \sigma_p)$ for a C^∞-function F on $\mathbb{R}^p$. We investigate this representation within the framework of Denjoy–Carleman classes. One can in general not expect that f and F lie in the same Denjoy–Carleman class C^M (with $M = (M_k)$). For finite groups G and (more generally) for polar representations V we show that for each G-invariant f of class C^M there is an F of class C^N such that $f = F(\sigma_1, \ldots, \sigma_p)$, if N is strongly regular and satisfies

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty,$$

where m is an (explicitly known) integer depending only on the representation. In particular, each G-invariant $(1+\delta)$-Gevrey function f (with $\delta > 0$) has the form $f = F(\sigma_1, \ldots, \sigma_p)$ for a $(1+\delta m)$-Gevrey function F. Applications to equivariant functions and basic differential forms are given.

1. Introduction

Let V be a real finite dimensional representation of a compact Lie group G. By a classical theorem due to Hilbert the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is finitely generated. Choose a system of homogeneous generators $\sigma_1, \ldots, \sigma_p$ of $\mathbb{R}[V]^G$ and define $\sigma := (\sigma_1, \ldots, \sigma_p) : V \to \mathbb{R}^p$. Schwarz [38] proved a smooth analog of Hilbert's theorem for orthogonal representations V of compact Lie groups G: the induced mapping $\sigma^* : C^\infty(\mathbb{R}^p) \to C^\infty(V)^G$ is surjective. Mather [27] showed that this mapping is even split surjective.

The finitely differentiable case was studied, too: $\sigma^* : C^n(\mathbb{R}^p) \to C^n(V)^G$ is in general not surjective, but $\sigma^* C^n(\mathbb{R}^p)$ contains $C^{nq}(V)^G$ for a suitable integer q. See [1], [3], [2], [37].

In this paper we treat Schwarz's theorem in the framework of Denjoy–Carleman classes. These classes of smooth functions play an important role in harmonic analysis and various branches of differential equations (especially Gevrey classes). Let $M = (M_k)_{k\in\mathbb{N}}$ be a non-decreasing sequence of real numbers with $M_0 = 1$. A smooth function f in an open subset $U \subseteq \mathbb{R}^n$ belongs to the Denjoy–Carleman class $C^M(U)$ if for any compact subset $K \subseteq U$ there exist positive constants C and ϱ such that

$$|\partial^\alpha f(x)| \le C \varrho^{|\alpha|} |\alpha|! M_{|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$ and $x \in K$. See section 2 for more on Denjoy–Carleman classes. As examples ([8], see also 3.3) show, one cannot expect in general that a smooth

Date: May 8, 2008.
2000 *Mathematics Subject Classification.* 26E10, 58C25, 57S15.
Key words and phrases. Denjoy–Carleman class, invariant functions.
The author was supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P19392 & Projekt J2771'.

G-invariant function f on V of class C^M has the form $f = F \circ \sigma$ for a function F of the same class C^M.

For finite groups G and (more generally) for polar representations V we prove that the representation $f = F \circ \sigma$ holds in the context of Denjoy–Carleman classes, where F has lower regularity than f. More precisely: Let G be a subgroup of finite order m of $\mathrm{GL}(V)$. Let M and N be sequences satisfying some mild conditions which guarantee stability under composition and derivation for C^M and C^N (see 2.1). Assume that N is strongly regular (see 2.6) and that

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then for any G-invariant function $f \in C^M(V)$ there exists a function $F \in C^N(\mathbb{R}^p)$ such that $f = F \circ \sigma$. In particular: Any G-invariant Gevrey function $f \in G^{1+\delta}(V)$ (with $\delta > 0$) has the form $f = F \circ \sigma$ for a Gevrey function $F \in G^{1+\delta m}(\mathbb{R}^p)$. See theorem 3.4. The result does not depend on the choice of generators σ_i, since any two choices differ only by a polynomial diffeomorphism and the involved Denjoy–Carleman classes are stable under composition.

Note that Thilliez [40] treats a very similar problem: For a compact subset $E \subseteq \mathbb{R}^n$, an analytic mapping $\Phi : U \to \mathbb{R}^n$ on an open neighborhood U of E, and a function $f \in C^M(U)$ of the form $f = g \circ \Phi$ with $g \in C^\infty(W)$ for an open neighborhood W of $\Phi(E)$, the existence of a sequence N such that $g \in C^N(W)$ is investigated. This is done by studying the complex setting: Now E is compact in $\mathbb{C}^n$, Φ is a $\mathbb{C}^n$-valued holomorphic mapping defined near E, and g is C^∞ on $\mathbb{C}^n$ and $\bar\partial$-flat on $\Phi(E)$. However, our results are not covered by Thilliez', since the minimal number of generators of $\mathbb{R}[V]^G$ does in general not coincide with the dimension of the representation space V.

We prove the main theorem in section 3. We shall deduce it from an analog theorem (see 3.3) due to Bronshtein [7, 8] which treats the standard representation of the symmetric group S_n in $\mathbb{R}^n$. This method is inspired by Barbançon and Raïs [3] deploying Weyl's account [43] of Noether's [30] proof of Hilbert's theorem.

The rest of the paper is devoted to several applications of this main theorem. In section 4 we treat the presentation in Denjoy–Carleman classes of equivariant functions between representations of a finite group.

In section 5 the main theorem 3.4 is generalized to polar representations, i.e., orthogonal finite dimensional representations V of a compact Lie group G allowing a linear subspace $\Sigma \subseteq V$ which meets each orbit orthogonally (see theorem 5.2). The trace of the G-action in Σ is the action of the generalized Weyl group W which is a finite group. In analogy with a result due to Palais and Terng [32], which states that restriction induces an isomorphism $I_1 : C^\infty(V)^G \cong C^\infty(\Sigma)^W$, we show that each W-invariant function on Σ of class C^M has a G-invariant extension to V of class C^N, where M and N are sequences with the aforementioned properties. More generally, Michor [28, 29] proved that restriction induces an isomorphism $I_2 : \Omega^p_{\mathrm{hor}}(V)^G \cong \Omega^p(\Sigma)^W$, where $\Omega^p_{\mathrm{hor}}(V)^G$ is the space of basic p-forms on V, i.e., G-invariant forms that kill each vector tangent to some orbit. Our main theorem 3.4 allows to conclude that each W-invariant p-form on Σ of class C^M has a basic extension to V of class C^N (with M and N as above).

In [32] and [28, 29] the isomorphisms I_1 and I_2 are established in the more general setting of smooth proper Riemannian G-manifolds X with sections, where there exist closed submanifolds $\Sigma \subseteq X$ meeting each orbit orthogonally. In section 6 we explain that our analog results in the framework of Denjoy–Carleman classes generalize to real analytic proper Riemannian G-manifolds X with sections.

2. Denjoy–Carleman classes

2.1. **Denjoy–Carleman classes of differentiable functions.** We mainly follow [42] (see also the references therein). We use $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$.

Let $M = (M_k)_{k\in\mathbb{N}}$ be an increasing sequence ($M_{k+1} \ge M_k$) of real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ϱ such that

$$|\partial^\alpha f(x)| \le C\, \varrho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|} \tag{2.1.1}$$

for all $\alpha \in \mathbb{N}^n$ and $x \in K$. The set $C^M(U)$ is the *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$.

We assume that $M = (M_k)$ is *logarithmically convex*, i.e.,

$$M_k^2 \le M_{k-1}\, M_{k+1} \quad \text{for all } k, \tag{2.1.2}$$

or, equivalently, M_{k+1}/M_k is increasing. Considering $M_0 = 1$, we obtain that also $(M_k)^{1/k}$ is increasing and

$$M_l\, M_k \le M_{l+k} \quad \text{for all } l, k \in \mathbb{N}. \tag{2.1.3}$$

Hypothesis (2.1.2) implies that $C^M(U)$ is a ring, for all open subsets $U \subseteq \mathbb{R}^n$, which can easily be derived from (2.1.3) by means of Leibniz's rule. Note that definition (2.1.1) makes sense also for functions $U \to \mathbb{R}^p$. For C^M-mappings, (2.1.2) guarantees stability under composition ([35], see also [4, 4.7]).

A further consequence of (2.1.2) is the inverse function theorem for C^M ([22]; for a proof see also [4, 4.10]): Let $f : U \to V$ be a C^M-mapping between open subsets $U, V \subseteq \mathbb{R}^n$. Let $x_0 \in U$. Suppose that the Jacobian matrix $(\partial f/\partial x)(x_0)$ is invertible. Then there are neighborhoods U' of x_0, V' of $y_0 := f(x_0)$ such that $f : U' \to V'$ is a C^M-diffeomorphism.

Moreover, (2.1.2) implies that C^M is closed under solving ODEs (due to [23]).

Suppose that $M = (M_k)$ and $N = (N_k)$ satisfy $M_k \le C^k\, N_k$, for all k and a constant C, or equivalently,

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} < \infty. \tag{2.1.4}$$

Then, evidently $C^M(U) \subseteq C^N(U)$. The converse is true as well (if (2.1.2) is assumed): One can prove that there exists $f \in C^M(\mathbb{R})$ such that $|f^{(k)}(0)| \ge k!\, M_k$ for all k (see [42, Theorem 1]). So the inclusion $C^M(U) \subseteq C^N(U)$ implies (2.1.4).

Setting $N_k = 1$ in (2.1.4) yields that $C^\omega(U) = C^M(U)$ if and only if

$$\sup_{k\in\mathbb{N}_{>0}} (M_k)^{\frac{1}{k}} < \infty.$$

Since $(M_k)^{1/k}$ is increasing (by logarithmic convexity), the strict inclusion $C^\omega(U) \subsetneq C^M(U)$ is equivalent to

$$\lim_{k\to\infty} (M_k)^{\frac{1}{k}} = \infty.$$

We shall also assume that C^M is stable under derivation, which is equivalent to the following condition

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}} < \infty. \tag{2.1.5}$$

Note that the first order partial derivatives of elements in $C^M(U)$ belong to $C^{M^{+1}}(U)$, where M^{+1} denotes the shifted sequence $M^{+1} = (M_{k+1})_{k\in\mathbb{N}}$. So the equivalence follows from (2.1.4), by replacing M with M^{+1} and N with M.

Definition. By a *DC-weight sequence* we mean a sequence $M = (M_k)_{k\in\mathbb{N}}$ of positive numbers with $M_0 = 1$ which is monotone increasing ($M_{k+1} \geq M_k$), logarithmically convex (2.1.2), and satisfies (2.1.5). Then $C^M(U,\mathbb{R})$ is a differential ring, and the class of C^M-functions is stable under compositions, as above.

2.2. **Quasianalytic function classes.** Let $\mathcal{F}_n$ denote the ring of formal power series in n variables (with real or complex coefficients). We denote by $\mathcal{F}_n^M$ the set of elements $F = \sum_{\alpha\in\mathbb{N}^n} F_\alpha\, x^\alpha$ of $\mathcal{F}_n$ for which there exist positive constants C and ϱ such that

$$|F_\alpha| \leq C\, \varrho^{|\alpha|}\, M_{|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$. A class C^M is called *quasianalytic* if, for open connected $U \subseteq \mathbb{R}^n$ and all $a \in U$, the Taylor series homomorphism

$$T_a : C^M(U) \to \mathcal{F}_n^M,\; f \mapsto T_a f(x) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, \partial^\alpha f(a)\, x^\alpha$$

is injective. By the Denjoy–Carleman theorem ([14], [10]), C^M is quasianalytic if and only if

$$\sum_{k=0}^{\infty} \frac{M_k}{(k+1)\, M_{k+1}} = \infty, \quad \text{or, equivalently,} \quad \sum_{k=1}^{\infty} \Big(\frac{1}{k!\, M_k}\Big)^{\frac{1}{k}} = \infty. \tag{2.2.1}$$

For contemporary proofs see for instance [19, 1.3.8] or [36, 19.11].

Suppose that $C^\omega(U) \subsetneq C^M(U)$ and $C^M(U)$ is quasianalytic. Then $T_a : C^M(U) \to \mathcal{F}_n^M$ is not surjective. This is due to Carleman [10]; an elementary proof can be found in [42, Theorem 3].

2.3. **Non-quasianalytic function classes.** If M is a DC-weight sequence which is not quasianalytic, then there are C^M partitions of unity. Namely, there exists a C^M function f on $\mathbb{R}$ which does not vanish in any neighborhood of 0 but which has vanishing Taylor series at 0. Let $g(t) = 0$ for $t \leq 0$ and $g(t) = f(t)$ for $t > 0$. From g we can construct C^M bump functions as usual.

2.4. **Strong non-quasianalytic function classes.** Let M be a DC-weight sequence with $C^\omega(U,\mathbb{R}) \subsetneq C^M(U,\mathbb{R})$. Then the mapping $T_a : C^M(U,\mathbb{R}) \to \mathcal{F}_n^M$ is surjective, for all $a \in U$, if and only if there is a constant C such that

$$\sum_{k=j}^{\infty} \frac{M_k}{(k+1)\, M_{k+1}} \leq C \frac{M_j}{M_{j+1}} \quad \text{for any integer } j \geq 0. \tag{2.4.1}$$

See [34] and references therein. (2.4.1) is called *strong non-quasianalyticity* condition.

2.5. **Moderate growth.** A DC-weight sequence M has *moderate growth* if

$$\sup_{j,k\in\mathbb{N}_{>0}} \Big(\frac{M_{j+k}}{M_j\, M_k}\Big)^{\frac{1}{j+k}} < \infty. \tag{2.5.1}$$

2.6. **Strong regularity.** Moderate growth (2.5.1) together with strong non-quasianalyticity (2.4.1) is called *strong regularity*: Then a version of Whitney's extension theorem holds for the corresponding function classes.

2.7. **Whitney's extension theorem.** Let $K \subseteq \mathbb{R}^n$ be compact. Denote by $J^\infty(K)$ the C^∞ Whitney jets on K. We say that $F = (F_\alpha)_{\alpha \in \mathbb{N}^n} \in J^\infty(K)$ is a C^M*-jet* on K, or belongs to $J^M(K)$, if there exist positive constants C and ϱ such that

$$|F_\alpha(x)| \le C \varrho^{|\alpha|} |\alpha|!\, M_{|\alpha|} \tag{2.7.1}$$

for all $\alpha \in \mathbb{N}^n$ and $x \in K$ and

$$|F_\beta(x) - \partial^\beta T_a^p F(x)| \le C \varrho^p |\beta|!\, M_{p+1} |x-a|^{p+1-|\beta|} \tag{2.7.2}$$

for all $p \in \mathbb{N}$, all $\beta \in \mathbb{N}^n$ with $|\beta| \le p$ and all $x \in K$, where

$$T_a^p F(x) = \sum_{|\beta| \le p} \frac{1}{\beta!} F_\beta(a)(x-a)^\beta.$$

If M is strongly regular then a version of Whitney's extension theorem holds (see [9], [5], and [11]): the mapping $J_K : C^M(\mathbb{R}^n) \to J^M(K), f \mapsto (\partial^\alpha f|_K)_{\alpha \in \mathbb{N}^n}$ is surjective.

Note that, if $f \in C^\infty(\mathbb{R}^n)$ such that $F = J_K f$ satisfies (2.7.1) and if K is Whitney 1-regular, then (2.7.2) is automatically fulfilled (see [5, 3.12]). Recall that K is *Whitney* 1*-regular* if any two points x and y in K can be connected by a path in K of length $\le C|x-y|$, where the constant C depends only on K.

2.8. **Gevrey functions.** Let $\delta > 0$ and put $M_k = (k!)^\delta$, for $k \in \mathbb{N}$. Then $M = (M_k)$ is strongly regular. The corresponding class C^M of functions is the *Gevrey class* $G^{1+\delta}$.

2.9. **More examples.** Let $\delta > 0$ and put $M_k = (\log(k+e))^{\delta k}$, for $k \in \mathbb{N}$. Then $M = (M_k)$ is quasianalytic for $0 < \delta \le 1$ and non-quasianalytic (but not strongly) for $\delta > 1$.

Let $q > 1$ and put $M_k = q^{k^2}$, for $k \in \mathbb{N}$. The corresponding C^M-functions are called *q-Gevrey regular*. Then $M = (M_k)$ is strongly non-quasianalytic but not of moderate growth, thus not strongly regular.

2.10. **Spaces of C^M-functions.** Let $U \subseteq \mathbb{R}^n$ be open. For any $\varrho > 0$ and $K \subseteq U$ compact with smooth boundary, define

$$C^M_\varrho(K) := \{f \in C^\infty(K) : \|f\|_{\varrho,K} < \infty\}$$

with

$$\|f\|_{\varrho,K} := \sup\Big\{\frac{|\partial^\alpha f(x)|}{\varrho^{|\alpha|}\,|\alpha|!\, M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K\Big\}.$$

It is easy to see that $C^M_\varrho(K)$ is a Banach space. In the description of $C^M_\varrho(K)$, instead of compact K with smooth boundary, we may also use open $K \subset U$ with $\overline{K}$ compact in U, like [42]. Or we may work with Whitney jets on compact K, like [21].

The space $C^M(U)$ carries the projective limit topology over compact $K \subseteq U$ of the inductive limit over $\varrho \in \mathbb{N}_{>0}$:

$$C^M(U) = \varprojlim_{K \subseteq U} \big(\varinjlim_{\varrho \in \mathbb{N}_{>0}} C^M_\varrho(K) \big).$$

One can prove that, for $\varrho < \varrho'$, the canonical injection $C^M_\varrho(K) \to C^M_{\varrho'}(K)$ is a compact mapping (see [21]). Hence $\varinjlim_\varrho C^M_\varrho(K)$ is a Silva space, i.e., an inductive limit of Banach spaces such that the canonical mappings are compact.

2.11. **Polynomials are dense in $C^M(U)$.** Let M be a DC-weight sequence and let $U \subseteq \mathbb{R}^n$ be open. It is proved in [20, 3.2] (see also [17, 3.2]) that the space of entire functions $\mathcal{H}(\mathbb{C}^n)$ is dense in $C^M(U)$. Since the polynomials are dense in $\mathcal{H}(\mathbb{C}^n)$ and the inclusion $\mathcal{H}(\mathbb{C}^n) \to C^M(U)$ is continuous, we obtain that the polynomials are dense in $C^M(U)$. For convenience we give a proof.

Lemma. *Let M be a DC-weight sequence and let $U \subseteq \mathbb{R}^n$ be open. Then $\mathcal{H}(\mathbb{C}^n)$ is dense in $C^M(U)$.*

Proof. Let $f \in C^M(U)$ and $K \subseteq U$ compact. Let $0 < c < 1$ such that $Q = K + B_c(0) \subseteq U$, where $B_c(0) = \{x \in \mathbb{R}^n : |x| \le c\}$. Let $\chi \in C^\infty(U)$ with $0 \le \chi \le 1$, $\chi|_Q = 1$, and compact support $Q_1 = \operatorname{supp}(\chi) \subseteq U$. We define for $j \in \mathbb{N}_{>0}$

$$f_j := E_j * \chi f \in \mathcal{H}(\mathbb{C}^n), \quad \text{where} \quad E_j : \mathbb{C}^n \to \mathbb{C},\ z \mapsto \big(\tfrac{j}{\pi}\big)^{\frac{n}{2}} e^{-j\langle z|z\rangle}.$$

Induction shows

$$\partial_{i_N} \cdots \partial_{i_1}(E_j * \chi f) = E_j * (\chi \partial_{i_N} \cdots \partial_{i_1} f) + \sum_{\nu=1}^{N} (\partial_{i_N} \cdots \partial_{i_{\nu+1}} E_j) * (\partial_{i_\nu} \chi)(\partial_{i_{\nu-1}} \cdots \partial_{i_1} f),$$

for all $N \in \mathbb{N}$ and $j \in \mathbb{N}_{>0}$, and hence

$$\begin{aligned} |\partial_{i_N} \cdots \partial_{i_1}(f - f_j)| &\le |\partial_{i_N} \cdots \partial_{i_1} f - E_j * (\chi \partial_{i_N} \cdots \partial_{i_1} f)| \\ &\quad + \sum_{\nu=1}^{N} |(\partial_{i_N} \cdots \partial_{i_{\nu+1}} E_j) * (\partial_{i_\nu} \chi)(\partial_{i_{\nu-1}} \cdots \partial_{i_1} f)|. \end{aligned} \tag{2.11.1}$$

We have for $x \in K$ and $\alpha \in \mathbb{N}^n$

$$\begin{aligned} |E_j * (\chi \partial^\alpha f)(x) - \partial^\alpha f(x)| &= \Big| \int E_j(y)\big(\chi(x-y)\partial^\alpha f(x-y) - \partial^\alpha f(x)\big)dy \Big| \\ &\le \int_{B_c(0)} E_j(y)|\partial^\alpha f(x-y) - \partial^\alpha f(x)|dy \\ &\quad + \int_{\mathbb{R}^n \setminus B_c(0)} E_j(y)\big(\chi(x-y)|\partial^\alpha f(x-y)| + |\partial^\alpha f(x)|\big)dy. \end{aligned}$$

By the generalized mean value theorem we have for $x \in K$, $y \in B_c(0)$, and $\alpha \in \mathbb{N}^n$

$$|\partial^\alpha f(x-y) - \partial^\alpha f(x)| \le \sqrt{n}\,|y| \sup_{\substack{1 \le i \le n \\ 0 \le t \le 1}} |\partial^{\alpha + e_i} f(x - ty)|.$$

Choose $\varrho_1 > 0$ such that $\|f\|_{\varrho_1, Q_1} < \infty$. Then for $x \in K$, $y \in B_c(0)$, and $\alpha \in \mathbb{N}^n$

$$\begin{aligned} |\partial^\alpha f(x-y) - \partial^\alpha f(x)| &\le \sqrt{n}\,|y| \|f\|_{\varrho_1, Q_1} \varrho_1^{|\alpha|+1} (|\alpha|+1)!\, M_{|\alpha|+1} \\ &\le 2\sqrt{n}\,|y| \varrho_1 \|f\|_{\varrho_1, Q_1} (2\varrho_1 C)^{|\alpha|} |\alpha|!\, M_{|\alpha|} \quad \text{(by (2.1.5))}, \end{aligned}$$

where C is a positive constant. For all $j \in \mathbb{N}_{>0}$ we have

$$\int_{\mathbb{R}^n} |y| E_j(y)dy \le \frac{C_1}{\sqrt{j}} \quad \text{and} \quad \int_{\mathbb{R}^n \setminus B_c(0)} E_j(y)dy \le \frac{C_1}{\sqrt{j}}$$

for a constant C_1 independent of j. Thus there exist positive constants C_2 and ϱ_2 independent of x, α, and j such that

$$|E_j * (\chi \partial^\alpha f)(x) - \partial^\alpha f(x)| \le \frac{C_2}{\sqrt{j}} \varrho_2^{|\alpha|} |\alpha|!\, M_{|\alpha|}. \tag{2.11.2}$$

We have for $x \in K$

$$
\begin{aligned}
&|(\partial_{i_N} \cdots \partial_{i_{\nu+1}} E_j) * (\partial_{i_\nu}\chi)(\partial_{i_{\nu-1}} \cdots \partial_{i_1} f)(x)| \\
&\quad \le |Q_1| \sup_{\substack{1\le i\le n\\ y\in U}} |\partial_i \chi(y)| \sup_{y\notin B_c(0)} |\partial_{i_N} \cdots \partial_{i_{\nu+1}} E_j(y)| \sup_{u\in Q_1} |\partial_{i_{\nu-1}} \cdots \partial_{i_1} f(u)|,
\end{aligned}
$$

where $|Q_1|$ denotes the Lebesgue measure of Q_1. Cauchy's inequalities imply for each $\alpha \in \mathbb{N}^n$ and $r > 0$

$$|\partial^\alpha E_j(y)| \le \frac{\alpha!}{r^{|\alpha|}} \sup_{z\in D_r(y)} |E_j(z)|,$$

where $D_r(y) = \{z \in \mathbb{C}^n : |z_i - y_i| \le r \text{ for all } i\}$. Choosing $r = \frac{c}{4\sqrt{n}}$ we get for $y \in \mathbb{R}^n \setminus B_c(0)$

$$|\partial^\alpha E_j(y)| \le \frac{\alpha!}{r^{|\alpha|}} \Big(\frac{j}{\pi}\Big)^{\frac{n}{2}} e^{-\frac{jc^2}{2}}.$$

Hence with $C_3 = |Q_1| \|f\|_{\varrho_1,Q_1} \sup_{\substack{1\le i\le n\\ y\in U}} |\partial_i\chi(y)|$ we obtain for $x \in K$

$$
\begin{aligned}
|(\partial_{i_N} \cdots \partial_{i_{\nu+1}} E_j)*&(\partial_{i_\nu}\chi)(\partial_{i_{\nu-1}} \cdots \partial_{i_1} f)(x)| \\
&\le C_3 \frac{(N-\nu)!}{r^{N-\nu}} \Big(\frac{j}{\pi}\Big)^{\frac{n}{2}} e^{-\frac{jc^2}{2}} \varrho_1^{\nu-1} (\nu-1)!\, M_{\nu-1} \\
&\le C_3 \Big(\frac{j}{\pi}\Big)^{\frac{n}{2}} e^{-\frac{jc^2}{2}} \varrho_3^N (N-1)!\, M_N, \qquad (2.11.3)
\end{aligned}
$$

where $\varrho_3 = \max\{\frac{1}{r}, \varrho_1\}$.

It follows from (2.11.1), (2.11.2), and (2.11.3) that for $\varrho_4 = \max\{\varrho_2, \varrho_3\}$

$$\|f - f_j\|_{\varrho_4,K} \le \frac{C_2}{\sqrt{j}} + C_3 \Big(\frac{j}{\pi}\Big)^{\frac{n}{2}} e^{-\frac{jc^2}{2}}.$$

That implies the assertion. □

2.12. **Closed ideals.** Let $U \subseteq \mathbb{R}^n$ be open. Let $\varphi \in C^\omega(U)$. Consider the principal ideal $\varphi C^M(U)$ generated by φ.

Proposition. *Assume that C^M is stable under derivation* (2.1.5). *Let φ be a linear form on $\mathbb{R}^n$. Then the ideal $\varphi C^M(\mathbb{R}^n)$ is closed in $C^M(\mathbb{R}^n)$. More generally, assume that $\psi = \varphi_1^{p_1} \cdots \varphi_l^{p_l}$ is a finite product of linear forms φ_i. Then $\psi C^M(\mathbb{R}^n)$ is closed in $C^M(\mathbb{R}^n)$.*

Proof. Let $f \in \overline{\varphi C^M(\mathbb{R}^n)}$. Then $f|_{\varphi^{-1}(0)} = 0$, since evaluation at points is continuous. As C^M is stable under derivation, the standard integral formula (after suitable linear coordinate change) implies that $f = \varphi g$ for a unique $g \in C^M(\mathbb{R}^n)$. The same reasoning shows that $\psi C^M(\mathbb{R}^n)$ is closed in $C^M(\mathbb{R}^n)$, where $\psi = \varphi_1^{p_1}$.

For the general statement it suffices to show: *Let ψ_1 be a polynomial and ψ_2 a power of a linear form. If ψ_1 and ψ_2 are relatively prime and both generate closed ideals in $C^M(\mathbb{R}^n)$, then $\psi_1\psi_2 C^M(\mathbb{R}^n)$ is closed in $C^M(\mathbb{R}^n)$.* For $f \in \overline{\psi_1\psi_2 C^M(\mathbb{R}^n)}$ we find functions $g_1, g_2 \in C^M(\mathbb{R}^n)$ with $f = \psi_1 g_1 = \psi_2 g_2$. Since ψ_1 and ψ_2 are relatively prime, we have $g_1|_{\psi_2^{-1}(0)} = 0$. By the standard integral formula we obtain as above $g_1 = \psi_2 h$ with $h \in C^M(\mathbb{R}^n)$. Hence the assertion. □

Remark. Note that for any hyperbolic polynomial φ the principal ideal $\varphi C^M(\mathbb{R}^n)$ is closed in $\varphi C^M(\mathbb{R}^n)$ (e.g. [42, 4.2]). This follows from the fact (due to [12]) that Weierstrass division holds in C^M for hyperbolic divisors. A polynomial $\varphi(x', x_n) = x_n^d + \sum_{j=1}^d a_j(x') x_n^{d-j}$ with $a_j \in C^M(\mathbb{R}^{n-1})$ and $a_j(0) = 0$, for $1 \le j \le d$, is called *hyperbolic* if, for each $x' \in \mathbb{R}^{n-1}$, all roots of $\varphi(x', \cdot)$ are real.

But in general the principal ideal $\varphi C^M(U)$ generated by a real analytic function φ need not be closed (see [41] and [42, part 4]). Compare this with the famous results on the division of distributions due to Hörmander [18] and Lojasiewicz [25, 26].

2.13. Let M be a DC-weight sequence, and let X be a real analytic manifold. We can define the space $C^M(X)$ of functions of Denjoy–Carleman class C^M on X by means of local coordinate systems, since C^M contains the real analytic functions and is stable under composition. Similarly, we may consider the space $(\Omega^M)^p(X)$ of p-forms of class C^M on X.

3. Invariant functions in Denjoy–Carleman classes

Throughout this paper we consider a compact Lie group G acting smoothly on a manifold X. A function f on X is said to be G-invariant if $f(g.x) = f(x)$ for all $g \in G$ and all $x \in X$. If $\mathcal{F}$ is a set of functions on X, then $\mathcal{F}^G$ denotes the subset of G-invariant elements in $\mathcal{F}$.

3.1. **Hilbert's theorem.** (e.g. [43]) Let G be a compact Lie group and let V be a real finite dimensional G-module. Then, by a theorem due to Hilbert, the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is finitely generated. The generators can be chosen homogeneous and with positive degree.

3.2. **Schwarz's theorem.** Suppose that the representation of G in V is orthogonal. Let $\sigma_1, \ldots, \sigma_p$ be a system of generators of $\mathbb{R}[V]^G$ and put $\sigma = (\sigma_1, \ldots, \sigma_p) : V \to \mathbb{R}^p$. Schwarz [38] proved that $\sigma^* : C^\infty(\mathbb{R}^p) \to C^\infty(V)^G$ is surjective, which is the smooth analog of 3.1. Mather [27] showed that $\sigma^* : C^\infty(\mathbb{R}^p) \to C^\infty(V)^G$ is even split surjective, i.e., it allows a continuous linear section.

3.3. **Symmetric functions in Denjoy–Carleman classes.** In the case that the symmetric group S_n acts in $\mathbb{R}^n$ by permuting the coordinates, the statement of Schwarz's theorem 3.2 is due to Glaeser [16]. In that case σ_i is the i-th elementary symmetric function, i.e., $\sigma_i(x) = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i}$, and $\sigma = (\sigma_1, \ldots, \sigma_n) : \mathbb{R}^n \to \mathbb{R}^n$.

The representation of symmetric functions in Denjoy–Carleman (Gevrey) classes was treated by Bronshtein [7, 8]. Since we shall need it later, we present a more general version and we sketch a proof. Let $\prod_{j=1}^p \mathrm{S}_n$ act in $\bigoplus_{j=1}^p \mathbb{R}^n$ by permuting the coordinates. Since $\mathbb{R}[\bigoplus_{j=1}^p \mathbb{R}^n]^{\prod_{j=1}^p \mathrm{S}_n} \cong \bigotimes_{j=1}^p \mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$, a $\prod_{j=1}^p \mathrm{S}_n$-invariant function f on $\mathbb{R}^{pn}$ has the form $f = F \circ \theta$ with $\theta = (\sigma, \ldots, \sigma)$.

Theorem. *Assume that M and N are increasing logarithmically convex sequences with $M_0 = N_0 = 1$. Then for any function $f \in C^M(\mathbb{R}^{pn})^{\prod_{j=1}^p \mathrm{S}_n}$ there exists a function $F \in C^N(\theta(\mathbb{R}^{pn}))$ such that $f = F \circ \theta$ if and only if*

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{kn}}{N_k}\Big)^{\frac{1}{k}} < \infty. \tag{3.3.1}$$

Sketch of proof. We indicate and adapt the main steps in Bronshtein's proof. The necessity of (3.3.1) is shown by considering the symmetric function $f \in C^M(\mathbb{R}^n)$ (for $n > 2$) given by

$$f(x) = \sum_{k=0}^\infty c_k \big(1 - \prod_{j=1}^n (\rho_k x_j e^{-\rho_k^2 x_j^2})\big)^{-1},$$

where $\rho_k = \frac{M_{kn+1}}{M_{kn}}$ and $c_k = \frac{M_{kn}}{2^k \rho_k^{kn}}$. Then $f = F \circ \sigma$ with

$$F(\sigma) = \sum_{k=0}^\infty c_k \big(1 - \rho_k^n \sigma_n e^{-\rho_k^2(\sigma_1^2 - 2\sigma_2)}\big)^{-1},$$

and hence

$$|(\partial_{\sigma_n})^m F(0)| = \sum_{k=0}^{\infty} c_k \rho_k^{mn} m! \ge c_m \rho_m^{mn} m! = \frac{m! M_{mn}}{2^m}.$$

Since $F \in C^N$ this implies (3.3.1). For $n = 2$ one can find a similar example.

Without loss suppose that $f \in C^M(\mathbb{R}^{2n})^{\mathrm{S}_n \times \mathrm{S}_n}$. Instead of the elementary symmetric polynomials σ_i we use the Newton polynomials $\nu_i(x) = \sum_{j=1}^n x_j^i$ and put $\nu = (\nu_1, \dots, \nu_n)$ (see remark 3.4(3)). Then we may write $f(x,y) = F(\nu(x), \nu(y)) = F(u,v)$ where $u = \nu(x)$, $v = \nu(y)$, and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. A direct computation gives

$$\partial_{u_k} F(u,v) = \frac{(-1)^{k+1}}{k} \sum_{i=1}^n \frac{\sigma_{n-k}(x_i') \partial_{x_i} f(x,y)}{\prod_{j \ne i}(x_j - x_i)} = \sum_{i=1}^n \frac{g_{ki}(x,y)}{\prod_{j \ne i}(x_j - x_i)},$$

$$\partial_{v_k} F(u,v) = \frac{(-1)^{k+1}}{k} \sum_{i=1}^n \frac{\sigma_{n-k}(y_i') \partial_{y_i} f(x,y)}{\prod_{j \ne i}(y_j - y_i)} = \sum_{i=1}^n \frac{h_{ki}(x,y)}{\prod_{j \ne i}(y_j - y_i)},$$

where $x_i' = (x_1, \dots, \widehat{x_i}, \dots, x_n)$, $\sigma_j(x_i')$ is the elementary symmetric function of degree j in $n-1$ variables ($\sigma_0 = 1$), respectively for y, and

$$g_{ki}(x,y) = \frac{(-1)^{k+1}}{k} \sigma_{n-k}(x_i') \partial_{x_i} f(x,y),$$

$$h_{ki}(x,y) = \frac{(-1)^{k+1}}{k} \sigma_{n-k}(y_i') \partial_{y_i} f(x,y).$$

One shows (see [7, 8]) that

(3.3.2) $$\partial_{u_k} F = \Big(\prod_{j=1}^{n-1} A_j^x\Big) g_{kn} \quad \text{and} \quad \partial_{v_k} F = \Big(\prod_{j=1}^{n-1} A_j^y\Big) h_{kn},$$

where the operators A_j^x and A_j^y are defined by

$$(A_j^x h)(x,y) = \int_0^1 [(\partial_{x_j} - \partial_{x_{j+1}})h](tP_{j,j+1}x + (1-t)x, y)\,dt,$$

$$(A_j^y h)(x,y) = \int_0^1 [(\partial_{y_j} - \partial_{y_{j+1}})h](x, tP_{j,j+1}y + (1-t)y)\,dt,$$

with $P_{j,j+1}$ the linear operator $\mathbb{R}^n \to \mathbb{R}^n$ which interchanges the j-th and the $(j+1)$-st coordinate.

We consider

$$L_x^\alpha = \prod_{i=1}^n \partial_{x_i}^{\alpha_i} \prod_{1 \le p < q \le n} (\partial_{x_p} - \partial_{x_q})^{\alpha_{pq}}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^n \times \mathbb{N}^{\binom{n}{2}},$$

and likewise L_y^α. Let $K, L \subseteq \mathbb{R}^n$ be convex, compact, and S_n-invariant. For non-negative m and μ we write

$$\|f\|_{\varrho, K \times L}^{m,\mu} = \sup_{\substack{\alpha,\beta \\ (x,y) \in K \times L}} \frac{|L_x^\alpha L_y^\beta f(x,y)|}{\varrho^{|\alpha|+|\beta|+m} |\alpha_1|!|\beta_1|!\alpha_2!\beta_2!(|\alpha|+|\beta|+m+1)^\mu M_{|\alpha|+|\beta|+m}}.$$

If $f \in C^M(\mathbb{R}^{2n})$ then $\|f\|_{\varrho, K \times L}^{m,\mu} < \infty$ for sufficiently large ϱ. We have the following estimates

(3.3.3) $$\|\partial_{x_i} f\|_{\varrho, K \times L}^{m+1,\mu+1} \le \|f\|_{\varrho, K \times L}^{m,\mu} \quad \text{and} \quad \|\partial_{y_i} f\|_{\varrho, K \times L}^{m+1,\mu+1} \le \|f\|_{\varrho, K \times L}^{m,\mu},$$

(3.3.4) $$\|x_i f\|_{\varrho, K \times L}^{m,\mu} \le C \|f\|_{\varrho, K \times L}^{m,\mu} \quad \text{and} \quad \|y_i f\|_{\varrho, K \times L}^{m,\mu} \le C \|f\|_{\varrho, K \times L}^{m,\mu},$$

(3.3.5) $$\|A_j^x f\|_{\varrho, K \times L}^{m+1,\mu} \le C \|f\|_{\varrho, K \times L}^{m,\mu} \quad \text{and} \quad \|A_j^y f\|_{\varrho, K \times L}^{m+1,\mu} \le C \|f\|_{\varrho, K \times L}^{m,\mu}.$$

It is easy to verify (3.3.3) and (3.3.4). For the proof of (3.3.5) we refer to [7, 8].

It follows from (3.3.2) and from (3.3.3), (3.3.4), and (3.3.5) that

$$\|\partial_u^\alpha \partial_v^\beta F\|_{\varrho,\nu(K)\times\nu(L)}^{m+n(|\alpha|+|\beta|),\mu+|\alpha|+|\beta|} \le C_1^{|\alpha|+|\beta|} \|f\|_{\varrho,K\times L}^{m,\mu}$$

for all $\alpha,\beta \in \mathbb{N}^n$. Hence for $\alpha,\beta \in \mathbb{N}^n$ and $(u,v) \in \nu(K)\times\nu(L)$ we find

$$\begin{aligned}
&|\partial_u^\alpha \partial_v^\beta F(u,v)| \\
&\quad \le \|\partial_u^\alpha \partial_v^\beta F\|_{\varrho,\nu(K)\times\nu(L)}^{n(|\alpha|+|\beta|),|\alpha|+|\beta|} \varrho^{n(|\alpha|+|\beta|)} (n(|\alpha|+|\beta|)+1)^{|\alpha|+|\beta|} M_{n(|\alpha|+|\beta|)} \\
&\quad \le \|f\|_{\varrho,K\times L}^{0,0} C_1^{|\alpha|+|\beta|} \varrho^{n(|\alpha|+|\beta|)} (n(|\alpha|+|\beta|)+1)^{|\alpha|+|\beta|} M_{n(|\alpha|+|\beta|)} \\
&\quad \le C_2 \varrho_1^{|\alpha|+|\beta|} (|\alpha|+|\beta|)!\, N_{|\alpha|+|\beta|},
\end{aligned}$$

for suitable constants C_2 and ϱ_1. That implies $F \in C^N(\nu(\mathbb{R}^n)\times\nu(\mathbb{R}^n))$. □

It was proved by Kostov [24] that $\sigma(\mathbb{R}^n)$ is Whitney 1-regular. Hence $\theta(\mathbb{R}^{pn}) = \sigma(\mathbb{R}^n)\times\cdots\times\sigma(\mathbb{R}^n)$ is Whitney 1-regular as well. It follows that, if N is strongly regular, then F can be extended to a function in $C^N(\mathbb{R}^{pn})$ (by Whitney's extension theorem; see 2.7):

Corollary. *Assume that M is an increasing logarithmically convex sequences with $M_0 = 1$. Let N be a strongly regular DC-weight sequence. For any function $f \in C^M(\mathbb{R}^{pn})^{\prod_{j=1}^p \mathrm{S}_n}$ there exists a function $F \in C^N(\mathbb{R}^{pn})$ such that $f = F\circ\theta$ if and only if*

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{kn}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

In particular: Any Gevrey function $f \in G^{1+\delta}(\mathbb{R}^{pn})^{\prod_{j=1}^p \mathrm{S}_n}$ (with $\delta > 0$) has the form $f = F\circ\theta$ with $F \in G^{1+\gamma}(\mathbb{R}^{pn})$, where the exponent $\gamma = \delta n$ is minimal possible.

3.4. **Invariant functions in Denjoy–Carleman classes.**

Theorem. *Let G be subgroup with finite order m of $\mathrm{GL}(V)$. Let $\sigma_1,\ldots,\sigma_p$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ and put $\sigma = (\sigma_1,\ldots,\sigma_p) : V \to \mathbb{R}^p$. Assume that M and N are DC-weight sequences. Suppose that N is strongly regular and that*

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty. \tag{3.4.1}$$

Then for any G-invariant function $f \in C^M(V)^G$ there exists a function $F \in C^N(\mathbb{R}^p)$ such that $f = F\circ\sigma$. In particular: Any G-invariant Gevrey function $f \in G^{1+\delta}(V)^G$ (with $\delta > 0$) has the form $f = F\circ\sigma$ with $F \in G^{1+\gamma}(\mathbb{R}^p)$, where $\gamma = \delta m$.

The proof of the theorem uses 3.3 and occupies the rest of the section. It is inspired by Barbançon and Raïs [3] deploying Weyl's account [43] of Noether's [30] proof of Hilbert's theorem.

Remarks. (1) The condition (3.4.1) implies that $C^M(U) \subseteq C^N(U)$ by (2.1.4). If additionally $\lim_{k\to\infty}(M_k/N_k)^{1/k} = 0$ then $C^M(U) \ne C^N(U)$, so there is a real loss of regularity.

(2) The loss of regularity announced in the theorem is not minimal. For a particular group G, one may find much better Denjoy–Carleman regularity for F.

(3) The result is independent of the choice of generators σ_i, since any two choices differ by a polynomial diffeomorphism and the involved Denjoy–Carleman classes are stable under composition.

3.5. **Reduction to the symmetric case.** Let V be a real vector space of finite dimension n and let G be a subgroup with finite order m of $\mathrm{GL}(V)$. The symmetric group S_m acts in a natural way on G by permuting the elements. This induces an action of S_m on the space $F(G,\mathbb{R})$ of functions defined in G with values in $\mathbb{R}$ (for $\sigma \in \mathrm{S}_m$ and $f \in F(G,\mathbb{R})$ we have $\sigma.f = f \circ \sigma^{-1}$). It can be identified with the standard representation ρ of S_m in $\mathbb{R}^m$. We obtain a natural action of S_m on $E = F(G,\mathbb{R}) \otimes V$, the vector space of functions defined in G with values in V. The corresponding representation π is given by $\pi = n\rho$.

Let $L : V \to E$ be the linear injective mapping defined by $L(v) : g \mapsto g.v$ for $v \in V$. We consider the pullback $L^* : F(E,\mathbb{R}) \to F(V,\mathbb{R})$ (where $F(X,Y)$ denotes the space of functions defined in X with values in Y). It is linear and maps S_m-invariant functions to G-invariant functions. Hence it drops to a mapping $L^* : F(E,\mathbb{R})^{\mathrm{S}_m} \to F(V,\mathbb{R})^G$. We define a linear mapping $J : F(V,\mathbb{R}) \to F(E,\mathbb{R})$ by putting

$$J(f)(h) = \frac{1}{m} \sum_{g \in G} f(h(g))$$

for $f \in F(V,\mathbb{R})$ and $h \in E = F(G,\mathbb{R}) \otimes V$. If we denote by $\mathrm{ev}_g : E \to V$ the evaluation at $g \in G$, i.e., $\mathrm{ev}_g(h) = h(g)$ for $h \in E$, then $J(f) = \frac{1}{m}\sum_{g\in G} \mathrm{ev}_g^* f$. Thus, J maps polynomials on V to polynomials on E. It is easy to check that $L^* \circ J|_{F(V,\mathbb{R})^G} = \mathrm{id}$, so $J|_{F(V,\mathbb{R})^G}$ is a section for $L^* : F(E,\mathbb{R})^{\mathrm{S}_m} \to F(V,\mathbb{R})^G$.

Let M be a DC-weight sequence. It is easily seen that L^* and J are both continuous as mappings $L^* : C^M(E)^{\mathrm{S}_m} \to C^M(V)^G$ and $J : C^M(V)^G \to C^M(E)^{\mathrm{S}_m}$.

Let $(\tau_1, \ldots, \tau_p)$ be a system of generators of the algebra $\mathbb{R}[E]^{\mathrm{S}_m}$. Let $f \in C^M(V)^G$. If theorem 3.4 holds for π, there exists $F \in C^N(\mathbb{R}^p)$ (with suitable strongly regular DC-weight sequence N, see 3.8) such that

$$J(f)(h) = F(\tau_1(h), \ldots, \tau_p(h))$$

for all $h \in E$. Then

$$f(v) = J(f)(L(v)) = F(\sigma_1(v), \ldots, \sigma_p(v))$$

for all $v \in V$, where $\sigma_i = L^*\tau_i$ for $1 \le i \le p$. It is clear from the above that the $\sigma_i = L^*\tau_i$ generate $\mathbb{R}[V]^G$. This shows theorem 3.4 under the assumption that it holds for the representation π (with suitable N).

3.6. Let $W \subseteq \mathrm{GL}(V)$ be a finite reflection group. Let H be a W-invariant graded linear subspace of $\mathbb{R}[V]$ which is complementary to the ideal generated by the W-invariant polynomials with strictly positive degree. The bilinear mapping $(h, f) \mapsto hf$ induces an isomorphism of W-modules $H \otimes \mathbb{R}[V]^W \to \mathbb{R}[V]$ (see [6, Ch. 5, 5.2, Thm. 2]). So $\mathbb{R}[V]$ is a free $\mathbb{R}[V]^W$-module of rank $|W|$.

Choose a basis $h_1, \ldots, h_{|W|}$ of H consisting of homogeneous elements. Let $w_1, \ldots, w_{|W|}$ denote the elements of W (in some ordering). Since $\mathbb{R}[V] = H\mathbb{R}[V]^W$, we find that, for each $v \in V$, the cardinality of the orbit $W.v$ equals the rank of the matrix $(h_j(w_i.v))_{i,j}$. Since there are $v \in V$ with $|W.v| = |W|$, the polynomial

$$\Delta(v) := \det(h_j(w_i.v))_{i,j}$$

is not $0 \in \mathbb{R}[V]$.

Lemma. *Let $W = \mathrm{S}_{m_1} \times \cdots \times \mathrm{S}_{m_n}$ act in $V = \mathbb{R}^{m_1} \oplus \cdots \oplus \mathbb{R}^{m_n}$ by permuting the coordinates. Then, for $v = (x_{1,1}, \ldots, x_{1,m_1}, \ldots, x_{n,1}, \ldots, x_{n,m_n})$, we have*

$$\Delta(v) = c \prod_{i=1}^{n} \prod_{1 \le j_i < k_i \le m_i} (x_{i,j_i} - x_{i,k_i})^{p_{i,j_i,k_i}} \tag{3.6.1}$$

for some non-zero constant c and positive integers p_{i,j_i,k_i}.

Proof. By definition, $\Delta(v) = 0$ if and only if v belongs to some reflecting hyperplane of W. It follows that each of the linear forms

$$\mathcal{L} := \{x_{i,j_i} - x_{i,k_i} : 1 \le i \le n, 1 \le j_i < k_i \le m_i\} \tag{3.6.2}$$

divides Δ. Since they are relatively prime, their product divides Δ. Suppose, for contradiction, there is a non-constant polynomial P which is relatively prime with any of the linear forms in $\mathcal{L}$ and divides Δ. Without loss we switch to the complexification of the W-module V. By Hilbert's Nullstellensatz, there is a positive integer r such that $(\prod_{l \in \mathcal{L}} l)^r$ belongs to the ideal generated by Δ, a contradiction. Hence the assertion. □

Remark. Actually, more is true: For any finite reflection group $W \subseteq \mathrm{GL}(V)$ we have $\Delta = cJ^{|W|/2}$, where c is a non-zero constant and $J = \prod_{l \in \mathcal{L}_W} l$ with $\mathcal{L}_W$ the set of linear forms with kernel a reflection hyperplane of W. See [3, 4.2 + Appendix]. For us the above lemma will suffice.

3.7. Let H and $h_1, \ldots, h_{|W|}$ be as in 3.6. The following proposition is a modification of [3, 3.3].

Proposition. *Let M be a DC-weight sequence. Let $W = \mathrm{S}_{m_1} \times \cdots \times \mathrm{S}_{m_n}$ act in $V = \mathbb{R}^{m_1} \oplus \cdots \oplus \mathbb{R}^{m_n}$ by permuting the coordinates. Then $h_1, \ldots, h_{|W|}$ constitutes a basis of $C^M(V)$ considered as $C^M(V)^W$-module.*

Proof. Let $f \in C^M(V)$. There exists a sequence (P_k) of polynomials which converges to f in $C^M(V)$ (by 2.11). Since $h_1, \ldots, h_{|W|}$ is a basis of $\mathbb{R}[V]$ as $\mathbb{R}[V]^W$-module, we can write $P_k = \sum_j h_j P_{k,j}$ with $P_{k,j} \in \mathbb{R}[V]^W$. For each $v \in V$, we obtain a system of $|W|$ equations

$$P_k(w_i.v) = \sum_j h_j(w_i.v) P_{k,j}(v) \quad (1 \le i \le |W|).$$

Cramer's rule implies

$$\Delta(v) P_{k,j}(v) = \sum_i \Delta_{ij}(v) P_k(w_i.v) \quad (1 \le j \le |W|),$$

where the Δ_{ij} denote the cofactors of the matrix $(h_j(w_i.v))_{i,j}$. The right-hand side of the single equations converges in $C^M(V)$ to the function

$$v \mapsto \sum_i \Delta_{ij}(v) f(w_i.v) \quad (1 \le j \le |W|),$$

respectively (a straightforward computation shows that multiplication by a polynomial is continuous). Hence, each sequence $(\Delta P_{k,j})_k$ converges in $C^M(V)$. By proposition 2.12 and lemma 3.6, the ideal $\Delta C^M(V)$ generated by Δ is closed in $C^M(V)$. Thus, there exist unique functions $f_j \in C^M(V)$ such that, for each v and each j,

$$\Delta(v) f_j(v) = \sum_i \Delta_{ij}(v) f(w_i.v). \tag{3.7.1}$$

The f_j are W-invariant: For each $w \in W$ there is $\epsilon_w \in \{0,1\}$ such that $\Delta(w.v) = (-1)^{\epsilon_w} \Delta(v)$ for all $v \in V$. Since the polynomials $P_{k,j}$ are W-invariant and evaluation at points is continuous, we find

$$(-1)^{\epsilon_w} \Delta(v) f_j(w.v) = (-1)^{\epsilon_w} \Delta(v) f_j(v)$$

and thus

$$f_j(w.v) = f_j(v)$$

on the open dense subset $\{v : \Delta(v) \neq 0\}$, and hence everywhere. From (3.7.1) we obtain

$$f(v) = \sum_j h_j(v) f_j(v)$$

on the open dense subset $\{v : \Delta(v) \neq 0\}$, and hence everywhere. □

Remark. Using remark 3.6, we find that this proposition is true for any finite reflection group $W \subseteq \mathrm{GL}(V)$.

3.8. **Theorem 3.4 for the representation $\pi : \mathrm{S}_m \to \mathrm{GL}(\mathbb{R}^{nm})$.** Let G be a subgroup of $W = \mathrm{S}_{m_1} \times \cdots \times \mathrm{S}_{m_n}$ acting in $V = \mathbb{R}^{m_1} \oplus \cdots \oplus \mathbb{R}^{m_n}$ by permuting the coordinates. Let H be (as in 3.6) a W-invariant graded linear subspace of $\mathbb{R}[V]$ which is complementary to the ideal generated by the W-invariant polynomials with strictly positive degree. Consider a basis $(h_1, \ldots, h_r)$ of H^G. By proposition 3.7, we find that $(h_1, \ldots, h_r)$ constitutes a basis of $C^M(V)^G$ considered as $C^M(V)^W$-module.

By the reduction in 3.5, in order to prove theorem 3.4 it suffices to consider the representation $\pi : \mathrm{S}_m \to \mathrm{GL}(\mathbb{R}^{nm})$. Let $\tau_1, \ldots . \tau_p$ and $\theta_1, \ldots, \theta_{nm}$ be systems of homogeneous generators of $\mathbb{R}[\mathbb{R}^{nm}]^{\mathrm{S}_m}$ and $\mathbb{R}[\bigoplus_{j=1}^n \mathbb{R}^m]^{\prod_{j=1}^n \mathrm{S}_m}$, respectively, and consider $\tau = (\tau_1, \ldots, \tau_p) : \mathbb{R}^{nm} \to \mathbb{R}^p$ and $\theta = (\theta_1, \ldots, \theta_{nm}) : \mathbb{R}^{nm} \to \mathbb{R}^{nm}$. By the previous paragraph and corollary 3.3, each $f \in C^M(\mathbb{R}^{nm})^{\mathrm{S}_m}$ has the form

$$f = \sum_{j=1}^r h_j f_j = \sum_{j=1}^r (H_j \circ \tau)(F_j \circ \Theta \circ \tau),$$

where $h_j \in \mathbb{R}[\mathbb{R}^{nm}]^{\mathrm{S}_m}$, $f_j \in C^M(\mathbb{R}^{nm})^{\prod_{j=1}^n \mathrm{S}_m}$, $H_j \in \mathbb{R}[\mathbb{R}^p]$, $F_j \in C^N(\mathbb{R}^{nm})$, and Θ is the polynomial mapping given by $\theta = \Theta \circ \tau$. Note that N is a strongly regular DC-weight sequence satisfying

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

This completes the proof of theorem 3.4.

4. Equivariant mappings in Denjoy–Carleman classes

We give an application of theorem 3.4 to the representation of equivariant mappings in Denjoy–Carleman classes. We follow standard techniques.

4.1. Let V_1 and V_2 be real finite dimensional representations of a compact Lie group G. It is well-known that the set $\mathrm{Pol}(V_1, V_2)^G$ of G-equivariant polynomial mappings from V_1 to V_2 is finitely generated as module over $\mathbb{R}[V_1]^G$.

Let M be a DC-weight sequence. We denote by $C^M(V_1, V_2)^G$ the set of G-equivariant C^M-mappings $f : V_1 \to V_2$.

Theorem. *Let V_1 and V_2 be representations of a finite group G with order m. Let $\sigma_1, \ldots, \sigma_p$ be a system of homogeneous generators of $\mathbb{R}[V_1]^G$ and put $\sigma = (\sigma_1, \ldots, \sigma_p)$. Let $P_1, \ldots, P_l$ be a system of generators of the $\mathbb{R}[V_1]^G$-module $\mathrm{Pol}(V_1, V_2)^G$. Assume that M and N are DC-weight sequences. Suppose that N is strongly regular and that*

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then for each $f \in C^M(V_1, V_2)^G$ there exists an $L(f) \in (C^N(\mathbb{R}^p))^l$ such that $f = \sum_{j=1}^l (L(f)_j \circ \sigma) P_j$.

Proof. The dual V_2^* of V_2 carries the dual G-action given by $g.l = l \circ g^{-1}$. Let $f \in C^M(V_1, V_2)^G$ and consider the G-invariant function $H_f : V_1 \times V_2^* \to \mathbb{R}$ given by $H_f(v,l) = l(f(v))$. So $H_f \in C^M(V_1 \times V_2^*)^G$ and, by theorem 3.4, there exists $L_f \in C^N(\mathbb{R}^q)$ such that $H_f = L_f \circ \tau$, where $\tau = (\tau_1, \ldots, \tau_q)$ and $\tau_1, \ldots, \tau_q$ generate $\mathbb{R}[V_1 \times V_2^*]^G$. Taking the derivative with respect to the second component gives

$$f(v) = \sum_{i=1}^q \partial_i L_f(\tau(v,0)) d_2\tau_i(v,0).$$

Since $v \mapsto d_2\tau_i(v,0)$ is a G-equivariant polynomial mapping, there exist $h_{ij} \in \mathbb{R}[V_1]^G$ such that $d_2\tau_i(v,0) = \sum_{j=1}^l h_{ij}(v) P_j(v)$. Since $v \mapsto \tau(v,0)$ is G-invariant, there is a polynomial mapping $\theta : \mathbb{R}^p \to \mathbb{R}^q$ with $\tau(v,0) = \theta(\sigma(v))$. Then

$$L(f) := \Big(\sum_{i=1}^q (\partial_i L_f \circ \theta) h_{ij} \Big)_{1 \le j \le l}$$

has the required properties. □

5. Polar representations

5.1. **Polar representations.** [13], [33], [39] A real finite dimensional orthogonal representation $\rho : G \to \mathrm{O}(V)$ of a Lie group G is called *polar*, if there exists a linear subspace $\Sigma \subseteq V$, called a *section*, which meets each orbit orthogonally. The trace of the G-action in Σ is the action of the *generalized Weyl group* $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$, where $N_G(\Sigma) := \{g \in G : \rho(g)(\Sigma) = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : \rho(g)(s) = s \text{ for all } s \in \Sigma\}$. The generalized Weyl group is a finite group. If Σ' is a different section, then there is an isomorphism $W(\Sigma) \to W(\Sigma')$ induced by an inner automorphism of G.

The following generalization of Chevalley's restriction theorem is due to Dadok and Kac [13] and independently to Terng [39].

Theorem. *Assume that G is a compact Lie group. Then restriction induces an isomorphism of algebras between $\mathbb{R}[V]^G$ and $\mathbb{R}[\Sigma]^{W(\Sigma)}$.*

5.2. **Invariant functions in Denjoy–Carleman classes.** We generalize theorem 3.4 to polar representations..

Theorem. *Let $G \to \mathrm{O}(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Write $m = |W|$. Let $\sigma_1, \ldots, \sigma_p$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ and put $\sigma = (\sigma_1, \ldots, \sigma_p)$. Assume that M and N are DC-weight sequences. Suppose that N is strongly regular and that*

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then for any G-invariant function $f \in C^M(V)^G$ there exists a function $F \in C^N(\mathbb{R}^p)$ such that $f = F \circ \sigma$. In particular: Any G-invariant Gevrey function $f \in G^{1+\delta}(V)^G$ (with $\delta > 0$) has the form $f = F \circ \sigma$ with $F \in G^{1+\gamma}(\mathbb{R}^p)$, where $\gamma = \delta m$.

Proof. Let $f \in C^M(V)^G$. By theorem 5.1, the restrictions $\sigma_1|_\Sigma, \ldots, \sigma_p|_\Sigma$ generate $\mathbb{R}[\Sigma]^W$ and $\sigma(V) = \sigma|_\Sigma(\Sigma)$. Since $f|_\Sigma \in C^M(\Sigma)^W$, theorem 3.4 implies that there is a $F \in C^N(\mathbb{R}^p)$ such that $f|_\Sigma = F \circ \sigma|_\Sigma$, and, hence, $f = F \circ \sigma$. □

5.3. In the situation of 5.2 we have:

Theorem. *Each* $f \in C^M(\Sigma)^W$ *(resp.* $G^{1+\delta}(\Sigma)^W$*) has an extension in* $C^N(V)^G$ *(resp.* $G^{1+\gamma}(V)^G$*).*

Proof. Let $f \in C^M(\Sigma)^W$. Choose a system of homogeneous generators $\tau_1, \dots, \tau_p$ of $\mathbb{R}[\Sigma]^W$. By theorem 3.4, there is an $F \in C^N(\mathbb{R}^p)$ such that $f = F \circ (\tau_1, \dots, \tau_p)$. Each τ_i extends to a polynomial $\tilde{\tau}_i \in \mathbb{R}[V]^G$, by theorem 5.1. So $\tilde{f} := F \circ (\tilde{\tau}_1, \dots, \tilde{\tau}_p)$ is a G-invariant extension of f belonging to $C^N(V)$. □

5.4. **Basic differential forms in Denjoy–Carleman classes.** Let $G \to \mathrm{O}(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. A differential form $\omega \in \Omega^p(V)$ is called *G-invariant* if $(l_g)^*\omega = \omega$ for all $g \in G$, where $l_g(x) = g.x$, and *horizontal* if it kills each vector tangent to a G-orbit, i.e., $i_{\zeta_X}\omega = 0$ for all $X \in \mathfrak{g} := \mathrm{Lie}(G)$, where ζ is the fundamental vector field mapping ($\zeta_X(x) = T_e(l^x).X$ with $l^x(g) = g.x$). Denote by $\Omega^p_{\mathrm{hor}}(V)^G$ the space of all horizontal G-invariant p-forms on V. Its elements are also called *basic p-forms*.

It is proved in [28, 29] that the restriction of differential forms induces an isomorphism between $\Omega^p_{\mathrm{hor}}(V)^G$ and $\Omega^p(\Sigma)^W$.

Let M be a DC-weight sequence. We may consider p-forms ω on V of Denjoy–Carleman class C^M. Let us denote the space of such forms ω by $(\Omega^M)^p(V)$. A careful inspection of the proofs in [28, 29] shows that we can deduce the following theorem in an analog manner:

(i) The statement in [28, 3.2] is true in Denjoy–Carleman classes C^M as well: *Let $l \in V^*$ and let $f \in C^M(V)$ with $f|_{l^{-1}(0)} = 0$. Then there exists a unique $h \in C^M(V)$ such that $f = l \cdot h$.* See the proof of proposition 2.12.

(ii) In [28, 3.7] instead of Schwarz's theorem we use theorem 3.4.

The rest works without change and yields:

Theorem. *Let $G \to \mathrm{O}(V)$ be a polar representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Put $m = |W|$. Assume that M and N are DC-weight sequences. Suppose that N is strongly regular and that*

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then each $\omega \in (\Omega^M)^p(\Sigma)^W$ has an extension in $(\Omega^N)^p_{\mathrm{hor}}(V)^G$. □

Remark. Obviously, restriction of differential forms does in general not map forms in $(\Omega^N)^p_{\mathrm{hor}}(V)^G$ to forms in $(\Omega^M)^p(\Sigma)^W$. So we cannot expect to obtain an isomorphism as in the smooth case.

6. Proper G-manifolds with sections

In this section X always denotes a connected complete Riemannian G-manifold, with effective and isometric G-action.

6.1. **Sections.** [33] Let X be a proper Riemannian G-manifold. A connected closed submanifold Σ of X is called a *section* for the G-action, if it meets all G-orbits orthogonally. Each section is a totally geodesic submanifold. Analogously with 5.1 we define the *generalized Weyl group* $W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$ which turns out to be a discrete group acting properly on Σ. If Σ' is a different section, then there is an isomorphism $W(\Sigma) \to W(\Sigma')$ induced by an inner automorphism of G.

6.2. **Invariant functions in Denjoy–Carleman classes.** In the smooth case, restriction induces an isomorphism $C^\infty(X)^G \cong C^\infty(\Sigma)^{W(\Sigma)}$, by [32]. We show an analog result in Denjoy–Carleman classes. From now on all manifolds are real analytic.

Theorem. *Let X be a real analytic proper Riemannian G-manifold with section Σ and Weyl group $W = W(\Sigma)$. Suppose that*

$$m := \sup_{x\in\Sigma} |W_x| < \infty.$$

Assume that M and N are DC-weight sequences. Suppose that N is strongly regular and that

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then each $f \in C^M(\Sigma)^W$ (resp. $G^{1+\delta}(\Sigma)^W$) has an extension in $C^N(X)^G$ (resp. $G^{1+\delta m}(X)^G$).

Proof. Let $f \in C^M(\Sigma)^W$. It is well-known (e.g. [33]) that each W-invariant continuous (smooth) function in Σ has a unique continuous (smooth) G-invariant extension. Let $\tilde f$ be the extension of f. We show that $\tilde f$ represents an element in $C^N(X)^G$. Let $x \in X$. Without loss we may assume that $x \in \Sigma$ (since the action is real analytic). Let S_x be a normal slice at x. Then, by the slice theorem, $G.S_x$ and $G \times_{G_x} S_x$ are real analytically G-isomorphic and $G \times S_x \to G \times_{G_x} S_x$ is a real analytic surjective submersion. Thus, it suffices to show that $\tilde f|_{S_x}$ belongs to $C^N(S_x)$. We can choose a ball $B \subseteq T_x S_x$ around 0_x such that $B \cong S_x$ and $T_x\Sigma \cap B \cong \Sigma \cap S_x$. Then the G_x-action on S_x is up to a real analytic isomorphism a polar representation with section $T_x\Sigma$ and Weyl group W_x (e.g. [33]). So the assertion follows from theorem 5.3. □

6.3. **Basic differential forms in Denjoy–Carleman classes.** In the smooth case, the restriction of differential forms induces an isomorphism between $\Omega^p_{\text{hor}}(X)^G$ and $\Omega^p(\Sigma)^W$, by [28, 29]. This is derived from the analog result for polar representations with the help of the slice theorem.

Let X be a real analytic proper Riemannian G-manifold with section Σ and Weyl group $W = W(\Sigma)$. Suppose that

$$m := \sup_{x\in\Sigma} |W_x| < \infty.$$

Let M and N be a DC-weight sequences. Suppose that N is strongly regular and that

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

It turns out that we are able to apply the same arguments as in [28, 29] in order to deduce a similar generalized statement from theorem 5.4. All mappings occurring while applying the slice theorem in [28, part 4] are real analytic and may, therefore, be taken over without change. Hence we may reduce to the slice representations $G_x \to \mathrm{O}(T_x S_x)$ which are polar with Weyl group W_x and we may apply theorem 5.4.

Following the final step of the proof [28, 4.2] we glue local differential forms $\omega^{x_n} \in (\Omega^N)^p_{\text{hor}}(G.S_{x_n})^G$ to a form $\tilde\omega \in (\Omega^N)^p_{\text{hor}}(X)^G$. This is done, using a method of Palais [31, 4.3.1], by constructing a suitable partition of unity consisting of G-invariant functions. More precisely: There exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points in Σ and open neighborhoods of x_n in Σ whose projections form a locally finite open covering of the orbit space $X/G \cong \Sigma/W$, and there exists a partition of unity f_n consisting of G-invariant functions with $\operatorname{supp}(f_n) \subseteq G.S_{x_n}$. The construction of the f_n is as follows: There exist neighborhoods $x_n \in K_n$ with compact closure in S_{x_n} such that their projection forms a covering of X/G. Let f_n be a non-negative function on S_{x_n} positive on K_n and with compact support in S_{x_n}. By averaging we may assume that f_n is G_{x_n}-invariant. Define $f_n(g.s) = f_n(s)$ for $g \in G$ and $s \in S_{x_n}$ and $f_n(x) = 0$ for $x \notin G.S_{x_n}$. Since there are C^N partitions of unity (by 2.3) and since averaging over the slice representation $G_{x_n} \to \mathrm{O}(T_{x_n}S_{x_n})$ (which is G_{x_n}-equivariantly real analytically isomorphic to the G_{x_n}-manifold S_{x_n}) preserves the Denjoy–Carleman class (by lemma 6.4 below), the functions f_n can be chosen in $C^N(X)^G$. Thus $\tilde\omega = \sum_n f_n \omega^{x_n} \in (\Omega^N)^p_{\mathrm{hor}}(X)^G$, and we obtain:

Theorem. *Let X be a real analytic proper Riemannian G-manifold with section Σ and Weyl group $W = W(\Sigma)$. Suppose that*

$$m := \sup_{x\in\Sigma} |W_x| < \infty.$$

Let M and N be a DC-weight sequences. Suppose that N is strongly regular and that

$$\sup_{k\in\mathbb{N}_{>0}} \Big(\frac{M_{km}}{N_k}\Big)^{\frac{1}{k}} < \infty.$$

Then each $\omega \in (\Omega^M)^p(\Sigma)^W$ has an extension in $(\Omega^N)^p_{\mathrm{hor}}(X)^G$. □

Lemma 6.4. *Let $G \to \mathrm{O}(V)$ be a real finite dimensional representation of a compact Lie group G. Let M be a DC-weight sequence. If $f \in C^M(V)$ then*

$$\tilde f(x) = \int_G f(g.x)dg$$

(where dg denotes Haar measure) belongs to $C^M(V)^G$.

Proof. We write $l_g : V \to V, x \mapsto g.x$ for the linear action of $g \in G$. By choosing a basis we identify $V = \mathbb{R}^n$. Let $K \subseteq V$ be compact. It suffices to show that for each positive $\varrho = \varrho(f, G.K)$ there exists a positive $\bar\varrho$ such that

$$\|f \circ l_g\|_{\bar\varrho,K} \le \|f\|_{\varrho,G.K} \tag{6.4.1}$$

for all $g \in G$. By Faà di Bruno ([15] for the 1-dimensional version)

$$\frac{\partial^\gamma (f\circ l_g)(x)}{\gamma!} = \sum_{\substack{\beta_i\in\mathbb{N}^n\setminus\{0\}\\ \alpha=\beta_1+\cdots+\beta_n\\ \gamma=(|\beta_1|,\ldots,|\beta_n|)}} \frac{1}{\beta_1!\cdots\beta_n!}\,\partial^\alpha f(g.x)\,(\partial_1 l_g(x))^{\beta_1}\cdots(\partial_n l_g(x))^{\beta_n},$$

where $\partial_i l_g(x) = (\partial_i (l_g)_1(x), \ldots, \partial_i (l_g)_n(x))$. So we find

$$\frac{|\partial^\gamma (f\circ l_g)(x)|}{|\gamma|!\,M_{|\gamma|}} \le \sum \frac{|\alpha|!}{\beta_1!\cdots\beta_n!}\,\frac{|\partial^\alpha f(g.x)|}{|\alpha|!\,M_{|\alpha|}}\,\|l_g\|^{|\alpha|},$$

where $\|l_g\|$ denotes the operator norm of l_g. Put

$$\mu := \max_{g\in G} \|l_g\|.$$

Then we obtain (6.4.1) by defining

$$\bar\varrho := n^2\mu\varrho.$$

This completes the proof. □

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