

THE TROUVÉ GROUP FOR SPACES OF TEST FUNCTIONS

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ABSTRACT. The Trouvé group $\mathcal{G}_{\mathcal{A}}$ from image analysis consists of the flows at a fixed time of all time-dependent vector fields of a given regularity $\mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$. For a multitude of regularity classes \mathcal{A} , we prove that the Trouvé group $\mathcal{G}_{\mathcal{A}}$ coincides with the connected component of the identity of the group of orientation preserving diffeomorphisms of \mathbb{R}^d which differ from the identity by a mapping of class \mathcal{A} . We thus conclude that $\mathcal{G}_{\mathcal{A}}$ has a natural regular Lie group structure. In many cases we show that the mapping which takes a time-dependent vector field to its flow is continuous. As a consequence we obtain that the scale of Bergman spaces on the polystrip with variable width is stable under solving ordinary differential equations.

1. INTRODUCTION

It is well-known that a time-dependent vector field $u \in L^1([0, 1], C_b^1(\mathbb{R}^d, \mathbb{R}^d))$, where $C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ denotes the space of C^1 -mappings which are globally bounded together with its first derivative, has a unique flow

$$(1.1) \quad \Phi(t, x) = x + \int_0^t u(s, \Phi(s, x)) ds, \quad x \in \mathbb{R}^d, t \in [0, 1]$$

such that $\Phi(t, \cdot) - \text{Id} \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ for all t . Given a locally convex space \mathcal{A} of mappings $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ which is continuously embedded in $C_b^1(\mathbb{R}^d, \mathbb{R}^d)$, one defines the associated *Trouvé group*

$$\mathcal{G}_{\mathcal{A}} := \{\Phi(1, \cdot) : u \in \mathcal{F}_{\mathcal{A}}\},$$

where $\mathcal{F}_{\mathcal{A}}$ is a suitable family of functions $[0, 1] \rightarrow \mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$; this construction is due to Trouvé [20], see also [22]. We require that the elements of $\mathcal{F}_{\mathcal{A}}$ are *integrable* functions in some sense, but the precise definition depends on the structure of the space $\mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$. If the latter space is a Fréchet space, then the elements of $\mathcal{F}_{\mathcal{A}}$ are *integrable by seminorm* (see Section 2.3), in particular, Bochner integrable in the Banach case.

There has been a recent interest in a precise description of the regularity properties of the elements of $\mathcal{G}_{\mathcal{A}}$; see [5] for the Sobolev and [16] for the Hölder case. In [6] similar problems are studied in the context of Banach and Fréchet Lie groups.

In this paper we prove that the Trouvé group $\mathcal{G}_{\mathcal{A}}$ is equal to the connected component of the identity $\text{Diff}_0 \mathcal{A}$ of

$$\text{Diff } \mathcal{A} := \{\Phi \in \text{Id} + \mathcal{A}(\mathbb{R}^d, \mathbb{R}^d) : \inf_{x \in \mathbb{R}^d} \det d\Phi(x) > 0\}$$

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for all of the following classes \mathcal{A} :

- Smooth functions with globally bounded derivatives \mathcal{B} ($= \mathcal{D}_{L^\infty}$ in [19]).
- Smooth functions with p -integrable derivatives $W^{\infty,p}$ ($= \mathcal{D}_{L^p}$ in [19]).
- Rapidly decreasing Schwartz functions \mathcal{S} .
- Smooth functions with compact support \mathcal{D} .
- Global Denjoy–Carleman classes $\mathcal{B}^{[M]}$.
- Sobolev–Denjoy–Carleman classes $W^{[M],p}$.
- Gelfand–Shilov classes $\mathcal{S}_{[L]}^{[M]}$.
- Denjoy–Carleman functions with compact support $\mathcal{D}^{[M]}$.

Hereby, $M = (M_k)$ is a (*strictly*) *regular* sequence (see Definition 3.1), while for the sequence $L = (L_k)$ we just assume $L_k \geq 1$ for all k .

In all these cases $\text{Diff } \mathcal{A}$ is a C^∞ (resp. $C^{[M]}$, see Section 3.2) regular Lie group, by [15] and [13], see also [14]. It thus follows that $\mathcal{G}_{\mathcal{A}}$ has a natural Lie group structure. In contrast to [15] and [13], where the vector fields were C^∞ (resp. $C^{[M]}$) in time t and hence the C^∞ (resp. $C^{[M]}$) exponential law applied, we are focused in this paper on vector fields which are just integrable in time.

For the unweighted spaces \mathcal{B} , $W^{\infty,p}$, \mathcal{S} , and \mathcal{D} we prove that the map which sends a time-dependent vector field $u \in L^1([0, 1], \mathcal{A}(\mathbb{R}^d, \mathbb{R}^d))$ to its flow $\Phi_u \in C([0, 1], \text{Diff } \mathcal{A})$ is continuous.

As an application we obtain that the scale of Bergman spaces on the polystrip with variable width is ODE-closed. More precisely: The L^p -Bergman space $A^p(S_{(r)})$ is the Banach space of p -integrable holomorphic functions on the polystrip $S_{(r)} := \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \text{Im}(z_i) < r, 1 \leq i \leq d\}$. We prove that the inductive limit

$$\underline{A}^p(\mathbb{R}^d) := \varinjlim_{r>0} A^p(S_{(r)})$$

is topologically isomorphic to the Sobolev–Denjoy–Carleman space $W^{\{\mathbf{1}\},p}(\mathbb{R}^d)$ of Roumieu type, where $\mathbf{1} = (1, 1, \dots)$. Similarly, the projective limit

$$\overline{A}^p(\mathbb{R}^d) := \varprojlim_{r>0} A^p(S_{(r)})$$

is topologically isomorphic to the Sobolev–Denjoy–Carleman space $W^{(\mathbf{1}),p}(\mathbb{R}^d)$ of Beurling type. This allows us to conclude that the flow of any time-dependent vector field $u : [0, 1] \rightarrow A^p(S_{(r)}, \mathbb{R}^d)$, for some $r > 0$, which is Bochner integrable in time is a continuous curve $t \mapsto \Phi_u(t)$ in $\text{Id} + \underline{A}^p(\mathbb{R}^d, \mathbb{R}^d)$. Analogously, if $u : [0, 1] \rightarrow \overline{A}^p(\mathbb{R}^d, \mathbb{R}^d)$ is integrable by seminorm, then $t \mapsto \Phi_u(t)$ is a continuous curve in $\text{Id} + \overline{A}^p(\mathbb{R}^d, \mathbb{R}^d)$.

The paper is organized as follows. We introduce notation and some necessary background in Section 2. The function spaces and diffeomorphism groups of interest in this paper are defined in Section 3. Our main result $\mathcal{G}_{\mathcal{A}} = \text{Diff}_0 \mathcal{A}$ is proved in Section 4. In Section 5 we show continuity of the flow map in the unweighted cases. The application for the scale of Bergman spaces on the polystrip with variable width is given in Section 6.

2. NOTATION AND PRELIMINARIES

In what follows $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{N}_{>0} = \mathbb{N}_{\geq 1} = \mathbb{N} \setminus \{0\}$. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| := \alpha_1 + \dots + \alpha_d$ and $\alpha! := \alpha_1! \cdots \alpha_d!$. By the multinomial theorem, for all $\alpha \in \mathbb{N}^d$,

$$(2.1) \quad |\alpha|! \leq d^{|\alpha|} \alpha!.$$

For a C^k -map $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$ and $|\alpha| \leq k$, $\partial^\alpha f$ denotes the α -th partial derivative of f . We also write ∂_x^α to make clear which variable is in the scope of differentiation.

For a C^k -function $f : U \rightarrow F$ defined on an open subset U of a Banach space E with values in some other Banach space F , we denote by $f^{(k)} = \partial^k f = d^k f : U \rightarrow L_k(E, F)$ the k -th Fréchet derivative, where $L_k(E, F)$ is the space of k -linear mappings $E^k \rightarrow F$ endowed with the operator norm.

For $f \in Z^{X \times Y}$ we consider $f^\vee \in (Z^Y)^X$ defined by $f^\vee(x)(y) = f(x, y)$, and with $g \in (Z^Y)^X$ we associate $g^\wedge \in Z^{X \times Y}$ given by $g^\wedge(x, y) = g(x)(y)$.

2.1. Sobolev spaces. For $k \in \mathbb{N}$ and $p \in [1, \infty]$, let $W^{k,p}(\mathbb{R}^d, \mathbb{R})$ denote the space of k -times weakly differentiable functions defined on \mathbb{R}^d with values in \mathbb{R} such that all partial derivatives up to order k lie in $L^p(\mathbb{R}^d, \mathbb{R})$. We endow $W^{k,p}(\mathbb{R}^d, \mathbb{R})$ with the norm

$$\|f\|_{W^{k,p}} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p}.$$

An important tool in the subsequent considerations is the following Sobolev inequality (e.g. [1]).

Lemma 2.1. *Let $k := \lfloor \frac{d}{p} \rfloor + 1$. Then $W^{k,p}(\mathbb{R}^d, \mathbb{R}) \subseteq C_b(\mathbb{R}^d, \mathbb{R})$ (where C_b denotes the space of bounded continuous functions) and there exists a constant $C = C(d, p)$ such that, for all $f \in W^{k,p}(\mathbb{R}^d, \mathbb{R})$,*

$$\|f\|_{L^\infty} \leq C \|f\|_{W^{k,p}}.$$

2.2. Faà di Bruno's formula. The Faà di Bruno formula is a generalization of the chain rule to higher order derivatives. The next proposition is a multivariable version, a proof can be found in [3, Proposition 4.3].

Proposition 2.2. *Let $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ and $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. We have, for all $\gamma \in \mathbb{N}^n \setminus \{0\}$,*

$$(2.2) \quad \frac{\partial^\gamma (f \circ g)(x)}{\gamma!} = \sum \frac{\alpha!}{k_1! \cdots k_\ell!} \frac{(\partial^\alpha f)(g(x))}{\alpha!} \left(\frac{\partial^{\delta_1} g(x)}{\delta_1!} \right)^{k_1} \cdots \left(\frac{\partial^{\delta_\ell} g(x)}{\delta_\ell!} \right)^{k_\ell},$$

where $\alpha = k_1 + \dots + k_\ell$ and the sum is taken over all sets $\{\delta_1, \dots, \delta_\ell\}$ of ℓ distinct elements of $\mathbb{N}^n \setminus \{0\}$ and all ordered ℓ -tuples $(k_1, \dots, k_\ell) \in (\mathbb{N}^m \setminus \{0\})^\ell$, $\ell = 1, 2, \dots$, such that $\gamma = \sum_{i=1}^\ell |k_i| \delta_i$.

We also need a version for composition of smooth functions between Banach spaces, cf. [21, Theorem 1].

Proposition 2.3. *Let $k \in \mathbb{N}_{\geq 1}$, E, F, G Banach spaces, $U \subseteq E$ and $V \subseteq F$ open subsets, and $f : U \rightarrow F$ and $g : V \rightarrow G$ C^k -functions with $f(U) \subseteq V$. Then, for $x \in U$,*

$$\frac{(f \circ g)^{(k)}(x)}{k!} = \text{sym}_k \left(\sum_{j=1}^k \sum_{\substack{\alpha \in \mathbb{N}_{\geq 1}^j \\ \alpha_1 + \dots + \alpha_j = k}} \frac{f^{(j)}(g(x))}{j!} \circ \left(\frac{g^{(\alpha_1)}(x)}{\alpha_1!} \times \cdots \times \frac{g^{(\alpha_j)}(x)}{\alpha_j!} \right) \right),$$

where sym_k denotes the symmetrization of k -linear mappings.

2.3. Integrability by seminorm. Integrability by seminorm is a notion of integration for functions with values in locally convex vector spaces. We briefly recall the necessary definitions and theorems. For a detailed treatment we refer to [4].

Let E be a locally convex vector space and \mathcal{P} the family of continuous seminorms on E . Let I be a bounded interval in \mathbb{R} . A function $f : I \rightarrow E$ is called *simple* if it is Lebesgue measurable and only takes finitely many values. It is called *measurable by seminorm* if for each $p \in \mathcal{P}$, there exists a nullset $N_p \subseteq I$ and a sequence $(f_n^p)_{n \in \mathbb{N}}$ of simple functions such that

$$p(f_n^p(t) - f(t)) \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } t \in I \setminus N_p.$$

It is actually enough to require that property for each $p \in \mathcal{P}_0$, where \mathcal{P}_0 is a subbase of the continuous seminorms. If N_p and (f_n^p) can be chosen independently from p , then f is called *strongly measurable*. Finally, f is called *weakly measurable* if $\ell \circ f : I \rightarrow \mathbb{R}$ is Lebesgue measurable for all $\ell \in E'$.

Theorem 2.4 ([4, Theorem 2.2]). *A function $f : I \rightarrow E$ is measurable by seminorm if and only if f is weakly measurable and for each $p \in \mathcal{P}$ there exists a nullset $N_p \subseteq I$ such that $f(I \setminus N_p)$ is separable.*

For a simple function $f : I \rightarrow E$, the integral over a measurable set $J \subseteq I$ is clearly defined as

$$\int_J f(t) dt := \sum_{y \in E} \lambda(f^{-1}(\{y\}) \cap J) \cdot y,$$

where λ is the Lebesgue measure on I . A function $f : I \rightarrow E$ which is measurable by seminorm is called *integrable by seminorm* if

$$\begin{aligned} \forall p \in \mathcal{P}, \forall n \in \mathbb{N} : p \circ (f_n^p - f) \in L^1(I), \\ \forall J \subseteq I \text{ measurable } \exists F_J \in E \forall p \in \mathcal{P} : p\left(F_J - \int_J f_n^p(t) dt\right) \xrightarrow{n \rightarrow \infty} 0; \end{aligned}$$

in this case $F_J =: \int_J f(t) dt$. For a complete locally convex space E , integrability of $p \circ f$ for each $p \in \mathcal{P}$ already implies integrability by seminorm. If E is a Banach space, then clearly integrability by seminorm coincides with Bochner integrability and the respective integrals coincide. It is easily seen that for each $p \in \mathcal{P}$,

$$p\left(\int_J f(t) dt\right) \leq \int_J p(f(t)) dt.$$

Let $\mathcal{L}^1(I, E)$ consist of functions $f : I \rightarrow E$ integrable by seminorm. For a continuous seminorm p on E , define $\tilde{p}(f) := \int_I p(f(t)) dt$ and let $\tilde{\mathcal{P}}$ be the family of such seminorms. Then $(\mathcal{L}^1(I, E), \tilde{\mathcal{P}})$ is a (non-Hausdorff) locally convex space. For $\mathcal{N} := \{f \in \mathcal{L}^1(I, E) : \tilde{p}(f) = 0 \text{ for all } \tilde{p} \in \tilde{\mathcal{P}}\}$, let

$$L^1(I, E) := \mathcal{L}^1(I, E) / \mathcal{N},$$

which is then clearly also Hausdorff.

3. FUNCTION SPACES AND THEIR ASSOCIATED DIFFEOMORPHISM GROUPS

Let us introduce the function spaces considered in this paper. We will define spaces $\mathcal{A}(\mathbb{R}^d, \mathbb{R})$ of real valued functions for different regularity classes \mathcal{A} , and set $\mathcal{A}(\mathbb{R}^d, \mathbb{R}^m) := (\mathcal{A}(\mathbb{R}^d, \mathbb{R}))^m$.

3.1. Classical spaces of test functions. For $p \in [1, \infty]$, we consider

$$W^{\infty,p}(\mathbb{R}^d, \mathbb{R}) := \bigcap_{k \in \mathbb{N}} W^{k,p}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \|f^{(\alpha)}\|_{L^p} < \infty \forall \alpha \in \mathbb{N}^d\}$$

and endow it with its natural Fréchet topology. For $p = \infty$ we also write $\mathcal{B}(\mathbb{R}^d, \mathbb{R}) := W^{\infty,\infty}(\mathbb{R}^d, \mathbb{R})$. We consider the Schwartz space

$$\mathcal{S}(\mathbb{R}^d, \mathbb{R}) := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \|f\|^{(p,\alpha)} < \infty \text{ for all } p \in \mathbb{N}, \alpha \in \mathbb{N}^d\},$$

where

$$\|f\|^{(p,\alpha)} := \sup_{x \in \mathbb{R}^d} (1 + |x|)^p |f^{(\alpha)}(x)|,$$

with its natural Fréchet topology. We denote by $\mathcal{D}(\mathbb{R}^d, \mathbb{R})$ the nuclear (LF)-space of smooth functions on \mathbb{R}^d with compact support.

3.2. Local Denjoy–Carleman classes. Let $M = (M_k)$ be a positive sequence. For an open subset $U \subseteq \mathbb{R}^d$ the *Denjoy–Carleman class* of *Beurling* type, denoted by $C^{(M)}(U, \mathbb{R})$, is the space of $f \in C^\infty(U, \mathbb{R})$ such that for all compact $K \subseteq U$ and all $\rho > 0$

$$\|f\|_{K,\rho}^M := \sup_{\alpha \in \mathbb{N}^d, x \in K} \frac{|f^{(\alpha)}(x)|}{\rho^{|\alpha|} |\alpha|! M_{|\alpha|!}} < \infty.$$

The Denjoy–Carleman class of *Roumieu* type $C^{\{M\}}(U, \mathbb{R})$ is the space of $f \in C^\infty(U, \mathbb{R})$ such that for all compact $K \subseteq U$ there exists $\rho > 0$ with $\|f\|_{K,\rho}^M < \infty$. They are endowed with their natural locally convex topologies. We write $C^{[M]}$ if we mean either $C^{(M)}$ or $C^{\{M\}}$.

Some regularity properties for the sequence $M = (M_k)$ guarantee that the fundamental results of analysis hold true for the classes $C^{[M]}$: stability under composition, differentiation, inversion, and solution of ODEs. This applies for sequences which satisfy the following definition. Here we do not strive for the utmost generality, but see [17] for a characterization of the stability properties.

Definition 3.1. We say that a positive sequence $M = (M_k)$ is *regular* if

- (1) $1 = M_0 \leq M_1 \leq M_2 \leq \dots$,
- (2) M is log-convex, i.e., $M_k^2 \leq M_{k-1} M_{k+1}$ for all k ,
- (3) M has moderate growth, i.e., there is $C > 0$ such that $M_{k+j} \leq C^{k+j} M_k M_j$ for all k, j .

It is easy to see that for a regular sequence M the Roumieu class $C^{\{M\}}$ contains the real analytic class C^ω . In the Beurling case, the inclusion $C^\omega \subsetneq C^{(M)}$ is equivalent to

- (4) $M_k^{1/k} \rightarrow \infty$ as $k \rightarrow \infty$, or equivalently $M_{k+1}/M_k \rightarrow \infty$.

A regular sequence $M = (M_k)$ which additionally satisfies (4) is called *strictly regular*.

Evidently, a regular sequence $M = (M_k)$ is *derivation closed*, i.e., there is a $C > 0$ such that $M_{k+1} \leq C^k M_k$ for all k . We say that a regular sequence $M = (M_k)$ is *quasianalytic* if

$$\sum_{k=1}^{\infty} \frac{1}{(k! M_k)^{1/k}} = \infty;$$

otherwise it is called *non-quasianalytic*. By the Denjoy–Carleman theorem, M is quasianalytic if and only if $C^{[M]}$ is quasianalytic, i.e., a function $f \in C^{[M]}(U)$,

where U is a connected open subset of \mathbb{R}^d , is uniquely determined by its Taylor series expansion at any point $a \in U$.

For regular sequences M the classes $C^{[M]}$ are stable under composition, indeed log-convexity implies the following inequality (cf. [3, Proposition 4.4])

$$(3.1) \quad M_1^k M_n \geq M_k M_1^{k_1} \cdots M_n^{k_n}, \quad \text{for } k_i \in \mathbb{N}, \quad \sum_{i=1}^n ik_i = n, \quad \sum_{i=1}^n k_i = k.$$

For later use we recall the following simple lemma.

Lemma 3.2 ([13, Lemma 2.2]). *Let $M = (M_k)$ satisfy (3.1) and let $A > 0$. Then there are positive constants B, C depending only on AM_1 , m , and n , and $C \rightarrow 0$ as $A \rightarrow 0$ such that*

$$\sum \frac{\alpha!}{k_1! \cdots k_\ell!} A^{|\alpha|} M_{|\alpha|} M_{|\delta_1|}^{k_1} \cdots M_{|\delta_\ell|}^{k_\ell} \leq BC^{|\gamma|} M_{|\gamma|}$$

where the sum is as in Proposition 2.2.

We refer to [10], [11], [12], [17], and [18] for a detailed exposition of the connection between these conditions on $M = (M_k)$ and the properties of $C^{[M]}$.

For regular sequences $M = (M_k)$, the classes $C^{[M]}$ can be extended to *convenient vector spaces* (i.e. Mackey-complete locally convex spaces, cf. [9]), and they then form cartesian closed categories. This has been developed in [10], [11], and [12].

3.3. Ultradifferentiable spaces of test functions. Let $M = (M_k)$ be a positive sequence. For $p \in [1, \infty]$ and $\sigma > 0$, we consider the Banach space

$$W_\sigma^{M,p}(\mathbb{R}^d, \mathbb{R}) := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \|f\|_\sigma^{M,p} < \infty\},$$

where

$$\|f\|_\sigma^{M,p} := \sup_{\alpha \in \mathbb{N}^d} \frac{\|f^{(\alpha)}\|_{L^p}}{\sigma^{|\alpha|} |\alpha|! M_{|\alpha|}}.$$

The corresponding *Beurling* class

$$W^{(M),p}(\mathbb{R}^d, \mathbb{R}) := \varprojlim_{n \in \mathbb{N}} W_{1/n}^{M,p}(\mathbb{R}^d, \mathbb{R}),$$

is a Fréchet space. The corresponding *Roumieu* class

$$W^{\{M\},p}(\mathbb{R}^d, \mathbb{R}) := \varinjlim_{n \in \mathbb{N}} W_n^{M,p}(\mathbb{R}^d, \mathbb{R}),$$

is a compactly regular (LB)-space, see [13, Lemma 4.9]. By writing $W^{[M],p}(\mathbb{R}^d, \mathbb{R})$ we mean either $W^{(M),p}(\mathbb{R}^d, \mathbb{R})$ or $W^{\{M\},p}(\mathbb{R}^d, \mathbb{R})$. For $p = \infty$, we also use $\mathcal{B}^{[M]}(\mathbb{R}^d, \mathbb{R}) := W^{[M],\infty}(\mathbb{R}^d, \mathbb{R})$ and

$$\|f\|_\sigma^M := \|f\|_\sigma^{M,\infty}.$$

Remark 3.3. The definition of $\mathcal{B}^{[M]}$ -mappings makes sense between arbitrary infinite dimensional Banach spaces E and F , or even an open subset $U \subseteq E$ of the domain. Then

$$\|f\|_{U,\sigma}^M := \sup_{k \in \mathbb{N}, x \in U} \frac{\|f^{(k)}(x)\|_{L_k(E,F)}}{\sigma^k k! M_k},$$

where $f^{(k)}$ denotes the k -th Fréchet derivative. The corresponding Beurling and Roumieu classes are defined analogously to the finite dimensional case. In the finite dimensional case both definitions yield the same function spaces and the respective norms are equivalent.

Let $L = (L_k)$ be another positive sequence. We consider the Banach space

$$\mathcal{S}_{L,\sigma}^M(\mathbb{R}^d, \mathbb{R}) := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \|f\|_\sigma^{L,M} < \infty\}.$$

with the norm

$$\|f\|_\sigma^{L,M} := \sup_{\substack{p \in \mathbb{N}, \alpha \in \mathbb{N}^d \\ x \in \mathbb{R}^d}} \frac{(1 + |x|)^p |f^{(\alpha)}(x)|}{\sigma^{p+|\alpha|} p! |\alpha|! L_p M_{|\alpha|}}.$$

The associated *Gelfand–Shilov* class of Beurling type is the Fréchet space

$$\mathcal{S}_{(L)}^{(M)}(\mathbb{R}^d, \mathbb{R}) := \varprojlim_{n \in \mathbb{N}} \mathcal{S}_{L,1/n}^M(\mathbb{R}^d, \mathbb{R}).$$

The Gelfand–Shilov class of Roumieu type

$$\mathcal{S}_{\{L\}}^{\{M\}}(\mathbb{R}^d, \mathbb{R}) := \varinjlim_{n \in \mathbb{N}} \mathcal{S}_{L,n}^M(\mathbb{R}^d, \mathbb{R})$$

is a compactly regular (LB)-space, see [13, Lemma 4.9]. By writing $\mathcal{S}_{[L]}^{[M]}(\mathbb{R}^d, \mathbb{R})$ we mean either $\mathcal{S}_{(L)}^{(M)}(\mathbb{R}^d, \mathbb{R})$ or $\mathcal{S}_{\{L\}}^{\{M\}}(\mathbb{R}^d, \mathbb{R})$.

Finally, we define

$$\mathcal{D}^{[M]}(\mathbb{R}^d, \mathbb{R}) := C^{[M]}(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{D}(\mathbb{R}^d, \mathbb{R}) = \mathcal{B}^{[M]}(\mathbb{R}^d, \mathbb{R}) \cap \mathcal{D}(\mathbb{R}^d, \mathbb{R})$$

which is non-trivial only if $M = (M_k)$ is non-quasianalytic. We equip $\mathcal{D}^{[M]}(\mathbb{R}^d, \mathbb{R})$ with the following topology,

$$\mathcal{D}^{[M]}(\mathbb{R}^d, \mathbb{R}) = \varinjlim_{K \in \mathbb{R}^d} \mathcal{D}_K^{[M]}(\mathbb{R}^d, \mathbb{R})$$

where

$$\mathcal{D}_K^{(M)}(\mathbb{R}^d, \mathbb{R}) := \varprojlim_{\ell \in \mathbb{N}} \mathcal{D}_{K,1/\ell}^M(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{D}_K^{\{M\}}(\mathbb{R}^d, \mathbb{R}) := \varinjlim_{\ell \in \mathbb{N}} \mathcal{D}_{K,\ell}^M(\mathbb{R}^d, \mathbb{R})$$

and

$$\mathcal{D}_{K,\rho}^M(\mathbb{R}^d, \mathbb{R}) := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \text{supp } f \subseteq K, \|f\|_\rho^M < \infty\}$$

is a Banach space. Then $\mathcal{D}^{(M)}(\mathbb{R}^d)$ is a (LFS)-space and $\mathcal{D}^{\{M\}}(\mathbb{R}^d)$ is a Silva space, see [7].

Remark 3.4. For $f = (f_1, \dots, f_m) \in W_\sigma^{M,p}(\mathbb{R}^d, \mathbb{R}^m)$, we also sometimes write $\|f\|_\sigma^{M,p}$ and actually mean $\max_{1 \leq i \leq m} \|f_i\|_\sigma^{M,p}$; similarly, for the other aforementioned function spaces. If domain and codomain are clear from the context, we sometimes omit mentioning them explicitly, e.g., $W^{[M],p}$ instead of $W^{[M],p}(\mathbb{R}^d, \mathbb{R}^m)$.

The next diagram taken from [13] describes the mutual inclusion relations of the above spaces. For $1 \leq p < q < \infty$, we have the following continuous inclusions:

$$\begin{array}{cccccccc} \mathcal{D} & \longrightarrow & \mathcal{S} & \longrightarrow & W^{\infty,p} & \longrightarrow & W^{\infty,q} & \longrightarrow & \mathcal{B} & \longrightarrow & C^\infty \\ \uparrow & & \uparrow \\ \mathcal{D}^{\{M\}} & \longrightarrow & \mathcal{S}_{\{L\}}^{\{M\}} & \longrightarrow & W^{\{M\},p} & \xrightarrow{*} & W^{\{M\},q} & \xrightarrow{*} & \mathcal{B}^{\{M\}} & \longrightarrow & C^{\{M\}} \\ \uparrow & & \uparrow \\ \mathcal{D}^{(M)} & \longrightarrow & \mathcal{S}_{(L)}^{(M)} & \longrightarrow & W^{(M),p} & \xrightarrow{*} & W^{(M),q} & \xrightarrow{*} & \mathcal{B}^{(M)} & \longrightarrow & C^{(M)} \end{array}$$

For the inclusions marked by $*$ we assume that $M = (M_k)$ is derivation closed. If the target is \mathbb{R} (or \mathbb{C}) then all spaces are algebras, provided that $(k!M_k)$ is log-convex in the weighted cases, and each space in

$$\mathcal{D}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d) \longrightarrow W^{\infty,p}(\mathbb{R}^d) \longrightarrow W^{\infty,q}(\mathbb{R}^d) \longrightarrow \mathcal{B}(\mathbb{R}^d)$$

is a $\mathcal{B}(\mathbb{R}^d)$ -module, and thus an ideal in each space on its right. Likewise each space in

$$\mathcal{D}^{[M]}(\mathbb{R}^d) \longrightarrow \mathcal{S}_{[L]}^{[M]}(\mathbb{R}^d) \longrightarrow W^{[M],p}(\mathbb{R}^d) \longrightarrow W^{[M],q}(\mathbb{R}^d) \longrightarrow \mathcal{B}^{[M]}(\mathbb{R}^d)$$

is a $\mathcal{B}^{[M]}(\mathbb{R}^d)$ -module, and thus an ideal in each space on its right.

3.4. Associated diffeomorphism groups. Let \mathcal{A} be any of the classes $\mathcal{B}, W^{\infty,p}, \mathcal{S}, \mathcal{D}, \mathcal{B}^{[M]}, W^{[M],p}, \mathcal{S}_{[L]}^{[M]}, \mathcal{D}^{[M]}$. Suppose that $M = (M_k)$ is a regular sequence, resp. strictly regular in the Beurling case, and let $L = (L_k)$ be a sequence with $L_k \geq 1$ for all k .

In [13] and [15] (see also [14]) it was shown that

$$\begin{aligned} \text{Diff } \mathcal{A} &:= \{ \Phi \in \text{Id} + \mathcal{A}(\mathbb{R}^d, \mathbb{R}^d) : \inf_{x \in \mathbb{R}^d} \det d\Phi(x) > 0 \} \\ &= \{ \Phi \in \text{Id} + \mathcal{A}(\mathbb{R}^d, \mathbb{R}^d) : \Phi \text{ is bijective, } \Phi^{-1} \in \text{Id} + \mathcal{A} \} \end{aligned}$$

is a manifold modelled on the open subset $\{ \Phi - \text{Id} : \Phi \in \text{Diff } \mathcal{A} \}$ of the convenient vector space $\mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$ with global chart $\Phi \mapsto \Phi - \text{Id}$ and actually that it is a smooth (resp. $C^{[M]}$) regular Lie group. We have C^∞ injective group homomorphisms

$$\text{Diff } \mathcal{D} \longrightarrow \text{Diff } \mathcal{S} \longrightarrow \text{Diff } W^{\infty,p} \xrightarrow{p < q} \text{Diff } W^{\infty,q} \longrightarrow \text{Diff } \mathcal{B}$$

and $C^{[M]}$ injective group homomorphisms

$$\text{Diff } \mathcal{D}^{[M]} \longrightarrow \text{Diff } \mathcal{S}_{[L]}^{[M]} \longrightarrow \text{Diff } W^{[M],p} \xrightarrow{p < q} \text{Diff } W^{[M],q} \longrightarrow \text{Diff } \mathcal{B}^{[M]},$$

where each group is a normal subgroup of the groups on its right.

Our main goal is now to give a different description of $\text{Diff } \mathcal{A}$ in terms of the so-called Trouvé group from image analysis, cf. [20] and [22]. This is outlined in the next section.

4. DIFFEOMORPHISM GROUPS GENERATED BY TIME-DEPENDENT VECTOR-FIELDS

4.1. ODE-closedness and the Trouvé group. Let I be some interval and u a time-dependent vector field $u : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ sufficiently regular, e.g., continuous in t and Lipschitz continuous in x with t -integrable global Lipschitz constant, to uniquely solve

$$(4.1) \quad x(t) = x_0 + \int_{s_0}^t u(s, x(s)) ds$$

for all $s_0, t \in I$ and $x_0 \in \mathbb{R}^d$.

Then one considers, for $t_0 \in I$, the *flow* $\Phi_u(s_0, t_0, x_0) := x(t_0)$ of u at time t_0 . From now on we assume, unless otherwise stated, that $I = [0, 1]$ and $s_0 = 0$ and simply write $\Phi_u(t_0, x_0)$ instead of $\Phi_u(0, t_0, x_0)$. In addition, we set $\phi_u(t_0, x_0) := \Phi_u(t_0, x_0) - x_0$.

In [22] it was shown that for $u^\vee \in L^1(I, C_0^1(\mathbb{R}^d, \mathbb{R}^d))$ the regularity with respect to the spatial variable is transferred to the flow in the sense that

$\Phi_u(t_0, \cdot) \in \text{Id} + C_0^1(\mathbb{R}^d, \mathbb{R}^d)$. In addition, $\Phi_u(t_0, \cdot)^{-1}$ exists and is again an element of $\text{Id} + C_0^1(\mathbb{R}^d, \mathbb{R}^d)$. This yields a way of constructing a multitude of diffeomorphisms of a certain type, namely such of the form $\text{Id} + C_0^1(\mathbb{R}^d, \mathbb{R}^d)$.

This way of regularity permanence with respect to ODEs is now generalized to arbitrary locally convex spaces E of maps $\mathbb{R}^d \rightarrow \mathbb{R}^d$ and dubbed *ODE-closedness*. To be more precise, let \mathcal{F} be a family of functions $\mathcal{F} \subseteq E^I$ containing all constant functions. Then E is called *\mathcal{F} -ODE-closed* if ϕ_{u^\wedge} exists and

$$\phi_{u^\wedge}(t, \cdot) \in E \text{ for all } u \in \mathcal{F}, t \in I.$$

If E is a Banach space, it is natural to choose \mathcal{F} to be the family of Bochner integrable functions $I \rightarrow E$. For general E , among the sensible choices for \mathcal{F} are functions integrable by seminorm, cf. Section 2.3.

If \mathcal{F} is a vector space and for $u, v \in \mathcal{F}$ the functions w_1, w_2 defined by

$$w_1(t) := \begin{cases} u(2t) & \text{if } t \in [0, 1/2] \\ v(2t - 1) & \text{if } t \in (1/2, 1] \end{cases} \quad \text{and} \quad w_2(t) := u(1 - t)$$

again belong to \mathcal{F} , then it is not hard to see that

$$\mathcal{G}_{\mathcal{F}} := \{\Phi_{u^\wedge}(1, \cdot) : u \in \mathcal{F}\}$$

is a group with respect to composition, the so-called *Trouvé group*; cf. [20] and [22]. In general not much is known about the Trouvé group. Clearly if E is \mathcal{F} -ODE-closed, then $\mathcal{G}_{\mathcal{F}} \subseteq \text{Id} + E$.

Remark 4.1. When it does not lead to confusion, we omit writing $(\cdot)^\vee$ and $(\cdot)^\wedge$, e.g., for $u : I \rightarrow E \subseteq (\mathbb{R}^d)^{\mathbb{R}^d}$ we simply write $u(t, x)$ instead of $u^\wedge(t, x)$. Similarly for functions of several variables $f : X_1 \times X_2 \rightarrow Y$, by writing $f(x_2)$, we actually mean $f(\cdot, x_2) : X_1 \rightarrow Y$. This notational inaccuracy will mainly occur, when writing $\Phi_u(t)$ instead of $\Phi_u(t, \cdot)$.

4.2. Admissible vector fields. Let \mathcal{A} be any of the classes $W^{\infty, p}$, \mathcal{S} , \mathcal{D} , $W^{[M], p}$, $\mathcal{S}_{[L]}^{[M]}$, $\mathcal{D}^{[M]}$, where $p \in [1, \infty]$ (in particular, the cases \mathcal{B} and $\mathcal{B}^{[M]}$ are included). Suppose that $M = (M_k)$ is a regular sequence, resp. strictly regular in the Beurling case, and let $L = (L_k)$ be a sequence with $L_k \geq 1$ for all k .

We will show that the Trouvé groups for the classes \mathcal{A} (with suitable choices of \mathcal{F}) coincide with the connected component of the identity in $\text{Diff } \mathcal{A}$ which we denote by $\text{Diff}_0 \mathcal{A}$. By the results of [13] and [15], see Section 3.4, we conclude that these Trouvé groups have a natural regular Lie group structure. So let us now fix $\mathcal{F}_{\mathcal{A}}$ for each \mathcal{A} mentioned above and write $\mathcal{G}_{\mathcal{A}}$ instead of $\mathcal{G}_{\mathcal{F}_{\mathcal{A}}}$ for the corresponding Trouvé group.

For $\mathcal{A} \in \{W^{\infty, p}, \mathcal{S}, W^{(M), p}, \mathcal{S}_{(L)}^{(M)}\}$, when $E := \mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$ is a Fréchet space, let $\mathcal{F}_{\mathcal{A}} := L^1(I, E)$ in the sense of Section 2.3. In particular, this means, for $u \in \mathcal{F}_{\mathcal{A}}$,

$$(4.2) \quad \forall \alpha \in \mathbb{N}^d : \int_0^1 \|\partial_x^\alpha u(t)\|_{L^p} dt < \infty, \quad \text{if } \mathcal{A} = W^{\infty, p},$$

$$(4.3) \quad \forall p \in \mathbb{N}, \forall \alpha \in \mathbb{N}^d : \int_0^1 \|u(t)\|^{(p, \alpha)} dt < \infty, \quad \text{if } \mathcal{A} = \mathcal{S},$$

$$(4.4) \quad \forall \sigma > 0 : \int_0^1 \|u(t)\|_\sigma^{M, p} dt < \infty, \quad \text{if } \mathcal{A} = W^{(M), p},$$

$$(4.5) \quad \forall \sigma > 0 : \int_0^1 \|u(t)\|_{\sigma}^{L,M} dt < \infty, \quad \text{if } \mathcal{A} = \mathcal{S}_{(L)}^{(M)}.$$

In the cases $\mathcal{A} \in \{W^{\{M\},p}, \mathcal{S}_{\{M\}}^{\{L\}}\}$, we take $\mathcal{F}_{\mathcal{A}}$ to be the set of those $u \in L^1(I, E)$ which factor into some step of the inductive limit which represents E and are (Bochner-)integrable therein. In particular,

$$(4.6) \quad \exists \sigma > 0 : \int_0^1 \|u(t)\|_{\sigma}^{M,p} dt < \infty, \quad \text{if } \mathcal{A} = W^{\{M\},p},$$

$$(4.7) \quad \exists \sigma > 0 : \int_0^1 \|u(t)\|_{\sigma}^{L,M} dt < \infty, \quad \text{if } \mathcal{A} = \mathcal{S}_{\{L\}}^{\{M\}}.$$

For the compactly supported classes $\mathcal{A} \in \{\mathcal{D}, \mathcal{D}^{(M)}\}$ we take $\mathcal{F}_{\mathcal{A}}$ to be the class of functions that map into some $\mathcal{D}_K(\mathbb{R}^d, \mathbb{R}^d)$, resp. $\mathcal{D}_K^{(M)}(\mathbb{R}^d, \mathbb{R}^d)$, and are integrable by seminorm therein. Then there exists a compact subset $K \subseteq \mathbb{R}^d$ such that $\text{supp } u(t) \subseteq K$ for all $t \in [0, 1]$, and

$$(4.8) \quad \forall \alpha \in \mathbb{N}^d : \int_0^1 \|\partial_x^{\alpha} u(t)\|_{L^{\infty}} dt < \infty, \quad \text{if } \mathcal{A} = \mathcal{D},$$

$$(4.9) \quad \forall \sigma > 0 : \int_0^1 \|u(t)\|_{\sigma}^M dt < \infty, \quad \text{if } \mathcal{A} = \mathcal{D}^{(M)}.$$

Finally in the Roumieu case $\mathcal{D}^{\{M\}}$, let $\mathcal{F}_{\mathcal{D}^{\{M\}}}$ be the class of functions that map into some $\mathcal{D}_{K,\sigma}^M(\mathbb{R}^d, \mathbb{R}^d)$ and are Bochner integrable therein. Then there exists a compact subset $K \subseteq \mathbb{R}^d$ such that $\text{supp } u(t) \subseteq K$ for all $t \in [0, 1]$ and

$$(4.10) \quad \exists \sigma > 0 : \int_0^1 \|u(t)\|_{\sigma}^M dt < \infty, \quad \text{if } \mathcal{A} = \mathcal{D}^{\{M\}}.$$

Remark 4.2. Since point evaluation is continuous on all spaces $\mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$, we get, by Theorem 2.4, that the function

$$I \rightarrow \mathbb{R}^d, \quad s \mapsto u(s, x),$$

is measurable for all $x \in \mathbb{R}^d$. Then it is straightforward to check that for all $\phi \in C(I, \mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the mapping

$$I \rightarrow \mathbb{R}^d, \quad s \mapsto u(s, x + \phi(s)),$$

is measurable; see [2, Lemma 2.2] for a detailed argument.

4.3. Main result.

Theorem 4.3. *For all classes \mathcal{A} and corresponding $\mathcal{G}_{\mathcal{A}}$ introduced in Section 4.2,*

$$\mathcal{G}_{\mathcal{A}} = \text{Diff}_0 \mathcal{A}.$$

For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{A}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{A}$ is continuous.

The proof of Theorem 4.3 is subdivided into several propositions. One inclusion can be proved uniformly for all classes \mathcal{A} . Abusing notation we shall denote by \mathcal{A} also the space $\mathcal{A}(\mathbb{R}^d, \mathbb{R}^d)$.

Proposition 4.4. $\mathcal{G}_{\mathcal{A}} \supseteq \text{Diff}_0 \mathcal{A}$.

Proof. Observe that $\text{Diff } \mathcal{A} - \text{Id}$ is open in \mathcal{A} , and hence so is $\text{Diff}_0 \mathcal{A} - \text{Id}$. Since $\text{Diff}_0 \mathcal{A} - \text{Id}$ is connected and locally path-connected, it is path-connected. Thus any $\Phi \in \text{Diff}_0 \mathcal{A}$ can be connected by a polygon with the identity.

Now take $\Phi = \text{Id} + \phi \in \text{Diff}_0 \mathcal{A}$ such that $\gamma(t) := \text{Id} + t\phi \in \text{Diff } \mathcal{A}$ for $0 \leq t \leq 1$. Since $\text{Diff } \mathcal{A}$ is a Lie group, $u(t) := \phi \circ \gamma(t)^{-1}$ is smooth. Hence in all cases u is in $\mathcal{F}_{\mathcal{A}}$ (in the Roumieu cases a smooth curve factors through a step in the defining inductive limit, since it is compactly regular or Silva). So $\Phi = \Phi_u(1, \cdot) \in \mathcal{G}_{\mathcal{A}}$.

For the general case, let $\Phi \in \text{Diff}_0 \mathcal{A}$. Take a polygon with vertices $\text{Id} = \Phi_1, \dots, \Phi_n = \Phi$ connecting Id with Φ . By the previous paragraph, we know already that $\Phi_2 \in \mathcal{G}_{\mathcal{A}}$. Now we may argue iteratively. By assumption, the segment between Φ_2 and Φ_3 lies in $\text{Diff } \mathcal{A}$ and therefore also the segment between Id and $\Phi_3 \circ \Phi_2^{-1}$ which shows that $\Phi_3 \circ \Phi_2^{-1} \in \mathcal{G}_{\mathcal{A}}$. Since $\mathcal{G}_{\mathcal{A}}$ is a group, we may conclude that $\Phi_3 \in \mathcal{G}_{\mathcal{A}}$. Applying the same argument for the remaining vertices finally shows that $\Phi \in \mathcal{G}_{\mathcal{A}}$. \square

The other inclusion $\mathcal{G}_{\mathcal{A}} \subseteq \text{Diff}_0 \mathcal{A}$ follows from the following assertions:

- (•) For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{A}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{A}$ is continuous.
- (••) $\inf_{x \in \mathbb{R}^d} \det d\Phi_u(t, x) > 0$ for all $t \in I$.

Let us check (••). Observe that there is some $v \in \mathcal{F}_{\mathcal{A}}$ such that $\Phi_u(t)^{-1} = \Phi_v(t)$ for all t . In particular, $\Phi_u(t)^{-1}$ is continuously differentiable on \mathbb{R}^d and the first derivative is invertible at any point $x \in \mathbb{R}^d$. For fixed $x \in \mathbb{R}^d$, the mapping $t \mapsto \gamma(t) := \det d\Phi_u(t, x)$ is continuous and has values in $\mathbb{R} \setminus \{0\}$. Since $\gamma(0) = 1$, it thus follows that $\gamma(t) = \det d\Phi_u(t, x) > 0$ for all $x \in \mathbb{R}^d$ and $t \in I$. Now suppose there exists a sequence x_n in \mathbb{R}^d such that $\det d\Phi_u(t, x_n) \rightarrow 0$ as $n \rightarrow \infty$. This would imply the existence of a sequence y_n such that $\det d\Phi_v(t, y_n) \rightarrow \infty$. But this contradicts (•), since all spaces \mathcal{A} are contained in \mathcal{B} .

The rest of the section is devoted to prove (•). Depending on \mathcal{A} , the techniques will be different.

4.4. Ultradifferentiable classes. First we treat the weighted Roumieu-type spaces.

Proposition 4.5. *Let $M = (M_k)$ be a regular sequence. For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{B}^{\{M\}}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{B}^{\{M\}}$ is continuous.*

Proof. Take $u \in \mathcal{F}_{\mathcal{B}^{\{M\}}}$. Due to (4.6), there exists $\rho > 0$ such that $\int_0^1 \|u(t)\|_{\rho}^M dt < \infty$. We show first that $t \mapsto \Phi_u(t, \cdot)$ has values in $\text{Id} + \mathcal{B}^{\{M\}}$ and later on continuity in t .

Let $\delta > 0$, $t_0 \in I$ and set $J := I \cap [t_0 - \delta, t_0 + \delta]$. For $\phi \in C(J, \mathbb{R}^d)$ and $x \in \mathbb{R}^d$, define a mapping $T : C(J, \mathbb{R}^d) \times \mathbb{R}^d \rightarrow C(J, \mathbb{R}^d)$ by setting

$$T(\phi, x)(t) := \int_{t_0}^t u(s, x + \phi(s)) ds.$$

We claim that $T(\cdot, x)$ is Lipschitz with Lipschitz constant less than 1 if δ is chosen sufficiently small. For,

$$|T(\phi, x)(t) - T(\psi, x)(t)| = \left| \int_{t_0}^t u(s, x + \phi(s)) - u(s, x + \psi(s)) ds \right|$$

$$\begin{aligned}
&\leq \int_{t_0}^t |u(s, x + \phi(s)) - u(s, x + \psi(s))| ds \\
&\leq \rho M_1 \int_{t_0}^t \|u(s)\|_\rho^M |\phi(s) - \psi(s)| ds \\
&\leq \rho M_1 \int_J \|u(s)\|_\rho^M ds \|\phi - \psi\|_{L^\infty},
\end{aligned}$$

and $\int_J \|u(s)\|_\rho^M ds$ gets arbitrarily small (independent of t_0) by choosing δ small enough. So let δ be chosen such that

$$(4.11) \quad \max(1, \rho) M_1 \int_J \|u(s)\|_\rho^M ds \leq 1/2.$$

Then, for each fixed x , $T(\cdot, x)$ is a contraction on $C(J, \mathbb{R}^d)$ and has a unique fixed point $\phi(x) \in C(J, \mathbb{R}^d)$, i.e., for all $|t - t_0| \leq \delta$,

$$(4.12) \quad \phi(x)(t) = T(\phi(x), x)(t) = \int_{t_0}^t u(s, x + \phi(x)(s)) ds.$$

Patching the solutions together appropriately (this will be carried out in more detail later, see Claim 3), we get a solution of (4.1) on I . Thus we can define, for each $t \in I$, $\Phi_u(t) = \text{Id} + \phi_u(t)$ and we have to show that

$$\phi_u(t) \in \mathcal{B}^{\{M\}}(\mathbb{R}^d, \mathbb{R}^d) \quad \text{for all } t \in I.$$

To this end we need some additional observations.

Claim 1. $T \in C^\infty(C(J, \mathbb{R}^d) \times \mathbb{R}^d, C(J, \mathbb{R}^d))$ and for $k \geq 0$ and $\alpha \in \mathbb{N}^d$, we get

$$(4.13) \quad \partial_\phi^k \partial_x^\alpha T(\phi, x)(\psi_1, \dots, \psi_k)(t) = \int_{t_0}^t d^k \partial_x^\alpha u(s, x + \phi(s))(\psi_1(s), \dots, \psi_k(s)) ds,$$

where ∂_ϕ^k denotes the k -th (Fréchet-)derivative with respect to $\phi \in C(J, \mathbb{R}^d)$, ∂_x^α the α -th partial derivative with respect to $x \in \mathbb{R}^d$, d^k the k -th total derivative with respect to the second variable (in \mathbb{R}^d).

We illustrate the inductive start, the induction step works along similar lines. For the existence (and continuity) of the first derivative it is enough to show existence and continuity of the first partial derivatives. Fix $x \in \mathbb{R}^d$ and $\phi, \psi \in C(J, \mathbb{R}^d)$. Then, for all $t \in J$,

$$\begin{aligned}
&|T(\phi + \psi, x)(t) - T(\phi, x)(t) - \int_{t_0}^t du(s, x + \phi(s)) \cdot \psi(s) ds| \\
&\leq \int_{t_0}^t |u(s, x + \phi(s) + \psi(s)) - u(s, x + \phi(s)) - du(s, x + \phi(s)) \cdot \psi(s)| ds \\
&\leq \int_{t_0}^t \sup_{y \in \mathbb{R}^d} \|d^2 u(s, y)\|_{L_2(\mathbb{R}^d, \mathbb{R}^d)} |\psi(s)|^2 ds \\
&\leq 2\rho^2 M_2 \|\psi\|_{L^\infty}^2 \int_0^1 \|u(s)\|_\rho^M ds.
\end{aligned}$$

Therefore $\partial_\phi T(\phi, x)(\psi)(t) = \int_{t_0}^t du(s, x + \phi(s)) \cdot \psi(s) ds$, which is easily seen to be continuous as a mapping from $C(J, \mathbb{R}^d) \times \mathbb{R}^d \rightarrow L(C(J, \mathbb{R}^d), C(J, \mathbb{R}^d))$. The existence and continuity of the derivative with respect to x can be proven similarly.

This implies (4.13) for $k = 1$ and $|\alpha| = 1$. Now proceed by induction to finish the proof of Claim 1.

Claim 2. $T \in \mathcal{B}^{\{M\}}(C(J, \mathbb{R}^d) \times \mathbb{R}^d, C(J, \mathbb{R}^d))$.

Using (4.13), we get for $\|\psi_i\|_{L^\infty} \leq 1$,

$$\|\partial_\phi^k \partial_x^\alpha T(\phi, x)(\psi_1, \dots, \psi_k)\|_{L^\infty} \leq \rho^{k+|\alpha|} (k+|\alpha|)! M_{k+|\alpha|} \int_J \|u(s)\|_\rho^M ds$$

which gives $\|T\|_{C(J, \mathbb{R}^d) \times \mathbb{R}^d, \rho}^M \leq 1/2$, by (4.11), and thus proves Claim 2. In particular,

$$(4.14) \quad \|\partial_\phi T(\phi, x)\|_{L(C(J, \mathbb{R}^d), C(J, \mathbb{R}^d))} \leq 1/2, \quad \phi \in C(J, \mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

Consider the mapping

$$S : C(J, \mathbb{R}^d) \times \mathbb{R}^d \rightarrow C(J, \mathbb{R}^d), \quad (\phi, x) \mapsto \phi - T(\phi, x).$$

Then $\partial_\phi S(\phi, x) = \text{Id} - \partial_\phi T(\phi, x) \in L(C(J, \mathbb{R}^d), C(J, \mathbb{R}^d))$ admits a bounded inverse due to (4.14). The inverse is given by the Neumann series $\sum_{k=0}^{\infty} \partial_\phi T(\phi, x)^k$, which thus gives

$$\|\partial_\phi S(\phi, x)^{-1}\|_{L(C(J, \mathbb{R}^d), C(J, \mathbb{R}^d))} \leq 2.$$

For all $(k, \alpha) \in \mathbb{N}^{1+d}$ except $(0, 0, \dots, 0)$ and $(1, 0, \dots, 0)$, we get

$$\|\partial_\phi^k \partial_x^\alpha S(\phi, x)\| \leq \rho^{k+|\alpha|} M_{k+|\alpha|} (k+|\alpha|)! \|T\|_{C(J, \mathbb{R}^d) \times \mathbb{R}^d, \rho}^M,$$

and for $(1, 0, \dots, 0)$ we clearly have

$$\|\partial_\phi S(\phi, x)\| \leq 1 + \rho M_1 \|T\|_{C(J, \mathbb{R}^d) \times \mathbb{R}^d, \rho}^M.$$

Since M is a regular sequence, we can apply the implicit function theorem for the class $\mathcal{B}^{\{M\}}$, cf. [21, Theorem 3]. Thus, $x \mapsto \phi(x)$ fulfills the $\mathcal{B}^{\{M\}}$ -estimates for derivatives of order ≥ 1 . Moreover, (4.12) and $\int_0^1 \|u(t)\|_{L^\infty} dt < \infty$ imply that $x \mapsto \phi(x)$ is globally bounded. So there exists (for each J) a $\tau > 0$ such that $\phi \in \mathcal{B}_\tau^M(\mathbb{R}^d, C(J, \mathbb{R}^d))$. Since we can cover $[0, 1]$ by finitely many J , we can choose one τ valid for all J . Observe however that ϕ (the fixed point of T) depends on the interval J .

Claim 3. *There is a global solution ϕ ,*

$$(4.15) \quad \phi(x)(t) = \int_0^t u(s, x + \phi(x)(s)) ds, \quad t \in I,$$

and there exists $\lambda > 0$ such that $\phi(\cdot)(t) \in \mathcal{B}_\lambda^M(\mathbb{R}^d, \mathbb{R}^d)$ and it is uniformly bounded therein with respect to $t \in I$.

At this point, we know that, for each $J_k := [(k-1)\delta, k\delta]$, there exists $\phi_k \in \mathcal{B}_\tau^M(\mathbb{R}^d, C(J_k, \mathbb{R}^d))$ such that

$$\phi_k(x)(t) = \int_{(k-1)\delta}^t u(s, x + \phi_k(x)(s)) ds, \quad t \in J_k, \quad x \in \mathbb{R}^d.$$

Since ev_t is a bounded linear operator on $C(J, \mathbb{R}^d)$, we have $\partial^l(\text{ev}_t \circ \phi_k)(x) = \text{ev}_t \partial^l \phi_k(x)$, and therefore $x \mapsto \phi_k(x)(t) = \text{ev}_t \circ \phi_k(x) \in \mathcal{B}_\tau^M(\mathbb{R}^d, \mathbb{R}^d)$ for any $t \in J_k$. Now we can define ϕ iteratively: For $t \in J_1$ and $x \in \mathbb{R}^d$, set $\phi(x)(t) := \phi_1(x)(t)$.

Suppose we have already defined ϕ on $[0, k\delta]$. Then set $y_k(x) := x + \phi(x)(k\delta) \in \text{Id} + \mathcal{B}^{\{M\}}(\mathbb{R}^d, \mathbb{R}^d)$. And for $t \in J_{k+1}$, let

$$\phi(x)(t) := \phi_{k+1}(y_k(x))(t).$$

Using composition-closedness of $\mathcal{B}^{\{M\}}$ (cf. [13, Theorem 6.1]), it is clear that $\phi(\cdot)(t) \in \mathcal{B}^{\{M\}}(\mathbb{R}^d, \mathbb{R}^d)$ for $t \in [0, (k+1)\delta]$. Iterating this procedure sufficiently many times, finally gives (4.15) and the construction also yields the additional boundedness condition. Claim 3 is proved.

We are left to prove continuity of $t \mapsto \Phi_u(t)$ as a mapping into $\text{Id} + \mathcal{B}^{\{M\}}$.

Claim 4. *There exists $\sigma \geq \rho$ such that $t \mapsto \phi_u(t)$ is continuous into $\mathcal{B}_\sigma^M(\mathbb{R}^d, \mathbb{R}^d)$.*

By Claim 3, there exists $\lambda > 0$ such that $\phi_u(t)$ is bounded (uniformly in t) in $\mathcal{B}_\lambda^M(\mathbb{R}^d, \mathbb{R}^d)$ by some constant C . And we have for $0 \leq r \leq t \leq 1$, $\gamma \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$,

$$(4.16) \quad \frac{|\partial_x^\gamma \phi_u(t, x) - \partial_x^\gamma \phi_u(r, x)|}{\sigma^{|\gamma|} |\gamma|! M_{|\gamma|}} \leq \int_r^t \frac{|\partial_x^\gamma (u(s, \Phi_u(s, x)))|}{\sigma^{|\gamma|} |\gamma|! M_{|\gamma|}} ds,$$

where we used (4.6) to justify interchanging differentiation and integration. Next we apply Faà di Bruno's formula (2.2) to the integrand and get

$$\begin{aligned} \frac{\|\partial_x^\gamma (u(s) \circ \Phi_u(s))\|_{L^\infty(\mathbb{R}^d)}}{\gamma! M_{|\gamma|}} &\leq \sum \frac{\alpha!}{k_1! \dots k_l!} \frac{\|\partial_x^\alpha u(s)\|_{L^\infty(\mathbb{R}^d)}}{\alpha! M_{|\alpha|}} \\ &\quad \cdot \left(\frac{\|\partial_x^{\delta_1} \Phi_u(s)\|_{L^\infty(\mathbb{R}^d)}}{\delta_1! M_{|\delta_1|}} \right)^{|k_1|} \dots \left(\frac{\|\partial_x^{\delta_l} \Phi_u(s)\|_{L^\infty(\mathbb{R}^d)}}{\delta_l! M_{|\delta_l|}} \right)^{|k_l|} \\ &\leq \sum \frac{\alpha!}{k_1! \dots k_l!} (d\rho(C+1))^{|\alpha|} (d\lambda)^{|\gamma|} \frac{\|\partial_x^\alpha u(s)\|_{L^\infty(\mathbb{R}^d)}}{\rho^{|\alpha|} |\alpha|! M_{|\alpha|}} \\ &\leq \sup_{\beta \in \mathbb{N}^d} \frac{\|\partial_x^\beta u(s)\|_{L^\infty(\mathbb{R}^d)}}{\rho^{|\beta|} |\beta|! M_{|\beta|}} (d\lambda)^{|\gamma|} \sum \frac{\alpha!}{k_1! \dots k_l!} (d\rho(C+1))^{|\alpha|}. \end{aligned}$$

where the summation is as in (2.2). For the first inequality we used (3.1), the second one follows from the boundedness condition and (2.1). The sum on the right-hand side is $\leq D\tau^{|\gamma|}$, for some $D, \tau > 0$, by Lemma 3.2. Choosing $\sigma = d\lambda\tau$ and plugging this into the right-hand side of (4.16) implies Claim 4.

The proof of the proposition is complete. \square

Proposition 4.6. *Let $M = (M_k)$ be a regular sequence and $p \in [1, \infty)$. For each time-dependent vector field $u \in \mathcal{F}_{W^{\{M\}, p}}$ the flow $t \mapsto \phi_u(t) \in W^{\{M\}, p}$ is continuous.*

Proof. Let $u \in \mathcal{F}_{W^{\{M\}, p}}$, i.e., there exists $\rho > 0$ such that $\int_0^1 \|u(t)\|_\rho^{M, p} dt < \infty$. Then $u \in \mathcal{F}_{\mathcal{B}^{\{M\}}}$, by Lemma 2.1, and thus by Proposition 4.5, $I \ni t \mapsto \Phi_u(t)$ is a curve of diffeomorphisms in $\text{Diff } \mathcal{B}^{\{M\}}$. Therefore, there is $B > 0$ such that

$$(4.17) \quad \begin{aligned} \int_{\mathbb{R}^d} |\partial_x^\alpha u(s, \Phi_u(s, x))|^p dx &= \int_{\mathbb{R}^d} \frac{|\partial_x^\alpha u(s, y)|^p}{|\det d\Phi_u(s, \Phi_u^{-1}(s, y))|} dy \\ &\leq B \int_{\mathbb{R}^d} |\partial_x^\alpha u(s, y)|^p dy. \end{aligned}$$

Using this, we may argue analogously as in the end of the proof of Proposition 4.5. First we apply Minkowski's integral inequality to get an analogue of (4.16). For $0 \leq r \leq t \leq 1$ and $\sigma \geq \rho$,

$$(4.18) \quad \sup_{\gamma} \frac{\|\partial_x^\gamma \phi_u(t) - \partial_x^\gamma \phi_u(r)\|_{L^p(\mathbb{R}^d)}}{\sigma^{|\gamma|} |\gamma|! M_{|\gamma|}} \leq \int_r^t \sup_{\gamma} \frac{\|\partial_x^\gamma (u(s) \circ \Phi_u(s))\|_{L^p(\mathbb{R}^d)}}{\sigma^{|\gamma|} |\gamma|! M_{|\gamma|}} ds.$$

For the integrand on the right-hand side, we use the Faà di Bruno formula (2.2) and (4.17) to get

$$\begin{aligned} \frac{\|\partial_x^\gamma (u(s) \circ \Phi_u(s))\|_{L^p(\mathbb{R}^d)}}{\gamma! M_{|\gamma|}} &\leq B^{1/p} \sum \frac{\alpha!}{k_1! \dots k_l!} \frac{\|\partial_x^\alpha u(s)\|_{L^p(\mathbb{R}^d)}}{\alpha! M_{|\alpha|}} \\ &\quad \cdot \left(\frac{\|\partial_x^{\delta_1} \Phi_u(s)\|_{L^\infty(\mathbb{R}^d)}}{\delta_1! M_{|\delta_1|}} \right)^{|k_1|} \dots \left(\frac{\|\partial_x^{\delta_l} \Phi_u(s)\|_{L^\infty(\mathbb{R}^d)}}{\delta_l! M_{|\delta_l|}} \right)^{|k_l|}, \end{aligned}$$

and then complete the proof as in Proposition 4.5. \square

Proposition 4.7. *Let $M = (M_k)$ be a regular sequence and $L = (L_k)$ satisfy $L_k \geq 1$. For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{S}_{\{L\}}^{\{M\}}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{S}_{\{L\}}^{\{M\}}$ is continuous.*

Proof. Let $u \in \mathcal{F}_{\mathcal{S}_{\{L\}}^{\{M\}}}$ and let σ be as in (4.7). Then clearly $u \in \mathcal{F}_{\mathcal{B}^{\{M\}}}$, it maps into the step \mathcal{B}_σ^M , and is integrable therein. Due to Proposition 4.5, we thus know that $t \mapsto \Phi_u(t)$ is continuous into $\text{Diff } \mathcal{B}^{\{M\}}$. In particular, there is $\rho > 0$ and $C_1 > 0$ such that

$$(4.19) \quad \sup_{t \in I} \|\phi_u(t)\|_\rho^M \leq C_1.$$

For $0 \leq r \leq t \leq 1$, we have

$$\frac{(1 + |x|)^p |\partial_x^\gamma \phi_u(t, x) - \partial_x^\gamma \phi_u(r, x)|}{p! |\gamma|! L_p M_{|\gamma|}} \leq \int_r^t \frac{(1 + |x|)^p |\partial_x^\gamma u(s) \circ \Phi_u(s, x)|}{p! |\gamma|! L_p M_{|\gamma|}} ds.$$

By Faà di Bruno's formula (2.2) and (4.19) (replacing C_1 by $1 + C_1$),

$$\frac{|\partial_x^\gamma (u(s) \circ \Phi_u(s, x))|}{|\gamma|! M_{|\gamma|}} \leq (d\rho)^{|\gamma|} \sum \frac{\alpha!}{k_1! \dots k_l!} (dC_1)^{|\alpha|} \frac{|\partial_x^\alpha u(s)(\Phi_u(s, x))|}{|\alpha|! M_{|\alpha|}}.$$

By (4.19), $(1 + |x|)(1 + |\Phi_u(s, x)|)^{-1}$ is uniformly bounded, say by C_2 , in s and x . Thus,

$$\begin{aligned} \frac{(1 + |x|)^p |\partial_x^\alpha u(s)(\Phi_u(s, x))|}{p! |\alpha|! L_p M_{|\alpha|}} &\leq C_2^p \frac{(1 + |\Phi_u(s, x)|)^p |\partial_x^\alpha u(s)(\Phi_u(s, x))|}{p! |\alpha|! L_p M_{|\alpha|}} \\ &\leq C_2^p \sigma^{p+|\alpha|} \|u(s)\|_\sigma^{L, M}. \end{aligned}$$

By Lemma 3.2, there are $C_3, C_4 > 0$ such that $\sum \frac{\alpha!}{k_1! \dots k_l!} (dC_1 \sigma)^{|\alpha|} \leq C_3 C_4^{|\gamma|}$. For $\tau = \max\{C_2 \sigma, C_4 d\rho\}$ we hence obtain

$$\|\phi_u(t, \cdot) - \phi_u(r, \cdot)\|_\tau^{L, M} \leq C_3 \int_r^t \|u(s)\|_\sigma^{L, M} ds,$$

which shows continuity of $t \mapsto \phi_u(t) \in \mathcal{S}_{\{L\}}^{\{M\}}$ since $s \mapsto \|u(s)\|_\sigma^{L, M}$ is integrable. \square

Proposition 4.8. *Let $M = (M_k)$ be a regular non-quasianalytic sequence. For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{D}^{\{M\}}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{D}^{\{M\}}$ is continuous.*

Proof. There is $R > 0$ such that $\bigcup_{t \in I} \text{supp } u(t)$ is contained in the ball $B_R(0)$ with center $0 \in \mathbb{R}^d$ and radius R . It is easily seen that $\bigcup_{t \in I} \text{supp } \phi_u(t) \subseteq B_{R+M}(0)$, where $M = \sup_{t \in I} \|\phi_u(t)\|_{L^\infty(\mathbb{R}^d)}$. The rest follows from Proposition 4.5. \square

Next we treat the Beurling analogues of the above spaces.

Proposition 4.9. *Let $M = (M_k)$ be a regular sequence and $p \in [1, \infty]$. For each time-dependent vector field $u \in \mathcal{F}_{W^{(M)}, p}$ the flow $t \mapsto \phi_u(t) \in W^{(M), p}$ is continuous.*

Proof. First let $p = \infty$. Take $u \in \mathcal{F}_{\mathcal{B}^{(M)}}$ and fix some $\rho_0 > 0$. Then u is integrable in $\mathcal{B}_{\rho_0}^M$. The proof of Proposition 4.5 (see Claim 3) implies that there are $C, \lambda > 0$ depending on ρ_0 such that $\sup_{t \in I} \|\varphi_u(t)\|_\lambda^M \leq C$. In the estimate of the integrand in (4.16) now take any $\rho < \rho_0$ and observe that the τ tends to 0 as ρ tends to 0, by Lemma 3.2. Therefore the σ may be chosen arbitrarily small, which yields continuity as a map into $\mathcal{B}^{(M)}$. The case $p < \infty$ works analogously. \square

Proposition 4.10. *Let $M = (M_k)$ be a regular sequence and let $L = (L_k)$ satisfy $L_k \geq 1$. For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{S}_{(L)}^{(M)}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{S}_{(L)}^{(M)}$ is continuous.*

Proof. Just repeat the proof of Proposition 4.7 and observe that for any choice of small $\sigma > 0$ in the beginning of the proof, ρ and C_1 can be chosen uniformly, and thus the τ in the end of the proof gets small as well. \square

Proposition 4.11. *Let $M = (M_k)$ be a regular non-quasianalytic sequence. For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{D}^{(M)}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{D}^{(M)}$ is continuous.*

Proof. That $\bigcup_{t \in I} \text{supp } \phi_u(t)$ is bounded follows as in Proposition 4.8. The rest follows from Proposition 4.9. \square

Remark 4.12. Let us sketch an alternative proof of Proposition 4.9, Proposition 4.10, and Proposition 4.11 for *strictly* regular M : First consider the case $\mathcal{B}^{(M)}$. Let $u \in \mathcal{F}_{\mathcal{B}^{(M)}}$ and set

$$L_k := \frac{1}{k!} \sum_{|\alpha|=k} \int_0^1 \|\partial_x^\alpha u(t, \cdot)\|_{L^\infty} dt.$$

By [8, Lemma 6], there is a strictly regular sequence $N \geq L$ such that $(N_k/M_k)^{1/k} \rightarrow 0$. Hence, Proposition 4.5 implies that $t \mapsto \phi_u(t)$ is a continuous map $I \rightarrow \mathcal{B}^{\{N\}}$, and thus a continuous map $I \rightarrow \mathcal{B}^{(M)}$, since $(N_k/M_k)^{1/k} \rightarrow 0$ entails that $\mathcal{B}^{\{N\}}$ is continuously included in $\mathcal{B}^{(M)}$.

If $p < \infty$ and $u \in \mathcal{F}_{W^{(M)}, p}$, then $t \mapsto \phi_u(t)$ is a continuous map $I \rightarrow \mathcal{B}^{(M)}$, by the previous paragraph. That $t \mapsto \phi_u(t)$ actually has values in $W^{(M), p}$ follows easily from

$$\phi_u(t) = \int_0^t u(s) \circ \Phi_u(s) ds,$$

and its continuity as map $I \rightarrow W^{(M), p}$ is shown similarly, since for $t \geq r$,

$$\phi_u(t) - \phi_u(r) = \int_r^t u(s) \circ \Phi_u(s) ds.$$

The case $\mathcal{S}_{(L)}^{(M)}$ is treated analogously and for $\mathcal{D}^{(M)}$ only a condition for the support has to be checked.

4.5. Unweighted classes.

Proposition 4.13. *Let \mathcal{A} be any of the classes $W^{\infty,p}$, for $1 \leq p \leq \infty$, \mathcal{S} , and \mathcal{D} . For each time-dependent vector field $u \in \mathcal{F}_{\mathcal{A}}$ the flow $t \mapsto \phi_u(t) \in \mathcal{A}$ is continuous.*

Proof. We may assume that there exists a unique continuous map ϕ_u such that

$$(4.20) \quad \phi_u(t, x) = \int_0^t u(s, x + \phi_u(s, x)) ds, \quad x \in \mathbb{R}^d, \quad t \in [0, 1].$$

Let us consider the case $\mathcal{A} = \mathcal{B}$. That $\phi_u(t)$ is uniformly bounded with respect to t in $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ is the content of [22, Theorem 8.9], at least for vector fields u vanishing at infinity together with all derivatives. But the proofs can be adjusted to work for the larger class of time-dependent \mathcal{B} -vector fields. Nevertheless we recall a proof of the uniform boundedness.

It is clear, from (4.20) and $\int_0^1 \|u(s)\|_{L^\infty} ds < \infty$, that $\phi_u(t)$ is globally bounded, uniformly for all t . In order to show that $\partial_x^k \phi_u(t)$ is globally bounded, uniformly in t , for each k , we use induction on k . So suppose that we already know that $\partial_x^h \phi_u(t)$ is globally bounded for each t and each $h < k$. Recall that $\Phi(s) = \Phi_u(s) = \text{Id} + \phi_u(s)$. By Proposition 2.3,

$$\frac{\partial_x^k (u(s) \circ \Phi(s))(x)}{k!} = (\partial_x u(s))(\Phi(s)(x)) \frac{\partial_x^k \Phi(s)(x)}{k!} + R(s, x),$$

where $R(s, x)$ is the rest of the Faà di Bruno formula which involves only derivatives $\partial_x^j u(s)$ of order $j \geq 2$ and derivatives $\partial_x^h \Phi_u(t)$ of order $h \leq k-1$. Using the induction hypothesis it is easy to see that

$$\int_0^1 \|R(s, x)\|_{L_k} ds =: C < \infty.$$

With (4.20) we have

$$\|\partial_x^k \phi_u(t, x)\|_{L_k} \leq \int_0^t \|(\partial_x u)(s, x + \phi_u(s, x))\|_{L_1} \frac{\|\partial_x^k (x + \phi_u(s, x))\|_{L_k}}{k!} ds + C.$$

Then Gronwall's inequality implies that $\partial_x^k \phi_u(t)$ is globally bounded, uniformly in t . That $t \mapsto \phi_u(t) \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ is continuous is easily shown in a manner similar to the proof of Claim 4 in Proposition 4.5 (the proof actually simplifies since each derivative can be treated separately).

The remaining cases $\mathcal{A} = W^{\infty,p}$, for $p < \infty$, $\mathcal{A} = \mathcal{S}$, and $\mathcal{A} = \mathcal{D}$ can be proved with slight modifications of the arguments used in the proofs of Proposition 4.6, Proposition 4.7, and Proposition 4.8, since \mathcal{A} is continuously included in \mathcal{B} . \square

5. CONTINUITY OF THE FLOW MAP

In this section we prove that the map $u \mapsto \phi_u$ is continuous for all classical test function spaces. We do not know if similar results hold for the ultradifferentiable classes.

For $f \in W^{\infty,p}(\mathbb{R}^d, \mathbb{R}^d)$ we will write

$$\|f\|_{W^{k,p}} := \sum_{j=0}^k \left(\int_{\mathbb{R}^d} \|f^{(j)}(x)\|_{L_j(\mathbb{R}^d, \mathbb{R}^d)}^p dx \right)^{1/p}$$

in the following.

Lemma 5.1. *Let $p \in [1, \infty]$. Let $g, f \in W^{\infty,p}(\mathbb{R}^d, \mathbb{R}^d)$ and assume that $\text{Id} + f$ is a diffeomorphism of \mathbb{R}^d with $c := \inf_{x \in \mathbb{R}^d} \det d(x + f(x)) > 0$. Then, for all $k \in \mathbb{N}$ there is a constant $C = C(c, k) > 0$, such that*

$$\|g \circ (\text{Id} + f)\|_{W^{k,p}} \leq C \|g\|_{W^{k,p}} (1 + \|f\|_{W^{k,\infty}})^k.$$

Proof. We prove the assertion by induction on k . By assumption, $\|g \circ (\text{Id} + f)\|_{L^p} \leq c^{-1/p} \|g\|_{L^p}$, thus the assertion holds for $k = 0$. For $k = 1$ we have,

$$\begin{aligned} \|d(g \circ (\text{Id} + f))\|_{L^p} &\leq \|dg \circ (\text{Id} + f)\|_{L^p} + \|dg \circ (\text{Id} + f) \cdot df\|_{L^p} \\ &\leq c^{-1/p} \|dg\|_{L^p} + c^{-1} \|dg\|_{L^p} \|df\|_{L^\infty} \\ &\leq c^{-1/p} \|g\|_{W^{1,p}} (1 + \|f\|_{W^{1,\infty}}). \end{aligned}$$

Now assume the statement holds for $k - 1$. Then

$$\begin{aligned} \|d(g \circ (\text{Id} + f))\|_{W^{k-1,p}} &\leq \|dg \circ (\text{Id} + f)\|_{W^{k-1,p}} + \|dg \circ (\text{Id} + f) \cdot df\|_{W^{k-1,p}} \\ &\leq C \|dg\|_{W^{k-1,p}} (1 + \|f\|_{W^{k-1,\infty}})^{k-1} (1 + \|df\|_{W^{k-1,\infty}}) \\ &\leq C \|g\|_{W^{k,p}} (1 + \|f\|_{W^{k,\infty}})^k. \quad \square \end{aligned}$$

Lemma 5.2. *Let $p \in [1, \infty]$. Let $g, f_1, f_2 \in W^{\infty,p}(\mathbb{R}^d, \mathbb{R}^d)$ and assume that $\text{Id} + f_i$ are diffeomorphisms of \mathbb{R}^d with $c_i := \inf_{x \in \mathbb{R}^d} \det d(x + f_i(x)) > 0$. Then, for all $k \in \mathbb{N}$ there is a constant $C = C(c_i, k) > 0$, such that*

$$\|g \circ (\text{Id} + f_1) - g \circ (\text{Id} + f_2)\|_{W^{k,p}} \leq C \|g\|_{W^{k+1,\infty}} (1 + \max_{i=1,2} \|f_i\|_{W^{k,\infty}})^k \|f_1 - f_2\|_{W^{k,p}}.$$

Proof. Induction on k . For $k = 0$,

$$|g \circ (\text{Id} + f_1) - g \circ (\text{Id} + f_2)| \leq \|dg\|_{L^\infty} |f_1 - f_2|$$

and hence $\|g \circ (\text{Id} + f_1) - g \circ (\text{Id} + f_2)\|_{L^p} \leq \|g\|_{W^{1,\infty}} \|f_1 - f_2\|_{L^p}$. Suppose that the claim holds for $k - 1$. Then, using the induction hypothesis and Lemma 5.1 for $p = \infty$,

$$\begin{aligned} &\|d(g \circ (\text{Id} + f_1)) - d(g \circ (\text{Id} + f_2))\|_{W^{k-1,p}} \\ &\leq \|(dg \circ (\text{Id} + f_1) - dg \circ (\text{Id} + f_2))(\mathbb{1} + df_2)\|_{W^{k-1,p}} \\ &\quad + \|dg \circ (\text{Id} + f_1)(df_1 - df_2)\|_{W^{k-1,p}} \\ &\leq C \|dg\|_{W^{k,\infty}} (1 + \max_{i=1,2} \|f_i\|_{W^{k-1,\infty}})^{k-1} \|f_1 - f_2\|_{W^{k-1,p}} (1 + \|f_2\|_{W^{k-1,\infty}}) \\ &\quad + \|dg \circ (\text{Id} + f_1)\|_{W^{k-1,\infty}} \|df_1 - df_2\|_{W^{k-1,p}} \\ &\leq C \|dg\|_{W^{k,\infty}} (1 + \max_{i=1,2} \|f_i\|_{W^{k-1,\infty}})^{k-1} \|f_1 - f_2\|_{W^{k-1,p}} (1 + \|f_2\|_{W^{k-1,\infty}}) \\ &\quad + C \|dg\|_{W^{k-1,\infty}} (1 + \|f_1\|_{W^{k-1,\infty}})^{k-1} \|df_1 - df_2\|_{W^{k-1,p}} \\ &\leq 2C \|g\|_{W^{k+1,\infty}} (1 + \max_{i=1,2} \|f_i\|_{W^{k,\infty}})^k \|f_1 - f_2\|_{W^{k,p}}. \quad \square \end{aligned}$$

Theorem 5.3. *Let $1 \leq p \leq \infty$. The mapping*

$$(5.1) \quad L^1([0, 1], W^{\infty,p}(\mathbb{R}^d, \mathbb{R}^d)) \ni u \mapsto \phi_u \in C([0, 1], W^{\infty,p}(\mathbb{R}^d, \mathbb{R}^d))$$

is continuous.

Proof. The case $p = \infty$ follows immediately from [16, Theorem 5.6] (the assumption that the vector fields vanish together with all its derivatives as $|x| \rightarrow \infty$ was not used in the proof).

Let $p \in [1, \infty)$. Then $L^1([0, 1], W^{\infty, p}(\mathbb{R}^d, \mathbb{R}^d))$ is continuously included in $L^1([0, 1], \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d))$. The result for $p = \infty$ together with [16, Proposition 3.6] shows that both $u \mapsto \Phi_u$ and $u \mapsto \Phi_u^{-1}$ are bounded into $C(I, \text{Diff}_0 \mathcal{B})$. And thus the arguments after (••) on p. 11 imply that for every bounded set $U \subseteq L^1(I, \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d))$ there is a constant $c > 0$ such that

$$(5.2) \quad \det d\Phi_u(t, x) \geq c$$

for all $x \in \mathbb{R}^d$, $t \in I$, and $u \in U$.

Let u, v be in a bounded subset of $L^1([0, 1], W^{\infty, p}(\mathbb{R}^d, \mathbb{R}^d))$ and let $\phi_u, \phi_v \in C([0, 1], W^{\infty, p}(\mathbb{R}^d, \mathbb{R}^d))$ be the corresponding flows. Thanks to (5.2) we may apply Lemma 5.1 and Lemma 5.2 to obtain

$$\begin{aligned} \|\phi_u(t) - \phi_v(t)\|_{W^{k, p}} &\leq \int_0^t \|u(s) \circ \Phi_u(s) - v(s) \circ \Phi_v(s)\|_{W^{k, p}} ds \\ &\leq \int_0^t \|u(s) \circ \Phi_u(s) - u(s) \circ \Phi_v(s)\|_{W^{k, p}} + \|(u(s) - v(s)) \circ \Phi_v(s)\|_{W^{k, p}} ds \\ &\leq C \int_0^t \|u(s)\|_{W^{k+1, \infty}} \|\phi_u(s) - \phi_v(s)\|_{W^{k, p}} + \|u(s) - v(s)\|_{W^{k, p}} ds \end{aligned}$$

for a constant C independent of u, v . Then Gronwall's inequality implies

$$\|\phi_u(t) - \phi_v(t)\|_{W^{k, p}} \leq C_1 \|u - v\|_{L^1(I, W^{k, p})} \exp(C_2 \|u\|_{L^1(I, W^{k+1, \infty})}),$$

and consequently, $\|\phi_u - \phi_v\|_{C(I, W^{k, p})} \leq C_3 \|u - v\|_{L^1(I, W^{k, p})}$. This implies the result. \square

Theorem 5.4. *The mappings*

$$\begin{aligned} L^1([0, 1], \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)) &\ni u \mapsto \phi_u \in C([0, 1], \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)), \\ L^1([0, 1], \mathcal{D}_K(\mathbb{R}^d, \mathbb{R}^d)) &\ni u \mapsto \phi_u \in C([0, 1], \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)) \end{aligned}$$

are continuous.

Proof. The case \mathcal{D} is immediate from the case \mathcal{B} .

It is easy to see that for $g, f, f_1, f_2 \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)$ we have the following analogues of Lemma 5.1 and Lemma 5.2:

$$\begin{aligned} \|g \circ (\text{Id} + f)\|^{(p, k)} &\leq C \|g\|^{(p, k)} (1 + \|f\|_{W^{k, \infty}})^k, \\ \|g \circ (\text{Id} + f_1) - g \circ (\text{Id} + f_2)\|^{(p, k)} &\leq C \|g\|_{W^{k+1, \infty}} (1 + \max_{i=1, 2} \|f_i\|_{W^{k, \infty}})^k \|f_1 - f_2\|^{(p, k)}. \end{aligned}$$

Then the case \mathcal{S} can be proved in a similar way as Theorem 5.3. \square

6. APPLICATION: THE BERGMAN SPACE ON THE POLYSTRIP IS ODE-CLOSED

In this section we prove that the p -Bergman space $A^p(S_{(r)})$ on the polystrip $S_{(r)} := \{|\text{Im}(z)| < r\}^d \subseteq \mathbb{C}^d$ is continuously included in $W_{a/r}^{1, p}(\mathbb{R}^d, \mathbb{C})$ and continuously contains $W_{b/r}^{1, p}(\mathbb{R}^d, \mathbb{C})$ if a, b are suitable constants, where $\mathbf{1} = (1, 1, \dots)$. As an application we obtain that the scale of p -Bergman spaces on the polystrip with variable width is ODE-closed.

6.1. Bergman spaces on polystrips. Let $S_r := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < r\}$ be the horizontal open strip centered at the real line with width $2r > 0$. For $r_1, \dots, r_d > 0$ consider the polystrip $S_{(r_i)} := S_{r_1} \times \dots \times S_{r_d}$, in particular, $S_r^d = S_{(r)}$. For $p \in [1, \infty]$, we consider the p -Bergman space

$$A^p(S_{(r_i)}) := \mathcal{H}(S_{(r_i)}) \cap L^p(S_{(r_i)}),$$

i.e., the space of holomorphic L^p -functions on $S_{(r_i)}$. Endowed with the L^p -norm, it is a complex Banach space, which is a direct consequence of the following Lemma.

Lemma 6.1. *Let $0 < l_i < r_i$ for $1 \leq i \leq d$. Then there exists $C > 0$ such that*

$$\|F\|_{L^\infty(S_{(l_i)})} \leq C \|F\|_{L^p(S_{(r_i)})} \quad \text{for } F \in A^p(S_{(r_i)}).$$

Proof. This is an easy consequence of the mean value inequality for subharmonic functions applied to the separately subharmonic function $|F|^p$. \square

In addition, we consider the *real p -Bergman space*

$$A_{\mathbb{R}}^p(S_{(r_i)}) := \{F \in A^p(S_{(r_i)}) : F(\mathbb{R}^d) \subseteq \mathbb{R}\}.$$

Endowed with the L^p -norm, it is a real Banach space, which is again a direct consequence of Lemma 6.1.

We are interested in the inductive limits of the Bergman spaces with respect to all positive widths of the underlying polystrip

$$\begin{aligned} \underline{\mathcal{A}}^p(\mathbb{R}^d) &:= \varinjlim_{r>0} A^p(S_{(r)}), \\ \underline{\mathcal{A}}_{\mathbb{R}}^p(\mathbb{R}^d) &:= \varinjlim_{r>0} A_{\mathbb{R}}^p(S_{(r)}), \end{aligned}$$

as well as in the projective limits

$$\begin{aligned} \overline{\mathcal{A}}^p(\mathbb{R}^d) &:= \varprojlim_{r>0} A^p(S_{(r)}), \\ \overline{\mathcal{A}}_{\mathbb{R}}^p(\mathbb{R}^d) &:= \varprojlim_{r>0} A_{\mathbb{R}}^p(S_{(r)}). \end{aligned}$$

We set $A_{\mathbb{R}}^p(S_{(r_i)}, \mathbb{R}^n) := A_{\mathbb{R}}^p(S_{(r_i)})^n$, $\underline{\mathcal{A}}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^n) := \underline{\mathcal{A}}_{\mathbb{R}}^p(\mathbb{R}^d)^n$, and $\overline{\mathcal{A}}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^n) := \overline{\mathcal{A}}_{\mathbb{R}}^p(\mathbb{R}^d)^n$.

6.2. The scale of Bergman spaces on polystrips with variable width is ODE-closed.

Theorem 6.2. *Let $0 < \sigma < 1 < \rho < \infty$, $p \in [1, \infty]$, and $r > 0$. Then we have continuous inclusions*

$$W_{\sigma/(2dr)}^{1,p}(\mathbb{R}^d, \mathbb{C}) \xrightarrow{\mathcal{S}} A^p(S_{(r)}) \xrightarrow{\mathcal{R}} W_{\rho/r}^{1,p}(\mathbb{R}^d, \mathbb{C}),$$

where \mathcal{R} assigns to each function in the Bergman space the restriction to \mathbb{R}^d , whereas \mathcal{S} assigns to each function its unique holomorphic extension to the polystrip. In particular,

$$W^{\{1\},p}(\mathbb{R}^d, \mathbb{C}) \cong \underline{\mathcal{A}}^p(\mathbb{R}^d) \quad \text{and} \quad W^{(1),p}(\mathbb{R}^d, \mathbb{C}) \cong \overline{\mathcal{A}}^p(\mathbb{R}^d).$$

Corollary 6.3. *In the setting of Theorem 6.2,*

$$W_{\sigma/(2dr)}^{1,p}(\mathbb{R}^d, \mathbb{R}) \xrightarrow{\mathcal{S}} A_{\mathbb{R}}^p(S(r)) \xrightarrow{\mathcal{R}} W_{\rho/r}^{1,p}(\mathbb{R}^d, \mathbb{R}),$$

in particular, $W^{\{1\},p}(\mathbb{R}^d, \mathbb{R}) \cong \underline{A}_{\mathbb{R}}^p(\mathbb{R}^d)$ and $W^{(1),p}(\mathbb{R}^d, \mathbb{R}) \cong \underline{A}_{\mathbb{R}}^p(\mathbb{R}^d)$.

As an application, we obtain that the spaces $\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)$ and $\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)$ are ODE-closed in the following sense: In analogy to Section 4, let $\mathcal{F}_{\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)}$ be the set of functions $u : I \rightarrow \underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)$, that actually map into some step $A_{\mathbb{R}}^p(S(r), \mathbb{R}^d)$ and are Bochner integrable therein, in particular,

$$\exists r > 0 : \int_0^1 \|u(t)\|_{L^p(S(r))} dt < \infty.$$

Moreover, let $\mathcal{F}_{\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)} := L^1(I, \underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d))$ so that

$$\forall r > 0 : \int_0^1 \|u(t)\|_{L^p(S(r))} dt < \infty.$$

Theorem 6.4. *We have:*

- (1) *For each $u \in \mathcal{F}_{\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)}$ the flow $t \mapsto \phi_u(t)$ is a continuous curve in $\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)$.*
- (2) *For each $u \in \mathcal{F}_{\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)}$ the flow $t \mapsto \phi_u(t)$ is a continuous curve in $\underline{A}_{\mathbb{R}}^p(\mathbb{R}^d, \mathbb{R}^d)$.*

Proof. (1) follows from Corollary 6.3, Proposition 4.5, and Proposition 4.6. (2) is a consequence of Corollary 6.3 and Proposition 4.9. \square

To prove Theorem 6.2, we need some auxiliary results.

Lemma 6.5. *There exists $D > 0$ such that for all $\alpha \in \mathbb{N}_{\geq 1}^d$,*

$$\int_{\mathbb{R}^d} \frac{1}{\prod_{j=1}^d |x_j + iy_j|^{\alpha_j+1}} d\lambda(x) \leq \frac{D}{\prod_{j=1}^d |y_j|^{\alpha_j}},$$

where λ denotes the Lebesgue measure in \mathbb{R}^d .

Proof. Substitute $\xi_j = x_j/y_j$. \square

Proof of Theorem 6.2. First we show continuity of \mathcal{S} . Fix $d \in \mathbb{N}$, $p \in [1, \infty]$, $r > 0$, and $0 < \sigma < 1$. By Lemma 2.1, there exists a constant $C_1 = C_1(p, d)$ such that for all $f \in C^\infty(\mathbb{R}^d)$, $\beta \in \mathbb{N}^d$, and $x \in \mathbb{R}^d$,

$$|\partial^\beta f(x)| \leq C_1 \sum_{|\alpha| \leq l} \|\partial^{\beta+\alpha} f\|_{L^p},$$

where $l := \lfloor \frac{d}{p} \rfloor + 1$. Therefore, for all $f \in W_{\sigma/(2dr)}^{1,p}(\mathbb{R}^d, \mathbb{C})$,

$$\begin{aligned} |\partial^\beta f(x)| &\leq C_1 \sum_{|\alpha| \leq l} \|f\|_{\sigma/(2dr)}^{1,p} \left(\frac{\sigma}{2dr}\right)^{|\beta|+|\alpha|} (|\beta| + |\alpha|)! \\ &\leq C_1 \|f\|_{\sigma/(2dr)}^{1,p} \left(\frac{\sigma}{dr}\right)^{|\beta|} |\beta|! \left(\sum_{|\alpha| \leq l} \left(\frac{\sigma}{dr}\right)^{|\alpha|} |\alpha|!\right). \end{aligned}$$

Thus f admits a holomorphic extension F on $S_{(r)}$ with

$$F(z) = \sum_{\alpha \in \mathbb{N}^d} \partial^\alpha f(x_1, \dots, x_d) \prod_{j=1}^d \frac{(iy_j)^{\alpha_j}}{\alpha_j}, \quad z = (x_j + iy_j)_{j=1}^d \in S_{(r)}.$$

Applying Hölder's inequality and (2.1), we thus get

$$|F(z)|^p \leq \left(\sum_{\alpha} \frac{|\partial^\alpha f(x)|}{|\alpha|!} (dr)^{|\alpha|} \right)^p \leq \left(\sum_{\alpha} \left(\frac{|\partial^\alpha f(x)|}{|\alpha|!} \right)^p \left(\frac{dr}{\sqrt{\sigma}} \right)^{p|\alpha|} \right) \left(\sum_{\alpha} \sqrt{\sigma}^{q|\alpha|} \right)^{p/q}$$

if $p \in (1, \infty)$ and $1/p + 1/q = 1$. Thus,

$$\int_{\mathbb{R}^d} |F(z)|^p d\lambda(x) \leq C_2 \sum_{\alpha \in \mathbb{N}^d} (\|f\|_{\sigma/(dr)}^{1,p})^p \left(\frac{\sigma}{dr} \right)^{p|\alpha|} \left(\frac{dr}{\sqrt{\sigma}} \right)^{p|\alpha|} \leq C_3 (\|f\|_{\sigma/(dr)}^{1,p})^p.$$

Then integration with respect to the y_j over $(-r, r)$ implies continuity of \mathcal{S} . The cases $p = 1, \infty$ are seen similarly.

Now let us consider \mathcal{R} . Let $0 < y_j < r$ for $1 \leq j \leq d$ and consider the box $B(t, u) := [-t, t] + i[-u, u]$. Let $F \in A^p(S_{(r)})$ and set $f := F|_{\mathbb{R}^d}$. Then for $x \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$, and $t > 0$ sufficiently large, we get (by iterating the Cauchy integral formula)

(6.1)

$$\partial^\alpha f(x) = \frac{\alpha!}{(2\pi i)^d} \int_{\partial B(t, y_1)} \cdots \int_{\partial B(t, y_d)} \frac{F(\zeta_1, \dots, \zeta_d)}{(\zeta_1 - x_1)^{\alpha_1+1} \cdots (\zeta_d - x_d)^{\alpha_d+1}} d\zeta_d \cdots d\zeta_1.$$

If we let t tend to ∞ the contribution of the integrals over the vertical boundary parts of the boxes $B(t, y_j)$ tends to 0. In order to estimate the contribution of the horizontal parts, observe that, by Lemma 6.5, for $1/p + 1/q = 1$ and $k \geq 1$,

$$\int_{-\infty}^{\infty} \frac{|G(t + iu)|}{|t - x + iu|^{k+1}} dt \leq \left(\int_{-\infty}^{\infty} \frac{|G(t + iu)|^p}{|t - x + iu|^{k+1}} dt \right)^{1/p} \left(\frac{D}{|u|^k} \right)^{1/q},$$

and an analogous estimate holds if we integrate with respect to x instead of t . Applying this to (6.1) we find, for $\alpha \in \mathbb{N}_{\geq 1}^d$,

$$\begin{aligned} & \|\partial^\alpha f\|_{L^p(\mathbb{R}^d)}^p \prod_{j=1}^d |y_j|^{p\alpha_j} \\ & \leq C_4 |\alpha|!^p \sum_{\tau \in \{0,1\}^d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |F((-1)^{\tau_j} \xi_j + i(-1)^{\tau_j+1} y_j)_{j=1}^d|^p d\xi_d \cdots d\xi_1. \end{aligned}$$

Integration with respect to y_j from 0 to r now yields

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^d)}^p \frac{r^{|\alpha|p+1}}{\prod_{j=1}^d (\alpha_j p + 1)} \leq C_4 |\alpha|!^p \|F\|_{L^p(S_{(r)})}^p,$$

and thus, for $\rho > 1$ so that $\rho^{-p|\alpha|} \prod_{j=1}^d (\alpha_j p + 1)$ is bounded for all α ,

$$\sup_{\alpha \in \mathbb{N}_{\geq 1}^d} \frac{\|\partial^\alpha f\|_{L^p(\mathbb{R}^d)}}{(\rho/r)^{|\alpha|} |\alpha|!} \leq C_5 \|F\|_{L^p(S_{(r)})}.$$

For $\alpha \in \mathbb{N}^d$ with some $\alpha_j = 0$, one uses the mean value inequalities for subharmonic functions for those zero-entries and the already established estimate for the non-zero entries to complete the proof. \square

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