

SELECTIONS OF BOUNDED VARIATION FOR ROOTS OF SMOOTH POLYNOMIALS

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ABSTRACT. We prove that the roots of a smooth monic polynomial with complex-valued coefficients defined on a bounded Lipschitz domain Ω in \mathbb{R}^m admit a parameterization by functions of bounded variation uniformly with respect to the coefficients. This result is best possible in the sense that discontinuities of the roots are in general unavoidable due to monodromy. We show that the discontinuity set can be chosen to be a finite union of smooth hypersurfaces. On its complement the parameterization of the roots is of optimal Sobolev class $W^{1,p}$ for all $1 \leq p < \frac{n}{n-1}$, where n is the degree of the polynomial. All discontinuities are jump discontinuities. For all this we require the coefficients to be of class $C^{k-1,1}(\bar{\Omega})$, where k is a positive integer depending only on n and m . The order of differentiability k is not optimal. However, in the case of radicals, i.e., for the solutions of the equation $Z^r = f$, where f is a complex-valued function and $r \in \mathbb{R}_{>0}$, we obtain optimal uniform bounds.

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1. INTRODUCTION

1.1. **The main results.** Let $\Omega \subseteq \mathbb{R}^m$ be an open set and let

$$P_a(x)(Z) = P_{a(x)}(Z) = Z^n + \sum_{j=1}^n a_j(x)Z^{n-j}, \quad x \in \Omega, \quad (1.1)$$

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be a monic polynomial with complex-valued coefficients $a = (a_1, \dots, a_n) : \Omega \rightarrow \mathbb{C}^n$. The roots of P_a form a multi-valued function $\Lambda : \Omega \rightsquigarrow \mathbb{C}$. If a is of Hölder class $C^{n-1,1}(\overline{\Omega})$, then Λ is of Sobolev class $W^{1,p}(\Omega)$, for all $1 \leq p < \frac{n}{n-1}$, in the sense of Almgren [2] (see also [16]), and this result is sharp. This follows from the main result of our recent paper [33]; see [33, Theorem 6].

In this paper we study the existence of regular selections and parameterizations of the multi-valued function Λ . The point-image $\Lambda(x)$, for $x \in \Omega$, is the unordered n -tuple consisting of the roots of $P_a(x)$ (with multiplicities). A *parameterization* of Λ is an n -tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ of single-valued functions such that $\lambda(x)$ represents $\Lambda(x)$ for all $x \in \Omega$. A *selection* of $\Lambda(x)$ is a single-valued function μ such that $\mu(x) \in \Lambda(x)$ for all $x \in \Omega$, or equivalently $P_a(x)(\mu(x)) = 0$.

The main result of [33] states that any **continuous** selection of the roots of P_a , where $a \in C^{n-1,1}(\overline{I}, \mathbb{C}^n)$ and I is an open bounded interval in \mathbb{R} , is of class $W^{1,p}(I)$, for all $1 \leq p < \frac{n}{n-1}$, uniformly with respect to coefficients. This result is optimal. It is not hard to see that in this one-dimensional case there always exist continuous parameterizations of the roots (e.g. [23, Ch. II Theorem 5.2]).

As a consequence, any continuous selection $\mu : V \rightarrow \mathbb{C}$ of a root of P_a , where $a \in C^{n-1,1}(\overline{\Omega}, \mathbb{C}^n)$, Ω is a Lipschitz domain, and $V \subseteq \Omega$ is an open subset, is of class $W^{1,p}(V)$, for all $1 \leq p < \frac{n}{n-1}$ (see Theorem A.1). But, for dimension $m \geq 2$, monodromy in general prevents the existence of continuous selections of roots on Ω . So it is natural to ask:

Can the roots of a polynomial (1.1) with coefficients in a differentiability class of sufficiently high order be represented by functions of bounded variation?

Functions of bounded variation (*BV*) are integrable functions whose distributional derivative is a vector-valued finite Radon measure. They form an algebra of discontinuous functions. Due to their ability to deal with discontinuities they are widely used in the applied sciences, see e.g. [24].

Our main result gives a positive answer to the above question:

Theorem 1.1. *For all integers $n, m \geq 2$ there exists an integer $k = k(n, m) \geq \max(n, m)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and let (1.1) be a monic polynomial with complex-valued coefficients $a = (a_1, \dots, a_n) \in C^{k-1,1}(\overline{\Omega}, \mathbb{C}^n)$.*

Then the roots of P_a admit a parameterization $\lambda = (\lambda_1, \dots, \lambda_n)$ by special functions of bounded variation (SBV) on Ω such that

$$\|\lambda\|_{BV(\Omega)} \leq C(n, m, \Omega) \max\{1, \|a\|_{L^\infty(\Omega)}\} \max\{1, \|a\|_{C^{k-1,1}(\overline{\Omega})}\}. \quad (1.2)$$

There is a finite collection of C^{k-1} -hypersurfaces E_j in Ω such that λ is continuous in the complement of $E := \bigcup_j E_j$. Any hypersurface E_j is closed in an open subset of Ω but possibly not in Ω itself. All discontinuities of λ are jump discontinuities.

Note that we do not claim that the discontinuity hypersurfaces E_j have finite $(m-1)$ -dimensional Hausdorff measure (\mathcal{H}^{m-1}). Actually we construct an example, see Example 2.4, where this is not possible.

If $n = 1$ then the problem has a trivial solution $\lambda = -a_1$. If $n \geq 2$ and $m = 1$, then the problem was solved in [33]. In both cases the roots admit continuous parameterizations.

A function of bounded variation is called *special* (*SBV*) if the Cantor part of its derivative vanishes; for precise definitions and background on *BV*-functions we refer to Section 4.

Remark 1.2. The dependence of k on m stems from the use of Sard's theorem. The dependence of the constant C on Ω originates from the use of Whitney's extension theorem (see Section 3.2) and also from the trivial bound $\|\lambda\|_{L^1(\Omega)} \leq |\Omega| \|\lambda\|_{L^\infty(\Omega)}$. It is well-known that $\max_{1 \leq i \leq n} |\lambda_i(x)| \leq 2 \max_{1 \leq j \leq n} |a_j(x)|^{1/j}$ for all x (cf. [27, p.56] or [34, (8.1.11)]).

Together with Theorem A.1 of Appendix A we immediately obtain the following supplement.

Corollary 1.3. *The parameterization λ satisfies*

$$\|\lambda\|_{W^{1,p}(\Omega \setminus \bar{E})} \leq C(n, m, p, \Omega) \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}(\bar{\Omega})}^{1/j} \quad (1.3)$$

for all $1 \leq p < \frac{n}{n-1}$.

For radicals, i.e., solutions of $Z^r = f$, where we allow $r \in \mathbb{R}_{>0}$, we have better bounds:

Theorem 1.4. *Let $r \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}$ and $\alpha \in (0, 1]$ be such that $k + \alpha \geq \max\{r, m\}$. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain. Let $f \in C^{k,\alpha}(\bar{\Omega})$. Then there exists a solution $\lambda \in SBV(\Omega)$ of the equation $Z^r = f$ such that*

$$\|\lambda\|_{BV(\Omega)} \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\bar{\Omega})}^{1/r}. \quad (1.4)$$

There is a C^k -hypersurface $E \subseteq \Omega$ (possibly empty) such that λ is continuous on $\Omega \setminus E$ and satisfies $\nabla \lambda \in L_w^p(\Omega \setminus \bar{E})$ for

$$p = \begin{cases} \frac{r}{r-1} & \text{if } r > 1, \\ \infty & \text{if } r \leq 1 \text{ and } r^{-1} \in \mathbb{N}, \\ \frac{\{r^{-1}\}^{-1}}{\{r^{-1}\}^{-1}-1} & \text{if } r < 1 \text{ and } r^{-1} \notin \mathbb{N}. \end{cases}$$

We have

$$\|\nabla \lambda\|_{L_w^p(\Omega \setminus \bar{E})} \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\bar{\Omega})}^{1/r}, \quad (1.5)$$

and

$$\int_E |f|^{1/r} d\mathcal{H}^{m-1} \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\bar{\Omega})}^{1/r}. \quad (1.6)$$

All discontinuities are jump discontinuities.

Here $L_w^p(V)$ denotes the weak Lebesgue space equipped with the quasinorm $\|\cdot\|_{L_w^p(V)}$ and $\{x\}$ is the fractional part of $x \in \mathbb{R}$. This result is optimal as follows from Remark 2.6: in general, $\nabla \lambda \notin L^p$, even if λ is continuous and f is real analytic, and λ need not have bounded variation if f is only of class $C^{\ell,\beta}$ whenever $\ell + \beta < r$.

Remark 1.5. In the case that $r = n$ is an integer, a complete parameterization of the roots of $Z^n = f$ is provided by $\theta^k \lambda$, $k = 0, \dots, n-1$, where $\theta = e^{2\pi i/n}$.

1.2. Background. The problem of determining the optimal regularity of the roots of univariate monic polynomials whose coefficients depend smoothly on parameters has a long and rich history. Its systematic investigation probably started with Rellich's work on the perturbation theory of symmetric operators in the 1930s [40, 41, 42, 42, 43, 44], see also his monograph [45]. This line of research culminated with Kato's monograph [23]. But the regularity problem of the eigenvalues of symmetric, Hermitian, and even normal matrices/operators behaves much better in many aspects than the related problem of choosing regular roots of smooth families of polynomials; see [38] for a survey of the known results.

The regularity of square roots of non-negative smooth functions was first studied by Glaeser [22]. The general case of *hyperbolic* polynomials (i.e. all roots are real) plays a crucial role for the Cauchy problem for hyperbolic partial differential equations with multiple characteristics. The central result is this connection is Bronshtein's theorem [8]: every continuous choice of the roots of a hyperbolic monic polynomial of degree n with $C^{n-1,1}$ -coefficients is locally Lipschitz. Note that there always is a continuous parameterization of the roots in this case, e.g., by ordering them increasingly. Variations on this fundamental result (and its proof) appeared in [29], [51], [1], [25], [5], [6], [50], [7], [13], [31].

The *complex* (i.e. not necessarily hyperbolic) counterpart, which is the problem at the center of this paper, was considered for the first time (for radicals) by Colombini, Jannelli, and Spagnolo [11]. Motivated by the analysis of certain systems of pseudo-differential equations Spagnolo [47] asked if the roots of a smooth *curve* of monic polynomials admit a parameterization by locally absolutely continuous functions. This conjecture was proved in our papers [32] and [33] which are based on the solution for radicals due to Ghisi and Gobino [21]. The optimal Sobolev regularity of the roots, which was already mentioned above, was established in [33] by elementary methods. Its precursor [32] in which the optimal bounds were still missing was based on Hironaka's resolution of singularities. We wish to mention that absolute continuity of the roots was also shown in [14] by different methods; for polynomials of degree $n \leq 3$ it is due to [46]. As already pointed out, a curve $I \ni t \mapsto P_a(t)$, where $I \subseteq \mathbb{R}$ is an interval, always admits a continuous parameterization of its roots. Further contributions with partial solutions appeared in [49], [12], [10], [35], [36], [37], [39].

The results of this paper complete this analysis and solve Open Problem 3 posed in [33].

1.3. Idea of the proof. The main difficulty of the problem is to make a *good* choice of the discontinuity set of the roots. On the complement of the discontinuity set the roots are of optimal Sobolev class $W^{1,p}$, for all $1 \leq p < \frac{n}{n-1}$, by the result of [33], see also Theorem A.1. In general, the discontinuity set has infinite codimension one Hausdorff measure, see Example 2.4. Thus, in order to have bounded variation it is crucial that the jump height of a selection of a root is integrable (with respect to \mathcal{H}^{m-1}) along its discontinuity set.

The proof of Theorem 1.1 is based on the radical case solved in Theorem 1.4 and on formulas for the roots of the universal polynomial P_a , $a \in \mathbb{C}^n$, which were found in [32]. Interestingly, the method of [32] seems to be better suited for the control of the discontinuities and integrability along them than a more elementary method of [33].

1.3.1. *The radical case.* Consider the equation $Z^n = f$, where f is a smooth complex-valued function. We choose the discontinuities of the solutions along the preimage of a regular value of the *sign* $\operatorname{sgn}(f) = f/|f| : \Omega \setminus f^{-1}(0) \rightarrow \mathbb{S}^1$ of f . The result of Ghisi and Gobbino [21] which we recall in Theorem 2.7 together with the coarea formula and Sard's theorem empowers us to show that

$$\int_{\operatorname{sgn}(f)^{-1}(y)} |f|^{1/n} d\mathcal{H}^{m-1} < \infty \quad \text{for } \mathcal{H}^1\text{-a.e. } y \in \mathbb{S}^1.$$

It is then not hard to complete the proof of Theorem 1.4 using the observation that any parameterization of the solutions of $Z^n = f$ is continuous (and zero) on the zero set of f , and hence a *BV*-parameterization on $\Omega \setminus f^{-1}(0)$ extends to a *BV*-parameterization on Ω with unchanged total variation.

1.3.2. *The general case.* The formulas for the roots of the universal polynomial P_a , $a \in \mathbb{C}^n$, which we recall in detail in Section 6, express the roots as finite sums of functions analytic in radicals of local coordinates on a resolution space (a blowing up of \mathbb{C}^n). We choose parameterizations of the involved radicals, using Theorem 1.4, and show that in this way we obtain *SBV*-parameterizations of these summands. But then a new difficulty arises which comes from the fact that these summands are defined only locally on the resolution space. (Actually, they cannot be defined neither globally nor canonically.) We solve this problem by cutting and pasting these locally defined summands which introduces new discontinuities. In order to stay in the class *SBV* we must ascertain integrability of the new jumps along these discontinuities. This is again based on a consequence of Ghisi and Gobbino's result for radicals and the coarea formula, see Lemma 7.4.

1.4. **Open problems.** The uniform bound in (1.2) is not scale-invariant. Nor is the degree of differentiability k sharp (in contrast, it is sharp in (1.3) and (1.4)). These deficiencies stem from the method of proof involving resolution of singularities. *Are there better bounds with lower differentiability requirements?*

The method of our proof is local. This forces us to deal with the global monodromy by cutting and pasting the local choices of the roots. It introduces additional discontinuities some of which are perhaps not necessary. *It would be interesting to have a global understanding of the monodromy and the discontinuities it necessitates.*

1.5. **Structure of the paper.** In Section 2 we investigate the discontinuities of radicals caused by monodromy and show in Example 2.4 that their codimension one Hausdorff measure is in general infinite. The main analytic ingredient for the proofs of Theorem 1.1 and Theorem 1.4 which allows us to control the integrability of the jump height of the roots along their discontinuity sets is developed in Section 3; it is presented in greater generality, see Theorem 3.3, since it might be of independent interest. In Section 4 we recall the required background on functions of bounded variation. The proof of Theorem 1.4 is completed in Section 5. The remaining sections are dedicated to the proof of Theorem 1.1. We recall the formulas for the roots of the universal polynomial in Section 6. In Section 7 we prove Theorem 1.1 modulo the local Proposition 7.2 which is then shown in Section 8. In Appendix A we refine a result from [33] on the Sobolev regularity of continuous roots.

Notation. We use $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{N}_{>m} := \{n \in \mathbb{N} : n > m\}$, $\mathbb{N}_{\geq m} := \mathbb{N}_{>m} \cup \{m\}$, $\mathbb{N}_+ := \mathbb{N}_{>0}$, and similarly, \mathbb{R}_+ , $\mathbb{R}_{>t}$, etc. For $r \in \mathbb{R}_+$ let $[r]$ be its integer part and $\{r\} := r - [r]$ its fractional part.

For $a \in \mathbb{R}^\ell$ and $b \in \mathbb{R}^m$, we denote by $a \otimes b$ the $\ell \times m$ matrix $(a_i b_j)_{i=1, j=1}^{\ell, m}$. By $B_r(x) = \{y \in \mathbb{R}^m : |x - y| < r\}$ we mean the open ball with center x and radius r . The open unit ball in \mathbb{C}^n is denoted by \mathbb{B} , the unit sphere in \mathbb{R}^m by \mathbb{S}^{m-1} . By $V(\mathcal{I})$ we denote the zero set of an ideal \mathcal{I} .

For a positive measure μ and a μ -measurable set E , let $\mu \llcorner E$ denote the restriction of μ to E , i.e., $(\mu \llcorner E)(F) = \mu(F \cap E)$. The m -dimensional Lebesgue measure is denoted by \mathcal{L}^m ; we also use $\mathcal{L}^m(E) = |E|$ and $d\mathcal{L}^m = dx$. We write $\int_E f dx$ for the average $|E|^{-1} \int_E f dx$. By \mathcal{H}^d we mean the d -dimensional Hausdorff measure.

For a mapping $f : X \rightarrow Y$ between metric spaces X, Y and $\alpha \in (0, 1]$, we set

$$\text{Höld}_{\alpha, X}(f) := \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{d(f(x_1), f(x_2))}{d(x_1, x_2)^\alpha} \quad \text{and} \quad \text{Lip}_X(f) := \text{Höld}_{1, X}(f).$$

Then f is said to be α -Hölder (or Lipschitz) if $\text{Höld}_{\alpha, X}(f) < \infty$ (or $\text{Lip}_X(f) < \infty$).

Let $\Omega \subseteq \mathbb{R}^m$ be open. We denote by $C^{k, \alpha}(\Omega)$ the space of complex-valued C^k -functions on Ω such that $\partial^\gamma f$ is locally α -Hölder for all $|\gamma| = k$. If Ω is bounded, then $C^{k, \alpha}(\overline{\Omega})$ is the subspace of functions f such that $\partial^\gamma f$ has a continuous extension to $\overline{\Omega}$ for all $0 \leq |\gamma| \leq k$ and $\text{Höld}_{\alpha, \Omega}(\partial^\gamma f) < \infty$ for all $|\gamma| = k$. Then $C^{k, \alpha}(\overline{\Omega})$ is a Banach space with the norm

$$\|f\|_{C^{k, \alpha}(\overline{\Omega})} := \sup_{|\gamma| \leq k, x \in \Omega} |\partial^\gamma f(x)| + \sup_{|\gamma| = k} \text{Höld}_{\alpha, \Omega}(\partial^\gamma f).$$

This norm makes also sense for $\Omega = \mathbb{R}^m$. If $f = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{C}^n$ is a vector-valued function, then we put

$$\|f\|_{C^{k, \alpha}(\overline{\Omega})} := \max_{1 \leq i \leq n} \|f_i\|_{C^{k, \alpha}(\overline{\Omega})}.$$

For real-valued functions f and g we write $f \lesssim g$ if $f \leq Cg$ for some universal constant C .

2. DISCONTINUITY DUE TO MONODROMY

In this section we investigate the first difficulty of the problem: discontinuities of the roots caused by the local monodromy. This difficulty is already present for radicals of smooth functions. Thus we concentrate on the solutions of

$$Z^r = f, \tag{2.1}$$

where $r > 1$ is a real number and f is a complex-valued smooth not identically equal to zero function defined in some open subset Ω of \mathbb{R}^m . Let us explain what we mean by a solution of (2.1) for $r \notin \mathbb{Z}$. Firstly, any solution should vanish on the zero set of f . On the set $\Omega_0 := \Omega \setminus f^{-1}(0)$, Equation (2.1) can be given an equivalent form $Z = \exp(r^{-1} \log f)$. Thus, by definition, a continuous function $\lambda(x)$ is a solution of (2.1) if there is a branch of logarithm $\log f$ such that $\lambda = \exp(r^{-1} \log f)$. Here by $\log f$ we mean a function defined on Ω_0 , not necessarily continuous, such that $\exp(\log f) = f$. For r irrational, if $\exp(r^{-1} \log f)$ is continuous then so is $\log f$. Note also that, if λ is a solution of (2.1) then so are $\lambda \exp(2\pi i k/r)$, $k \in \mathbb{Z}$. In particular, for r irrational, if there is a continuous r -th root of f , then there are

infinitely many of them. If $r = a/b$ with a, b being two relatively prime integers, then $Z = \exp(r^{-1} \log f)$ can be written equivalently as $Z^a = f^b$.

Consider the *sign* function of f defined by

$$\operatorname{sgn}(f) : \Omega_0 \rightarrow \mathbb{S}^1, \quad \operatorname{sgn}(f) := \frac{f}{|f|}. \quad (2.2)$$

The existence of a continuous selection of $f^{1/r}$ depends on the image $\operatorname{Im}(\pi_1(\operatorname{sgn} f))$ of the induced homomorphism of the fundamental groups $\pi_1(\operatorname{sgn} f) : \pi_1(\Omega_0) \rightarrow \pi_1(\mathbb{S}^1) = \mathbb{Z}$ (here we suppose for simplicity that Ω_0 is connected). More precisely, a continuous selection of $f^{1/r}$ exists if and only if

- (1) $\operatorname{Im}(\pi_1(\operatorname{sgn} f)) \subseteq n\mathbb{Z}$, if $r = n \geq 1$ is an integer,
- (2) $\operatorname{Im}(\pi_1(\operatorname{sgn} f)) \subseteq a\mathbb{Z}$, if $r = a/b$ with a, b being two relatively prime integers. (Indeed, the equation $Z^a = f^b$ admits a continuous solution if and only if the equation $Z^a = f$ has one. For instance, if $\lambda^a = f^b$ then $(\lambda^l f^k)^a = f$, for the integers k, l such that $ak + bl = 1$.)
- (3) $\operatorname{Im}(\pi_1(\operatorname{sgn} f)) = 0$, that is $\pi_1(\operatorname{sgn} f)$ is zero, if r is irrational.

If the above stated conditions are not satisfied, then every solution of (2.1) has to be discontinuous. We will see in Proposition 2.2 that in this case the discontinuity set can be chosen to be a smooth hypersurface provided that f is differentiable of sufficiently high order. But, in general, as shows Example 2.4, the \mathcal{H}^{m-1} -measure of the discontinuity set is infinite! Off its discontinuity set every solution of (2.1) is of Sobolev class $W^{1,p}$ for all $1 \leq p < \frac{r}{r-1}$ which follows from a result of Ghisi and Gobbino [21] which we recall in Section 2.3.

We shall use a version of Sard's theorem which we recall for convenience.

2.1. Sard's theorem. The following extension of Sard's theorem to Sobolev spaces is due to [17]; see also [20] for a different proof.

Theorem 2.1 (Sard's theorem). *Let $\Omega \subseteq \mathbb{R}^m$ be open. Let $f : \Omega \rightarrow \mathbb{R}^\ell$ be a $W_{\text{loc}}^{m-\ell+1,p}$ -function, where $p > m \geq \ell$. Then the set of critical values of f has \mathcal{L}^ℓ -measure zero.*

In the case $m = \ell$ the result follows from a theorem of Varberg and the fact that a $W^{1,p}$ self-mapping of \mathbb{R}^m satisfies the Luzin N-property if $p > m$; cf. the discussion in [17, Section 5].

In particular, the conclusion holds for each $f \in C^{m-\ell,1}(\Omega, \mathbb{R}^\ell)$, where $m \geq \ell$. See also [4] and [30].

2.2. On the discontinuity set of radicals. The roots of the equation $Z^n = x$, $x \in \mathbb{C}$, $n \in \mathbb{N}_{\geq 2}$ do not admit a continuous choice in any neighborhood of the origin. However, the roots can be chosen continuously on any set $\mathbb{C} \setminus \mathbb{R}_+ v$, where $v \in \mathbb{S}^1$. The next proposition generalizes this fact.

Proposition 2.2. *Let $r \in \mathbb{R}_{>1}$, $m \in \mathbb{N}_{\geq 2}$, and $k \in \mathbb{N}_{\geq m-1}$. Let $\Omega \subseteq \mathbb{R}^m$ be open and let $f \in C^{k,1}(\Omega)$. Then there exist a C^k -hypersurface $E \subseteq \Omega$ (possibly empty) and a continuous function $\lambda : \Omega \setminus E \rightarrow \mathbb{C}$ such that $\lambda^r = f$.*

Proof. Let $\Omega_0 := \Omega \setminus f^{-1}(0)$. Clearly $\operatorname{sgn}(f) : \Omega_0 \rightarrow \mathbb{S}^1$ is a $C^{k,1}$ -mapping.

If $\operatorname{sgn}(f)(\Omega_0) \neq \mathbb{S}^1$ define $E := \emptyset$ and choose $v \in \mathbb{S}^1 \setminus \operatorname{sgn}(f)(\Omega_0)$. If $\operatorname{sgn}(f)(\Omega_0) = \mathbb{S}^1$, then there exists a regular value $v \in \mathbb{S}^1$ of $\operatorname{sgn}(f)$, by Sard's theorem (cf. Theorem 2.1), and the set $E := \operatorname{sgn}(f)^{-1}(v)$ is a C^k -hypersurface.

In either case define $\lambda := f^{1/r} = \exp(r^{-1} \log f)$, where the logarithm is understood to have its branch cut along the ray $\mathbb{R}_+ v$. Then λ is continuous (even $C^{k,1}$) on $\Omega_0 \setminus E$ and satisfies $\lambda^r = f$. Clearly, λ extends continuously by 0 to the zero set of f . \square

Remark 2.3. Note that E is closed in Ω_0 but not necessarily in Ω .

The following example shows that in general we cannot choose the radicals of a smooth function with compact support in \mathbb{R}^m in such a way that its discontinuity set has finite \mathcal{H}^{m-1} -measure.

Example 2.4. There exists a C^∞ -function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ with compact support such that the discontinuity set S_λ of any function $\lambda : \mathbb{R}^2 \rightarrow \mathbb{C}$ with $\lambda^2 = f$ satisfies $\mathcal{H}^1(S_\lambda) = \infty$.

Proof. Consider a collection $\mathcal{D} = \{D_k\}_{k=1}^\infty$ of pairwise disjoint open disks $D_k = \{z \in \mathbb{C} : |z - p_k| < \frac{1}{k}\}$. The total area of this collection is

$$\left| \bigcup_{k=1}^\infty D_k \right| = \sum_{k=1}^\infty \frac{\pi}{k^2} = \frac{\pi^3}{6}.$$

We may assume that the disks D_k are distributed such that $\bigcup_{k=1}^\infty D_k$ is bounded. In fact, the entire collection \mathcal{D} fits in the rectangle $R = (0, 4) \times (0, 2)$. To see this let $\mathcal{D}_n := \{D_k \in \mathcal{D} : 2^{n-1} \leq k < 2^n\}$, for $n \in \mathbb{N}_+$. Any disk in \mathcal{D}_n fits in a square of side-length 2^{-n+2} ; there are 2^{n-1} such disks. Subdivide the rectangle R by the vertical lines $x = \sum_{j=1}^n 2^{-j+2}$, $n \in \mathbb{N}_+$, which provides a family of open disjoint rectangles $\{R_n\}_{n=1}^\infty$ of dimensions $2^{-n+2} \times 2$. Let us decompose each R_n into a collection \mathcal{C}_n of 2^{n-1} pairwise disjoint squares. By distributing the disks in \mathcal{D}_n to the squares in \mathcal{C}_n we achieve $\bigcup_{k=1}^\infty D_k \subseteq R$.

Let $h : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ -function such that $h(x) = 0$ if $x \leq 1/4$ and $h(x) = 1$ if $x \geq 1$. Then the C^∞ -function $h_k : \mathbb{C} \rightarrow [0, 1]$ given by

$$h_k(z) := 1 - h(k^2|z - p_k|^2)$$

vanishes outside of D_k and equals 1 on $D'_k := \{z \in \mathbb{C} : |z - p_k| < \frac{1}{2k}\}$. We claim that the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$f(z) := \sum_{k=1}^\infty h_k(z) \frac{z - p_k}{2^k} \quad (2.3)$$

is C^∞ . Indeed, the sum consists of at most one term at any point z . Set $c_k(z) := 2^{-k}(z - p_k)$, $H_\ell := \sup_{t \in \mathbb{R}, i \leq \ell} |h^{(i)}(t)|$, and for $\alpha \in \mathbb{N}^2$ with $|\alpha| = \ell$ consider

$$\begin{aligned} \sup_{z \in \mathbb{C}} |\partial^\alpha (h_k c_k)(z)| &= \sup_{z \in D_k} |\partial^\alpha (h_k c_k)(z)| \\ &\leq \sup_{z \in D_k} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta h_k(z)| |\partial^{\alpha-\beta} c_k(z)| \\ &\leq C_\ell H_\ell k^{2\ell} 2^{-k}. \end{aligned}$$

The right-hand side is summable and thus the series in (2.3) converges uniformly in each derivative and hence represents an element $f \in C^\infty(\mathbb{C})$.

Let $\lambda : \mathbb{C} \rightarrow \mathbb{C}$ be any function satisfying $\lambda^2 = f$. On the set D'_k we thus have $\lambda(z)^2 = 2^{-k}(z - p_k)$. For each $r \in (0, (2k)^{-1})$ there exists $q_r \in \{z : |z - p_k| = r\}$ such that λ restricted to $\{z : |z - p_k| = r\}$ is discontinuous at q_r . The set $S_k :=$

$\{q_r : r \in (0, (2k)^{-1})\}$ is a subset of the discontinuity set S_λ of λ . If S is a subset of \mathbb{R}^2 such that $\varphi : z \mapsto |z|$ maps S onto $(0, R)$, then

$$R = \mathcal{H}^1((0, R)) \leq \mathcal{H}^1(\varphi(S)) \leq \mathcal{H}^1(S),$$

since φ is Lipschitz with $\text{Lip}_{\mathbb{R}^2}(\varphi) = 1$. Therefore, for all $n \geq 1$,

$$\mathcal{H}^1(S_\lambda) \geq \mathcal{H}^1\left(\bigcup_{k=1}^n S_k\right) \geq \sum_{k=1}^n \frac{1}{2k},$$

since any two sets S_k and S_ℓ have positive distance if $k \neq \ell$. This implies the assertion. \square

2.3. The regularity of continuous radicals. The regularity of *continuous* radicals is fully understood thanks to a result of Ghisi and Gobbinò [21] which we recall next.

Theorem 2.5. *Let $k \in \mathbb{N}_+$, $\alpha \in (0, 1]$, and $r = k + \alpha$. Let $I \subseteq \mathbb{R}$ be an open bounded interval. Let $\lambda : I \rightarrow \mathbb{R}$ be continuous and assume that there exists $f \in C^{k,\alpha}(\bar{I}, \mathbb{R})$ such that*

$$|\lambda|^r = |f|. \quad (2.4)$$

Let p be defined by $1/p + 1/r = 1$. Then we have $\lambda' \in L_w^p(I)$ and

$$\|\lambda'\|_{p,w,I} \leq C(k) \max \left\{ (\text{Höld}_{\alpha,I}(f^{(k)}))^{1/r} |I|^{1/p}, \|f'\|_{L^\infty(I)}^{1/r} \right\}, \quad (2.5)$$

where $C(k)$ is a constant that depends only on k .

For an open set $\Omega \subseteq \mathbb{R}^m$, $L_w^p(\Omega)$ denotes the *weak Lebesgue space* of functions $f : \Omega \rightarrow \mathbb{R}$ such that

$$\|f\|_{p,w,\Omega} = \|f\|_{L_w^p(\Omega)} := \sup_{r>0} \left(r \left(\mathcal{L}^m(\{x \in \Omega : |f(x)| > r\}) \right)^{1/p} \right) < \infty.$$

Remark 2.6. The result of Theorem 2.5 is optimal in the following sense:

- In general $\lambda' \notin L^p(I)$ even if $f \in C^\omega(\bar{I})$, e.g., $f(t) = t$.
- The assumption $f \in C^{k,\alpha}(\bar{I}, \mathbb{R})$ cannot be relaxed to $f \in C^{k,\beta}(\bar{I}, \mathbb{R})$, for any $\beta < \alpha$. Indeed, there exists a non-negative function f contained in $C^{k,\beta}(\bar{I}, \mathbb{R}) \cap C^\infty(I)$ for all $\beta < \alpha$ such that any real solution λ of (2.4) has unbounded variation on I ; see [21, Example 4.4].

By a standard argument based on Fubini's theorem, also the following result for several variables was obtained in [21] (compare with Theorem A.1).

Theorem 2.7. *Let $k \in \mathbb{N}_+$, $\alpha \in (0, 1]$, and $r = k + \alpha$. Let $f : \Omega \rightarrow \mathbb{R}$ be a $C^{k,\alpha}$ -function defined on an open set $\Omega \subseteq \mathbb{R}^m$. Let $\lambda : \Omega \rightarrow \mathbb{R}$ be any continuous function satisfying (2.4). Then, for every relatively compact subset $V \Subset \Omega$, we have $\nabla \lambda \in L_w^p(V, \mathbb{R}^m)$, where $1/p + 1/r = 1$, and*

$$\|\nabla \lambda\|_{p,w,V} \leq C(m, k, \Omega, V) \max \left\{ (\text{Höld}_{\alpha,\Omega}(f^{(k)}))^{1/r}, \|\nabla f\|_{L^\infty(\Omega)}^{1/r} \right\}.$$

3. GENERIC INTEGRABILITY ALONG LEVEL SETS

The results of this section show that

$$\int_{\text{sgn}(f)^{-1}(y)} |f|^{1/r} d\mathcal{H}^{m-1} < \infty \quad \text{for } \mathcal{H}^1\text{-a.e. } y \in \mathbb{S}^1$$

if $f : \mathbb{R}^m \supseteq \Omega \rightarrow \mathbb{C}$ is of class $C^{k,\alpha}$ and $k+\alpha \geq r$. This is the main ingredient needed in the proof of Theorem 1.4 in Section 5. It can be understood as a complement of Sard's theorem for the sign function.

We believe that the results of this section are of independent interest and thus we formulate them in greater generality for maps $f : \mathbb{R}^m \supseteq \Omega \rightarrow \mathbb{R}^{\ell+1}$. Then

$$\text{sgn}(f) : \Omega \setminus f^{-1}(0) \rightarrow \mathbb{S}^\ell, \quad \text{sgn}(f) := \frac{f}{|f|}.$$

We will investigate the level sets of the sign $\text{sgn}(f)$ and the norm $|f|$. The proofs are based on Theorem 2.7 and the coarea formula.

3.1. The coarea formula. We will use the following version of the coarea formula due to [28, Theorem 1.1], see also [19].

Recall that a function \tilde{f} is a *precise representative* of $f \in L^1_{\text{loc}}(\Omega)$ if

$$\tilde{f}(x) = \lim_{r \downarrow 0} \int_{B_r(x)} f(y) dy$$

at all points x where this limit exists. In the following we say that $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^\ell)$ is *precisely represented* if each of the component functions of f is a precise representative. (In fact $f^{-1}(y)$ will depend on the representative of f .)

Theorem 3.1 (Coarea formula). *Suppose that $m \geq \ell \geq 1$. Let $\Omega \subseteq \mathbb{R}^m$ be open and let $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^\ell)$ be precisely represented, where either $p > \ell$ or $p \geq \ell = 1$. Then $f^{-1}(y)$ is countably $\mathcal{H}^{m-\ell}$ rectifiable for almost all $y \in \mathbb{R}^\ell$, and for all measurable $E \subseteq \Omega$,*

$$\int_E |J_\ell f(x)| dx = \int_{\mathbb{R}^\ell} \mathcal{H}^{m-\ell}(E \cap f^{-1}(y)) dy.$$

Recall that $|J_\ell f(x)|$ is the square root of the sum of squares of the determinants of the $\ell \times \ell$ minors of the Jacobian of f .

The following *change of variables formula* is an easy consequence of the coarea formula.

Corollary 3.2. *If f is as in Theorem 3.1 and $g : \Omega \rightarrow [0, \infty]$ is measurable, then*

$$\int_\Omega g(x) |J_\ell f(x)| dx = \int_{\mathbb{R}^\ell} \int_{f^{-1}(y)} g d\mathcal{H}^{m-\ell} dy.$$

We will apply these results only to continuous functions f which evidently are precisely represented.

3.2. Extension from Lipschitz domains. It will be sometimes helpful to assume that functions are defined on \mathbb{R}^m instead of on open subsets Ω and have compact support. If Ω is a bounded Lipschitz domain, then this is possible thanks to Whitney's extension theorem.

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and let $f \in C^{k,\alpha}(\overline{\Omega})$. By Whitney's extension theorem, f admits a $C^{k,\alpha}$ -extension \hat{f} to \mathbb{R}^m such that

$$\|\hat{f}\|_{C^{k,\alpha}(\mathbb{R}^m)} \leq C \|f\|_{C^{k,\alpha}(\overline{\Omega})}, \quad (3.1)$$

for some constant $C = C(m, k, \alpha, \Omega)$ (cf. [48, Theorem 4, p.177] and [9, Theorem 2.64]). Let $\Omega_1 := \bigcup_{x \in \Omega} B_1(x)$ be the open 1-neighborhood of Ω . By multiplying \hat{f} with a suitable cut-off function we may assume that $\text{supp}(\hat{f}) \subseteq \Omega_1$, and that

$$\|\hat{f}\|_{C^{k,\alpha}(\overline{\Omega_1})} \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\overline{\Omega})}. \quad (3.2)$$

Clearly, these observations generalize to vector-valued functions.

3.3. Level sets of the sign.

Theorem 3.3. *Let $k \in \mathbb{N}_+$, $\alpha \in (0, 1]$, and set $s = k + \alpha$. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and $f \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^{\ell+1})$, where $m \geq \ell \geq 1$. Then there is a constant $C = C(m, \ell, k, \alpha, \Omega)$ such that for each small $\epsilon > 0$*

$$\mathcal{H}^\ell \left(\left\{ y \in \mathbb{S}^\ell : \int_{\text{sgn}(f)^{-1}(y)} |f|^{\ell/s} d\mathcal{H}^{m-\ell} \geq \epsilon^{-1} C \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{\ell/s} \right\} \right) \leq \epsilon. \quad (3.3)$$

Proof. Let $f = (f_1, \dots, f_{\ell+1}) \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^{\ell+1})$. Without loss of generality we may assume $f \not\equiv 0$. For convenience set $g := \text{sgn}(f) = f/|f| : \Omega \setminus f^{-1}(0) \rightarrow \mathbb{S}^\ell$. Then, for all $j = 1, \dots, \ell + 1$ and all $i = 1, \dots, m$,

$$\partial_i g_j = \frac{\partial_i f_j}{|f|} - \frac{f_j}{|f|^3} \sum_{k=1}^{\ell+1} f_k \partial_i f_k \implies |\partial_i g_j| \leq 2(\ell + 1) \max_{1 \leq k \leq \ell+1} \frac{|\partial_i f_k|}{|f|},$$

and

$$\partial_i (|f_j|^{1/s}) = \frac{1}{s} \frac{f_j \partial_i f_j}{|f_j|^{2-1/s}}.$$

Let $h = (g_1, \dots, g_\ell)$ consist of the first ℓ components of g . Then

$$|J_\ell h| \leq C(m, \ell) \left(\max_{\substack{1 \leq j \leq \ell \\ 1 \leq i \leq m}} |\partial_i g_j| \right)^\ell \leq C(m, \ell) \left(\max_{\substack{1 \leq k \leq \ell+1 \\ 1 \leq i \leq m}} \frac{|\partial_i f_k|}{|f|} \right)^\ell$$

and consequently

$$|f|^{\ell/s} |J_\ell h| \leq C(m, \ell) \left(\max_{\substack{1 \leq k \leq \ell+1 \\ 1 \leq i \leq m}} \frac{|\partial_i f_k|}{|f|^{1-1/s}} \right)^\ell \leq C(m, \ell) s \left(\max_{\substack{1 \leq k \leq \ell+1 \\ 1 \leq i \leq m}} |\partial_i (|f_k|^{1/s})| \right)^\ell. \quad (3.4)$$

By Theorem 2.7 (applied to an extension of f as in Section 3.2) and by (3.4), we may conclude that

$$\int_{\Omega} |f(x)|^{\ell/s} |J_\ell h(x)| dx \leq C(m, \ell, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{\ell/s}. \quad (3.5)$$

By the coarea formula (Corollary 3.2),

$$\int_{\Omega \setminus f^{-1}(0)} |f|^{\ell/s} |J_\ell h(x)| dx = \int_{\mathbb{R}^\ell} \int_{h^{-1}(y)} |f|^{\ell/s} d\mathcal{H}^{m-\ell} dy. \quad (3.6)$$

Then (3.5) and (3.6) entail

$$\int_{[-1,1]^\ell} \int_{h^{-1}(y)} |f|^{\ell/s} d\mathcal{H}^{m-\ell} dy \leq C(m, \ell, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{\ell/s}. \quad (3.7)$$

It follows that, for all small enough $\epsilon > 0$,

$$\left| \left\{ y \in [-1, 1]^\ell : \int_{h^{-1}(y)} |f|^{\ell/s} d\mathcal{H}^{m-\ell} \geq \epsilon^{-1} C(m, \ell, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{\ell/s} \right\} \right| \leq \epsilon,$$

which entails the statement of the lemma, since $g^{-1}(z_1, \dots, z_{\ell+1}) \subseteq h^{-1}(z_1, \dots, z_\ell)$ for all $z = (z_1, \dots, z_{\ell+1}) \in \mathbb{S}^\ell$. \square

Corollary 3.4. *In the setting of Theorem 3.3,*

$$\int_{\text{sgn}(f)^{-1}(y)} |f|^{\ell/s} d\mathcal{H}^{m-\ell} < \infty, \quad \text{for } \mathcal{H}^\ell\text{-a.e. } y \in \mathbb{S}^\ell, \quad (3.8)$$

and for every relatively compact open $K \Subset \Omega \setminus f^{-1}(0)$,

$$\mathcal{H}^{m-\ell}(K \cap \text{sgn}(f)^{-1}(y)) < \infty, \quad \text{for } \mathcal{H}^\ell\text{-a.e. } y \in \mathbb{S}^\ell. \quad (3.9)$$

Proof. It is clear that (3.3) implies (3.8). Let $K \Subset \Omega \setminus f^{-1}(0)$ be open and relatively compact. By the coarea formula, where h is the map defined in the proof of Theorem 3.3,

$$\int_{\mathbb{R}^\ell} \mathcal{H}^{m-\ell}(K \cap h^{-1}(y)) dy = \int_K |J_\ell h(x)| dx \quad (3.10)$$

which is finite, since $|f| \geq \delta > 0$ on K . So there is a subset $A = A_{f,K} \subseteq \mathbb{R}^\ell$ with $|\mathbb{R}^\ell \setminus A| = 0$ such that $\mathcal{H}^{m-\ell}(K \cap h^{-1}(y)) < \infty$ for all $y \in A$. This entails (3.9). \square

3.4. Level sets of the norm. The result of this section will not be needed in this paper but we think it is interesting in its own right.

Theorem 3.5. *Let $k \in \mathbb{N}_+$, $\alpha \in (0, 1]$, and set $s = k + \alpha$. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and $f \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^{\ell+1})$, $f \neq 0$. Then there is a constant $C = C(m, \ell, k, \alpha, \Omega)$ such that for all $0 < \epsilon \leq 1$ and all small $\delta > 0$ we have*

$$\left| \left\{ y \in (0, \delta) : y^{1/s} \mathcal{H}^{m-1}(|f|^{-1}(y)) \geq \epsilon^{-1} C \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{1/s} \right\} \right| \leq \epsilon \delta.$$

Proof. For $\delta > 0$ consider

$$I_f(\delta) := \int_0^\delta y^{1/s} \mathcal{H}^{m-1}(|f|^{-1}(y)) dy.$$

(Note that $|f|$ is Lipschitz and has a Lipschitz extension to \mathbb{R}^m , and thus $y \mapsto y^{1/s} \mathcal{H}^{m-1}(|f|^{-1}(y))$ is \mathcal{L}^1 -measurable.) Then

$$\begin{aligned} I_f(\delta) &= \int_0^\delta \int_{|f|^{-1}(y)} |f|^{1/s} d\mathcal{H}^{m-1} dy \\ &= \int_{\mathbb{R}} \int_{|f|^{-1}(y)} |f|^{1/s} \mathbf{1}_{|f|^{-1}((0,\delta))} d\mathcal{H}^{m-1} dy \\ &= \int_{|f|^{-1}((0,\delta))} |f(x)|^{1/s} |\nabla |f|(x)| dx, \end{aligned}$$

where the last identity holds by the coarea formula (Corollary 3.2).

On the set $\{x : f(x) \neq 0\}$,

$$|\partial_i |f|| = \left| \frac{\langle f, \partial_i f \rangle}{|f|} \right| \leq |\partial_i f|.$$

Hence,

$$\begin{aligned}
|f|^{1/s} |\nabla|f|| &\leq \sqrt{m} \max_{1 \leq i \leq m} |f|^{1/s} |\partial_i|f|| \leq \sqrt{m} \max_{1 \leq i \leq m} |f| \frac{|\partial_i f|}{|f|^{1-1/s}} \\
&\leq \sqrt{m(\ell+1)} |f| \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq \ell+1}} \frac{|\partial_i f_j|}{|f|^{1-1/s}} \\
&\leq s \sqrt{m(\ell+1)} |f| \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq \ell+1}} |\partial_i(|f_j|^{1/s})|.
\end{aligned}$$

By Theorem 2.7 (applied to an extension of f as in Section 3.2),

$$\int_{\Omega} |\partial_i(|f_j|^{1/s})| dx \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k, \alpha}(\bar{\Omega})}^{1/s}.$$

Consequently, we have

$$I_f(\delta) \leq C \|f\|_{C^{k, \alpha}(\bar{\Omega})}^{1/s} \cdot \delta, \quad (3.11)$$

for a constant $C = C(m, \ell, k, \alpha, \Omega)$.

Set $A_{\epsilon, \delta} := \{y \in (0, \delta) : y^{1/s} \mathcal{H}^{m-1}(|f|^{-1}(y)) \geq \epsilon^{-1} C \|f\|_{C^{k, \alpha}(\bar{\Omega})}^{1/s}\}$. For small $\delta > 0$, we have the lower bound

$$I_f(\delta) \geq \epsilon^{-1} C \|f\|_{C^{k, \alpha}(\bar{\Omega})}^{1/s} |A_{\epsilon, \delta}|$$

which implies the assertion in view of (3.11). \square

Corollary 3.6. *In the setting of Theorem 3.5 let $A \subseteq [0, \infty)$ be such that $|A \cap [0, \epsilon)| = \epsilon$ for some $\epsilon > 0$. Then there is a sequence $A \ni y_j \rightarrow 0$ with*

$$\sup_j \left(y_j^{1/s} \mathcal{H}^{m-1}(|f|^{-1}(y_j)) \right) \leq C(m, \ell, k, \alpha, \Omega) \|f\|_{C^{k, \alpha}(\bar{\Omega})}^{1/s}.$$

4. BACKGROUND ON FUNCTIONS OF BOUNDED VARIATION

In this section we recall some facts on functions of bounded variation and fix notation. We follow the presentation in [3]. In Sections 4.6 and 4.7 we prove some simple statements we shall need later on. They are probably well-known, but we include proofs, since we could not find them in the literature.

4.1. Functions of bounded variation. Let $\Omega \subseteq \mathbb{R}^m$ be open. A real-valued function $f \in L^1(\Omega)$ is a *function of bounded variation in Ω* if the distributional derivative of f is representable by a finite Radon measure in Ω , i.e.,

$$\int_{\Omega} f \partial_i \varphi dx = - \int_{\Omega} \varphi dD_i f, \quad \text{for all } \varphi \in C_c^\infty(\Omega), \quad i = 1, \dots, m,$$

for some \mathbb{R}^m -valued measure $Df = (D_1 f, \dots, D_m f)$ in Ω . The functions of bounded variation in Ω form a vector space denoted by $BV(\Omega)$. The Sobolev space $W^{1,1}(\Omega)$ is strictly contained in $BV(\Omega)$; for $f \in W^{1,1}(\Omega)$, $Df = \nabla f \mathcal{L}^m$.

A real-valued function $f \in L^1_{\text{loc}}(\Omega)$ belongs to $BV_{\text{loc}}(\Omega)$ if $f \in BV(\Omega')$ for every relatively compact $\Omega' \Subset \Omega$. We define $BV(\Omega, \mathbb{R}^\ell) := BV(\Omega, \mathbb{R})^\ell$ and $BV(\Omega, \mathbb{C}) := BV(\Omega, \mathbb{R}^2)$.

An element $f = (f_1, \dots, f_\ell) \in L^1(\Omega, \mathbb{R}^\ell)$ belongs to $BV(\Omega, \mathbb{R}^\ell)$ if and only if the *variation*

$$\text{Var}(f, \Omega) := \sup \left\{ \sum_{j=1}^{\ell} \int_{\Omega} f_j \operatorname{div} \varphi_j \, dx : \varphi \in C_c^\infty(\Omega, \mathbb{R}^m)^\ell, \|\varphi\|_\infty \leq 1 \right\}$$

is finite. Then $\text{Var}(f, \Omega)$ coincides with the total variation measure $|Df|(\Omega)$. It is lower semicontinuous in the $L^1_{\text{loc}}(\Omega, \mathbb{R}^\ell)$ topology, i.e.,

$$\text{Var}(f, \Omega) \leq \liminf_{k \rightarrow \infty} \text{Var}(f_k, \Omega), \quad \text{for } f_k \rightarrow f \text{ in } L^1_{\text{loc}}(\Omega, \mathbb{R}^\ell).$$

The space $BV(\Omega, \mathbb{R}^\ell)$ endowed with the norm $\|f\|_{BV} := \|f\|_{L^1} + |Df|(\Omega)$ is a Banach space.

4.2. Approximate continuity and differentiability of functions of bounded variation. We say that $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^\ell)$ has an *approximate limit* at $x \in \Omega$ if there is $z \in \mathbb{R}^\ell$ such that

$$\lim_{r \downarrow 0} \int_{B_r(x)} |f(y) - z| \, dy = 0.$$

The *approximate discontinuity set* S_f is the set of $x \in \Omega$, where this property does not hold. For $x \in \Omega \setminus S_f$ the uniquely determined approximate limit z is denoted by $\tilde{f}(x)$. The function f is said to be *approximately continuous* at x if $x \notin S_f$ and $f(x) = \tilde{f}(x)$ (i.e., x is a Lebesgue point of f). The set S_f is an \mathcal{L}^m -negligible Borel set and $\tilde{f} : \Omega \setminus S_f \rightarrow \mathbb{R}^\ell$ is a Borel function which coincides \mathcal{L}^m -a.e. in $\Omega \setminus S_f$ with f .

We say that $x \in \Omega$ is an *approximate jump point* of f if there exist $a^\pm \in \mathbb{R}^\ell$ and $\nu \in \mathbb{S}^{m-1}$ such that $a^+ \neq a^-$ and

$$\lim_{r \downarrow 0} \int_{B_r^\pm(x, \nu)} |f(y) - a^\pm| \, dy = 0,$$

where $B_r^\pm(x, \nu) := \{y \in B_r(x) : \pm \langle y - x, \nu \rangle > 0\}$. The triplet (a^+, a^-, ν) is denoted by $(f^+(x), f^-(x), \nu_f(x))$. The set of approximate jump points, denoted by J_f , is a Borel subset of S_f , the functions $f^\pm : J_f \rightarrow \mathbb{R}^\ell$ and $\nu_f : J_f \rightarrow \mathbb{S}^{m-1}$ are Borel functions.

Let $x \in \Omega \setminus S_f$. Then f is *approximately differentiable* at x if there exists an $\ell \times m$ matrix T such that

$$\lim_{r \downarrow 0} \int_{B_r(x)} \frac{|f(y) - \tilde{f}(x) - T(y - x)|}{r} \, dy = 0.$$

The matrix T is uniquely determined. It is called the *approximate differential* of f at x and denoted by $\nabla f(x)$. The set of approximate differentiability points is denoted by D_f . It is a Borel set and $\nabla f : D_f \rightarrow \mathbb{R}^{\ell m}$ is a Borel function.

By the Federer–Vol’pert theorem (cf. [3, Theorem 3.78]), for every $f \in BV(\Omega, \mathbb{R}^\ell)$ the set S_f is countably \mathcal{H}^{m-1} -rectifiable, $\mathcal{H}^{m-1}(S_f \setminus J_f) = 0$, and

$$Df \llcorner J_f = ((f^+ - f^-) \otimes \nu_f) \mathcal{H}^{m-1} \llcorner J_f.$$

By the Calderón–Zygmund theorem (cf. [3, Theorem 3.83]), each $f \in BV(\Omega, \mathbb{R}^\ell)$ is approximately differentiable at \mathcal{L}^m -a.e. point of Ω , and the approximate differential ∇f is the density of the absolutely continuous part of Df with respect to \mathcal{L}^m .

4.3. Decomposition of Df . Let $f \in BV(\Omega, \mathbb{R}^\ell)$. The Lebesgue decomposition provides a decomposition

$$Df = D^a f + D^s f,$$

where $D^a f$ is the absolutely continuous and $D^s f$ is the singular part of Df with respect to \mathcal{L}^m . By defining

$$D^j f := D^s f \llcorner J_f, \quad D^c f := D^s \llcorner (\Omega \setminus S_f)$$

we obtain the decomposition

$$Df = D^a f + D^j f + D^c f,$$

noting that Df vanishes on the \mathcal{H}^{m-1} -negligible set $S_f \setminus J_f$ (cf. [3, Lemma 3.76]). Then $D^j f$ and $D^c f$ are called the *jump* and the *Cantor part* of Df , respectively. We have

$$\begin{aligned} D^a f &= \nabla f \mathcal{L}^m, \\ D^j f &= ((f^+ - f^-) \otimes \nu_f) \mathcal{H}^{m-1} \llcorner J_f. \end{aligned} \tag{4.1}$$

The Cantor part vanishes on sets which are σ -finite with respect to \mathcal{H}^{m-1} and on sets of the form $\tilde{f}^{-1}(E)$, where $E \subseteq \mathbb{R}^\ell$ with $H^1(E) = 0$ (cf. [3, Proposition 3.92]).

4.4. Special functions of bounded variation. A function $f \in BV(\Omega, \mathbb{R}^\ell)$ is said to be a *special function of bounded variation* if $D^c f = 0$; in this case we write $f \in SBV(\Omega, \mathbb{R}^\ell)$. Then $SBV(\Omega, \mathbb{R}^\ell)$ forms a closed subspace of $BV(\Omega, \mathbb{R}^\ell)$. We have strict inclusions $W^{1,1}(\Omega, \mathbb{R}^\ell) \subsetneq SBV(\Omega, \mathbb{R}^\ell) \subsetneq BV(\Omega, \mathbb{R}^\ell)$, in fact:

Proposition 4.1 ([3, Proposition 4.4]). *Let $\Omega \subseteq \mathbb{R}^m$ be open and bounded, and let $K \subseteq \mathbb{R}^m$ be closed with $\mathcal{H}^{m-1}(K \cap \Omega) < \infty$. Then any function $f : \Omega \rightarrow \mathbb{R}$ which belongs to $L^\infty(\Omega \setminus K) \cap W^{1,1}(\Omega \setminus K)$ belongs also to $SBV(\Omega)$ and satisfies $\mathcal{H}^{m-1}(S_f \setminus K) = 0$.*

It is not hard to conclude from this proposition that the solutions of $Z^n = x$, $x \in \mathbb{C}$, admit representations in SBV_{loc} ; but see Example 2.4.

4.5. The chain rule. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open set. It is not hard to see that the composite $h = f \circ g$ of a function $g \in BV(\Omega, \mathbb{R}^\ell)$ and a Lipschitz function $f : \mathbb{R}^\ell \rightarrow \mathbb{R}^k$ belongs to $BV(\Omega, \mathbb{R}^k)$ and that $|Dh| \leq \text{Lip}(f)|Dg|$. We shall need a more precise *chain rule*. For our purpose it is enough to assume that f is C^1 ; for the general case see [3, Theorem 3.101].

It is convenient to distinguish between the *diffuse* part $\tilde{D}g := D^a g + D^c g$ and the jump part $D^j g$ of Dg , since they behave differently.

Theorem 4.2 ([3, Theorem 3.96]). *Let $g \in BV(\Omega, \mathbb{R}^\ell)$ and let $f \in C^1(\mathbb{R}^\ell, \mathbb{R}^k)$ be a Lipschitz function satisfying $f(0) = 0$ if $|\Omega| = \infty$. Then $h = f \circ g$ belongs to $BV(\Omega, \mathbb{R}^k)$ and*

$$\begin{aligned} \tilde{D}h &= \nabla f(g) \nabla g \mathcal{L}^m + \nabla f(\tilde{g}) D^c g = \nabla f(\tilde{g}) \tilde{D}g, \\ D^j h &= (f(g^+) - f(g^-)) \otimes \nu_g \mathcal{H}^{m-1} \llcorner J_g. \end{aligned} \tag{4.2}$$

The following *product rule* is an immediate consequence. For $g_1, g_2 \in BV(\Omega)$, $g = (g_1, g_2)$, and $f(y_1, y_2) = y_1 y_2$, the product $g_1 g_2$ belongs to $BV(\Omega)$ and

$$\begin{aligned} \tilde{D}(g_1 g_2) &= \tilde{g}_1 \tilde{D}g_2 + \tilde{g}_2 \tilde{D}g_1, \\ D^j(g_1 g_2) &= (f(g^+) - f(g^-)) \nu_g \mathcal{H}^{m-1} \llcorner J_g. \end{aligned} \tag{4.3}$$

If $g \in W^{1,p}(\Omega, \mathbb{R}^\ell)$, $f : \mathbb{R}^\ell \rightarrow \mathbb{R}^k$ is Lipschitz, and $f \circ g \in L^p(\Omega, \mathbb{R}^\ell)$, then $f \circ g \in W^{1,p}(\Omega, \mathbb{R}^\ell)$ and the chain rule reduces to

$$\nabla(f \circ g)(x) = \nabla f(g(x)) \cdot \nabla g(x) \quad \text{for a.e. } x \in \Omega; \quad (4.4)$$

e.g. [52, Theorem 2.1.11].

4.6. Extension to zero sets. The *pointwise variation* of a function $f : I \rightarrow \mathbb{R}^\ell$ on an open interval $I = (a, b)$, for $a < b \in \mathbb{R}_{\pm\infty}$ is defined by

$$\text{pVar}(f, I) := \sup \left\{ \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| : n \geq 2, a < t_1 < \dots < t_n < b \right\}.$$

For open $\Omega \subseteq \mathbb{R}$ one sets

$$\text{pVar}(f, \Omega) := \sum_I \text{pVar}(f, I),$$

where I runs through all connected components of Ω . The *essential variation*

$$\text{eVar}(f, \Omega) := \inf \{ \text{pVar}(g, \Omega) : g = f \text{ } \mathcal{L}^1\text{-a.e. in } \Omega \}$$

coincides with the variation $\text{Var}(f, \Omega)$ if f is in $L^1_{\text{loc}}(\Omega)$; see [3, Theorem 3.27].

Lemma 4.3. *Let $\Omega_0 \subseteq \Omega \subseteq \mathbb{R}$ be bounded open subsets. Let $f : \Omega \rightarrow \mathbb{C}$ be a function in $L^1_{\text{loc}}(\Omega)$ such that $f|_{\Omega_0}$ has bounded (resp. pointwise) variation and f vanishes and is continuous at all points of $\Omega \setminus \Omega_0$. Then f has bounded (resp. pointwise) variation on Ω and $\text{Var}(f, \Omega_0) = \text{Var}(f, \Omega)$ (resp. $\text{pVar}(f, \Omega_0) = \text{pVar}(f, \Omega)$).*

Proof. We prove first the statement about the pointwise variation. Let $I = (a, b)$ be a connected component of Ω and let $\{J_k\}_k$ be the collection of all components of Ω_0 contained in I . Suppose $a < t_1 < \dots < t_n < b$ and let $\epsilon > 0$. We have a subdivision into a finite number m of maximal chains $t_i < t_{i+1} < \dots < t_{i+j}$ contained in some J_k and the remaining $t_p \in \Omega \setminus \Omega_0$. If $t_i < t_{i+1} < \dots < t_{i+j}$ is a chain contained in $J_k = (a_k, b_k)$, there exist t_i^- and t_{i+j}^+ such that $a_k < t_i^- < t_i < \dots < t_{i+j} < t_{i+j}^+ < b_k$ and $|f(t_i^-)| \leq \epsilon/(2m)$ and $|f(t_{i+j}^+)| \leq \epsilon/(2m)$. Thus,

$$\sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| \leq \sum_k \text{pVar}(f, J_k) + \epsilon$$

and consequently $\text{pVar}(f, I) \leq \sum_k \text{pVar}(f, J_k)$. This implies $\text{pVar}(f, \Omega_0) = \text{pVar}(f, \Omega)$.

Suppose that $\text{Var}(f, \Omega_0) < \infty$. Then

$$\text{Var}(f, \Omega_0) = \inf \{ \text{pVar}(g, \Omega_0) : g = f \text{ } \mathcal{L}^1\text{-a.e. in } \Omega_0 \}$$

and there exists g coinciding \mathcal{L}^1 -a.e. with f in Ω_0 such that $\text{pVar}(g, \Omega_0) < \infty$. If we extend such g by 0 on $\Omega \setminus \Omega_0$, then $\text{pVar}(g, \Omega_0) = \text{pVar}(g, \Omega)$ by the first part. (For the proof of the first part it is enough that, if $t_0 \in \Omega \setminus \Omega_0$, then for all $\epsilon > 0$ there is $\delta > 0$ such that $|f(t)| \leq \epsilon$ for \mathcal{L}^1 -a.e. t with $|t - t_0| \leq \delta$.) This implies $\text{Var}(f, \Omega_0) \geq \text{Var}(f, \Omega)$. The opposite inequality is trivial. The proof is complete. \square

The lemma implies a similar result for functions in several variables. If $\Omega \subseteq \mathbb{R}^m$ is an open set and $v \in \mathbb{S}^{m-1}$, we denote by Ω_v the orthogonal projection of Ω onto the hyperplane orthogonal to v . For each $y \in \Omega_v$ we have the section $\Omega_y^v := \{t \in$

$\mathbb{R} : y + tv \in \Omega\}$. If f is a function defined on Ω , then $f_y^v := f(y + tv)$ is a function defined on Ω_y^v .

Proposition 4.4. *Let $\Omega_0 \subseteq \Omega \subseteq \mathbb{R}^m$ be bounded open subsets. Let $f : \Omega \rightarrow \mathbb{C}$ be a function such that $f|_{\Omega_0}$ has bounded variation and f vanishes and is continuous at all points of $\Omega \setminus \Omega_0$. Then f has bounded variation on Ω and $\text{Var}(f, \Omega_0) = \text{Var}(f, \Omega)$.*

Proof. Since $f|_{\Omega_0} \in BV(\Omega_0)$, [3, Remark 3.104] implies that there exist m linearly independent vectors v_i such that $f_y^{v_i} \in BV(\Omega_{0,y}^{v_i})$ for \mathcal{L}^{m-1} -a.e. $y \in \Omega_{0,v_i}$ and

$$\int_{\Omega_{0,v_i}} |Df_y^{v_i}|(\Omega_{0,y}^{v_i}) dy < \infty, \quad \text{for } i = 1, \dots, m.$$

By Lemma 4.3, each such $f_y^{v_i}$ extends by 0 to a function $f_y^{v_i} \in BV(\Omega_y^{v_i})$ with $|Df_y^{v_i}|(\Omega_{0,y}^{v_i}) = |Df_y^{v_i}|(\Omega_y^{v_i})$. Since $f_y^{v_i} \equiv 0$ for all $y \in \Omega_{v_i} \setminus \Omega_{0,v_i}$, we have

$$\int_{\Omega_{v_i}} |Df_y^{v_i}|(\Omega_y^{v_i}) dy = \int_{\Omega_{0,v_i}} |Df_y^{v_i}|(\Omega_{0,y}^{v_i}) dy < \infty, \quad \text{for } i = 1, \dots, m.$$

Thus $f \in BV(\Omega)$, again by [3, Remark 3.104]. The identity $\text{Var}(f, \Omega_0) = \text{Var}(f, \Omega)$ follows from [3, Theorem 3.103]. \square

4.7. A sufficient condition for bounded variation. Let us recall a version of the Gauss–Green theorem which is a special case of the result given in [18, p. 314].

Theorem 4.5 (Gauss–Green theorem). *Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open set with $\mathcal{H}^{m-1}(\partial\Omega) < \infty$. Assume that there is a closed set $E \subseteq \mathbb{R}^m$ such that $\partial\Omega \setminus E$ is a C^1 -hypersurface and $\mathcal{H}^{m-1}(E) = 0$. Then for each $f \in C^1(\bar{\Omega}, \mathbb{R}^m)$,*

$$\int_{\Omega} \text{div } f \, dx = - \int_{\partial\Omega} \langle f, \nu_{\Omega} \rangle d\mathcal{H}^{m-1},$$

where ν_{Ω} is the inner unit normal to Ω .

The following consequence will be used several times.

Proposition 4.6. *Let $\Omega_0 \subseteq \mathbb{R}^m$ be an open set and let E be a closed C^1 -hypersurface in Ω_0 . Let $f = (f_1, \dots, f_{\ell}) \in L^1(\Omega_0, \mathbb{R}^{\ell})$ be such that $f \in W^{1,1}(\Omega_0 \setminus E)$, f is C^1 on $\Omega_0 \setminus E$ and extends together with its partial derivatives continuously to E from both sides, and*

$$\int_E |f| d\mathcal{H}^{m-1} < \infty.$$

Then $f \in BV(\Omega_0, \mathbb{R}^{\ell})$ and

$$\text{Var}(f, \Omega_0) \leq C(m, \ell) \left(\int_{\Omega_0 \setminus E} |\nabla f| \, dx + 2 \int_E |f| d\mathcal{H}^{m-1} \right).$$

Proof. Let $\varphi \in C_c^{\infty}(\Omega_0, \mathbb{R}^m)^{\ell}$ with $\|\varphi\|_{\infty} \leq 1$. There exists a open relatively compact subset $\Omega_{\varphi} \Subset \Omega_0$ which contains $\text{supp } \varphi$ and has smooth boundary such that $\mathcal{H}^{m-1}(\partial\Omega_{\varphi} \cap E) = 0$. Let \mathcal{C} be the set of connected components of the open

set $\Omega_\varphi \setminus E$. Then, by the Gauss–Green theorem 4.5, for each $j = 1, \dots, \ell$,

$$\begin{aligned} \int_{\Omega_0} f_j \operatorname{div} \varphi_j \, dx &= \int_{\Omega_\varphi} f_j \operatorname{div} \varphi_j \, dx = \sum_{C \in \mathcal{C}} \int_C f_j \operatorname{div} \varphi_j \, dx \\ &= \sum_{C \in \mathcal{C}} \left(- \int_C \langle \nabla f_j, \varphi \rangle \, dx - \int_{\partial C} f_j \langle \varphi, \nu \rangle \, d\mathcal{H}^{m-1} \right) \\ &= - \int_{\Omega_\varphi} \langle \nabla f_j, \varphi \rangle \, dx - \sum_{C \in \mathcal{C}} \int_{\partial C} f_j \langle \varphi, \nu \rangle \, d\mathcal{H}^{m-1} \\ &= - \int_{\Omega_0} \langle \nabla f_j, \varphi \rangle \, dx - \sum_{C \in \mathcal{C}} \int_{\partial C \cap E} f_j \langle \varphi, \nu \rangle \, d\mathcal{H}^{m-1}, \end{aligned}$$

where ν is the inner unit normal to C . The statement follows. \square

5. PROOF OF THEOREM 1.4

Let $r > 0$ and $m \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}$ and $\alpha \in (0, 1]$ be such that

$$k + \alpha \geq \max\{r, m\}.$$

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and let $f \in C^{k, \alpha}(\overline{\Omega})$. By Section 3.2, we may assume that f has compact support contained in Ω . Let

$$r = \ell + \beta$$

be the unique representation of r , where $\ell \in \mathbb{N}$ and $\beta \in (0, 1]$. Consider the equation

$$Z^r = f. \tag{5.1}$$

In order to single out some trivial cases we make the following distinction.

Case 1: $\ell = 0$, $1/\beta \in \mathbb{N}$. In this case $f^{1/\beta}$ is a solution of (5.1) in the same differentiability class as f .

Case 2: $\ell = 0$, $1/\beta \notin \mathbb{N}$. Then $1/\beta > 1$ and $f^{\lfloor 1/\beta \rfloor} \cdot \lambda$ is a solution of (5.1), provided that λ solves (5.1) for $r = 1/\{1/\beta\}$. The factor $f^{\lfloor 1/\beta \rfloor}$ is in the differentiability class of f . Since $\{1/\beta\} \in (0, 1)$, the existence and regularity of λ is covered by the next case.

Case 3: $r > 1$ or equivalently $\ell \in \mathbb{N}_+$, $\beta \in (0, 1]$.

Henceforth we restrict to the Case 3.

Let $\Omega_0 := \Omega \setminus f^{-1}(0)$, and consider the sign map $\operatorname{sgn}(f)$ defined in (2.2). By Proposition 2.2, for each regular value $y \in \mathbb{S}^1$ of $\operatorname{sgn}(f)$ the set $E := \operatorname{sgn}(f)^{-1}(y)$ is a closed C^k -hypersurface of Ω_0 (possibly empty) and there is a continuous function $\lambda : \Omega \setminus E \rightarrow \mathbb{C}$ such that $\lambda^r = f$; note that λ is of class C^k on $\Omega_0 \setminus E$. If we write $f = u + iv$, then, as in (3.4),

$$|\nabla \lambda| = \frac{1}{r} \frac{|\nabla f|}{|f|^{1-1/r}} \leq \left(|\nabla(|u|^{1/r})| + |\nabla(|v|^{1/r})| \right).$$

By Theorem 2.7, we may conclude that $\nabla \lambda \in L_w^p(\Omega_0 \setminus E, \mathbb{C}^m)$ with

$$\|\nabla \lambda\|_{L_w^p(\Omega_0 \setminus E)} \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k, \alpha}(\overline{\Omega})}^{1/r} \tag{5.2}$$

where $p = r/(r-1)$. Here we use the fact that $C^{k, \alpha}(\overline{\Omega})$ is continuously embedded in $C^{\ell, \beta}(\overline{\Omega})$, as Ω is quasiconvex; cf. [15, Proposition 3.7].

There exists $y \in \mathbb{S}^1$ which is a regular value of $\text{sgn}(f)$ and such that $E = \text{sgn}(f)^{-1}(y)$ satisfies

$$\int_E |f|^{1/r} d\mathcal{H}^{m-1} \leq C(m, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{1/r}, \quad (5.3)$$

$$\mathcal{H}^{m-1}(K \cap E) < \infty, \quad \text{for every relatively compact } K \Subset \Omega_0. \quad (5.4)$$

Here we apply Theorem 3.3 and Corollary 3.4, using $s := k + \alpha \geq r$ and the continuous inclusion of $C^{k,\alpha}(\overline{\Omega})$ in $C^{\ell,\beta}(\overline{\Omega})$.

Then Proposition 4.6 implies that $\lambda \in BV(\Omega_0)$ and

$$\begin{aligned} \text{Var}(\lambda, \Omega_0) &\leq C(m) \left(\int_{\Omega_0 \setminus E} |\nabla \lambda| dx + 2 \int_E |f|^{1/r} d\mathcal{H}^{m-1} \right) \\ &\leq C(m, k, \alpha, \Omega) \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{1/r}, \end{aligned}$$

by (5.2) and (5.3). By Proposition 4.4, we may conclude that $\lambda \in BV(\Omega)$ and we obtain the uniform bound (1.4), since $\|\lambda\|_{L^1(\Omega)} \leq |\Omega| \|f\|_{L^\infty(\Omega)}^{1/r}$.

By construction, $S_\lambda = J_\lambda = E$. To see that the Cantor part $D^c \lambda$ vanishes consider the disjoint union $\Omega = (\Omega_0 \setminus E) \cup E \cup f^{-1}(0)$. Now λ is of class $W^{1,1}$ in $\Omega_0 \setminus E$, E is σ -finite with respect to \mathcal{H}^{m-1} , and λ is continuous on the set $f^{-1}(0) = \lambda^{-1}(0)$. Thus $D^c \lambda$ is zero on each of these sets, by [3, Proposition 3.92]. The proof of Theorem 1.4 is complete. \square

6. FORMULAS FOR THE ROOTS

6.1. Idea of the proof of Theorem 1.1. The main idea of the proof of Theorem 1.1 lies in the reduction to the radical case using the formulas for the roots of the universal polynomial P_a , $a \in \mathbb{C}^n$, given in [32, Theorem 1.6]. These formulas express the roots of P_a as finite sums of functions analytic in the radicals of local coordinates on a resolution space (i.e. a blowing-up of \mathbb{C}^n), see [32] and Theorem 6.4 below. Thus using the radical case we may choose a parameterization of every such locally defined summand, by choosing a parameterization of the radicals, but then a new difficulty arises. It comes from the fact that these summands can be defined only locally on the resolution space. Therefore we have to cut and paste them, introducing new discontinuities and taking care of the integrability of the new jumps. This step will be done in Section 7.4.

Remark 6.1. This local decomposition of the roots into finitely many summands is obtained in [32] by a repeated “splitting” of P_a , a procedure that reflects how the roots are regrouped in clusters. The summands of Equation (6.5) represent the arithmetic means of such clusters.

6.2. Tower of smooth principalizations and formulas for the roots. Let

$$P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j}, \quad (6.1)$$

where $a_j \in \mathbb{C}$. We defined in [32] the generalized discriminant ideals $\mathcal{D}_\ell \subseteq \mathbb{C}[a] = \mathbb{C}[a_1, \dots, a_n]$, $\ell = 2, \dots, n$, that, in particular, satisfy the following properties. For each ℓ the zero set $V(\mathcal{D}_\ell)$ of \mathcal{D}_ℓ is exactly the set of those a for which P_a has at most $\ell - 1$ distinct roots and therefore $V(\mathcal{D}_{\ell-1}) \subseteq V(\mathcal{D}_\ell)$. The top ideal \mathcal{D}_n is principal

and generated by a power of the discriminant of P_a . The other discriminant ideals are not principal.

For the ideals \mathcal{D}_ℓ , $\ell = 2, \dots, n$, we constructed in [32] a tower of smooth principalizations

$$M_1 = \mathbb{C}^n \xleftarrow{\sigma_{2,1}} M_2 \xleftarrow{\sigma_{3,2}} M_3 \xleftarrow{\sigma_{4,3}} \dots \xleftarrow{\sigma_{n,n-1}} M_n, \quad (6.2)$$

where each $\sigma_{\ell,\ell-1}$ is the composition of blowing-ups with smooth centers and for every $\sigma_\ell = \sigma_{2,1} \circ \sigma_{3,2} \circ \dots \circ \sigma_{\ell,\ell-1}$ the ideal $\sigma_\ell^*(\mathcal{D}_\ell)$ is principal. We write $\sigma_{n,\ell} = \sigma_{\ell+1,\ell} \circ \dots \circ \sigma_{n,n-1}$. Such a tower of smooth principalizations exists by the classical results of resolution of singularities.

By construction each $p \in M_\ell$ comes with a privileged system of local analytic coordinates on M_ℓ at p . It is obtained as follows (see Definition 6.5). The map σ_ℓ is itself a blowing-up of an ideal $\mathcal{K}_\ell \subseteq \mathbb{C}[a]$ such that $V(\mathcal{K}_\ell) = V(\mathcal{D}_\ell)$. We fix such an ideal \mathcal{K}_ℓ as well as a system of its generators $\mathcal{K}_\ell = (h_1, \dots, h_k)$, where $h_i \in \mathbb{C}[a]$. The pull-back $\sigma_\ell^*(\mathcal{K}_\ell)$ is principal, so it is generated in a neighborhood of p by one of the $h_i \circ \sigma_\ell$. We denote this h_i simply by h . Also, in order to simplify the notation, we use the same symbol for a polynomial on \mathbb{C}^n and its pull-back to M_ℓ (in particular, we will write h instead of $h_i \circ \sigma_\ell$) if no confusion is possible.

Similarly, $\sigma_\ell^*(\mathcal{D}_\ell)$ is principal and we fix a polynomial $f \in \mathbb{C}[a]$ that generates $\sigma_\ell^*(\mathcal{D}_\ell)$ at p . Let $E = V(\sigma_\ell^*(\mathcal{D}_\ell))$. There is a neighborhood \mathcal{U} of p in M_ℓ , and a coordinate system y_1, \dots, y_n on \mathcal{U} , such that $y_i = P_i/h^s$, $P_i \in \mathbb{C}[a]$, $s \in \mathbb{N}$, $E \cap \mathcal{U} = \{y_1 \cdots y_r = 0\}$ and such that on \mathcal{U}

$$f = \text{unit} \cdot \prod_{i=1}^r y_i^{n_i}, \quad h = \text{unit} \cdot \prod_{i=1}^r y_i^{m_i}, \quad (6.3)$$

with $n_i > 0$ and $m_i > 0$. Here by a unit we mean an analytic function defined and nowhere vanishing on \mathcal{U} .

Remark 6.2. The choice of h , f , and P_i is not unique, but f and h are well defined at p up to a local analytic unit. Similarly, y_i , for $i \leq r$, are defined by their zero sets up to a unit. These zero sets, that is the components of the exceptional divisor of σ_ℓ , are well-defined.

Definition 6.3. By a *chain* $\mathbf{C} = (p_\ell, f_\ell, h_\ell, P_{\ell,i}, s_\ell, r_\ell)$ for $p_n \in M_n$ we mean the points $p_\ell := \sigma_{n,\ell}(p_n)$, $\ell = 1, \dots, n$, and the local data $(f_\ell, h_\ell, P_{\ell,i}, s_\ell, r_\ell)$ for p_ℓ . We complete this data for $\ell = 1$ by putting $f_1 = h_1 = 1$, $P_{1,i} = a_i$, and $s_1 = r_1 = 0$.

We pull back the polynomial P_a onto M_ℓ via σ_ℓ ,

$$P_{\sigma_\ell^*(a)}(Z) = Z^n + \sum_{i=1}^n (a_i \circ \sigma_\ell) Z^{n-i}.$$

The roots of $P_{\sigma_\ell^*(a)}$ are the pull-backs of the roots of P_a .

Theorem 6.4 ([32, Theorem 1.6]). *We may associate with every chain $\mathbf{C} = (p_\ell, f_\ell, h_\ell, P_{\ell,i}, s_\ell, r_\ell)$ convergent power series ψ_ℓ , integers $q_\ell \geq 1$, and positive exponents $\alpha_\ell \in \frac{1}{q_\ell} \mathbb{N}_{>0}$, such that the following holds. The roots of $P_{\sigma_n^*(a)}$ in a neighborhood of p_n are given by*

$$\sum_{\ell=1}^n A_\ell \varphi_\ell \circ \sigma_{n,\ell}, \quad (6.4)$$

where $A_\ell \in \mathbb{Q}$ and

$$\varphi_\ell = f_\ell^{\alpha_\ell} \psi_\ell(y_{\ell,1}^{1/q_\ell}, \dots, y_{\ell,r_\ell}^{1/q_\ell}, y_{\ell,r_\ell+1}, \dots, y_{\ell,n}). \quad (6.5)$$

The meaning of the radicals in (6.5) is explained in [32, Remark 1.7]. There are neighborhoods \mathcal{U}_ℓ of p_ℓ , satisfying $\sigma_{\ell,\ell-1}(\mathcal{U}_\ell) \subseteq \mathcal{U}_{\ell-1}$, and their branched covers $\tau_\ell : \tilde{\mathcal{U}}_\ell \rightarrow \mathcal{U}_\ell$, given by the formulas

$$y_{\ell,i} = \begin{cases} t_i^{q_\ell} & \text{if } i \leq r_\ell, \\ t_i & \text{if } i > r_\ell + 1, \end{cases} \quad (6.6)$$

such that φ_ℓ can be interpreted as an analytic function on $\tilde{\mathcal{U}}_\ell$. Since $\sigma_{\ell+1,\ell}^{-1}(f_\ell^{-1}(0)) \subseteq f_{\ell+1}^{-1}(0)$, the composite $y_{\ell,i} \circ \sigma_{\ell+1,\ell}$, for $i \leq r_\ell$, is a normal crossings in $y_{\ell+1,1}, \dots, y_{\ell+1,r_{\ell+1}}$ and therefore, we may suppose that $\sigma_{\ell+1,\ell} \circ \tau_{\ell+1}$ factors through τ_ℓ , changing $q_{\ell+1}$ if necessary. Thus we obtain a sequence of branched covers τ_i making the following diagram commutative.

$$\begin{array}{ccccccccccc} \tilde{\mathcal{U}}_1 & \xleftarrow{\tilde{\sigma}_{1,2}} & \tilde{\mathcal{U}}_2 & \xleftarrow{\tilde{\sigma}_{2,3}} & \tilde{\mathcal{U}}_3 & \xleftarrow{\tilde{\sigma}_{3,4}} & \dots & \xleftarrow{\tilde{\sigma}_{n-1,n-2}} & \tilde{\mathcal{U}}_{n-1} & \xleftarrow{\tilde{\sigma}_{n,n-1}} & \tilde{\mathcal{U}}_n \\ \parallel & & \downarrow \tau_2 & & \downarrow \tau_3 & & & & \downarrow \tau_{n-1} & & \downarrow \tau_n \\ \mathcal{U}_1 & \xleftarrow{\sigma_{1,2}} & \mathcal{U}_2 & \xleftarrow{\sigma_{2,3}} & \mathcal{U}_3 & \xleftarrow{\sigma_{3,4}} & \dots & \xleftarrow{\sigma_{n-1,n-2}} & \mathcal{U}_{n-1} & \xleftarrow{\sigma_{n,n-1}} & \mathcal{U}_n \end{array}$$

Then Theorem 6.4 says that the roots of $P_{\tilde{\sigma}_n^*(a)}$, where $\tilde{\sigma}_n = \sigma_n \circ \tau_n$, are combinations of analytic functions on $\tilde{\mathcal{U}}_n$ that are pull-backs of such φ_ℓ .

Definition 6.5. By an *extended chain* $\mathbf{E} = (p_\ell, f_\ell, h_\ell, P_{\ell,i}, s_\ell, r_\ell, \mathcal{U}_\ell)$ for $p_n \in M_n$ we mean a chain $\mathbf{C} = (p_\ell, f_\ell, h_\ell, P_{\ell,i}, s_\ell, r_\ell)$ and a system of neighborhoods \mathcal{U}_ℓ of p_ℓ as above. The $y_{\ell,i} = P_{\ell,i}/h_\ell^{s_\ell}$, $i = 1, \dots, n$, are called a *privileged system of coordinates* on \mathcal{U}_ℓ .

By (6.3), we may express φ_ℓ of (6.5) as follows

$$\varphi_\ell = h_\ell^{\tilde{\alpha}_\ell} \tilde{\psi}_\ell(y_{\ell,1}^{1/\tilde{q}_\ell}, \dots, y_{\ell,r_\ell}^{1/\tilde{q}_\ell}, y_{\ell,r_\ell+1}, \dots, y_{\ell,n}), \quad (6.7)$$

where $\tilde{\alpha}_k \in \frac{1}{q_\ell} \mathbb{N}_{>0}$ and \tilde{q}_ℓ is a positive integer possibly much bigger than q_ℓ .

7. PROOF OF THEOREM 1.1

We assume that the coefficients a_j , $j = 1, \dots, n$, have compact support in \mathbb{R}^m and that the image of a is contained in the closed unit ball $\overline{\mathbb{B}} \subseteq \mathbb{C}^n$. The general case will be reduced to this case in Section 7.5. We assume also that the a_j are of class $C^{k-1,1}$ with $k = k(n, m)$ defined in Section 7.3.

In the following we shall be dealing with multi-valued functions arising from complex radicals, their composition with single-valued functions, and their addition and multiplication.

7.1. In one blow-up chart. We will use the notation of Section 6. Let $(\mathcal{U}, \mathbf{y})$ be a chart on M_ℓ with a privileged system of coordinates $y_i = P_i/h^s$. We may assume that $\mathbf{y}(\mathcal{U})$ coincides with the open unit ball \mathbb{B} in \mathbb{C}^n .

Let $a : \mathbb{R}^m \rightarrow \mathbb{C}^n$ be sufficiently differentiable with compact support. We distinguish between the chart map \mathbf{y} and the composite map $\underline{y} = (y_1, \dots, y_n)$ given by

$$\underline{h} := h \circ a, \quad \underline{P}_i := P_i \circ a, \quad \underline{y}_i := y_i \circ a = \frac{P_i}{h^s}.$$

The map \underline{y} is defined and continuous on the set $\{x \in \mathbb{R}^m : \underline{h}(x) \neq 0\}$. Consider the open subset

$$\Omega_0 := \underline{y}^{-1}(\mathbb{B}) = \{x \in \mathbb{R}^m : |\underline{y}(x)| < 1, \underline{h}(x) \neq 0\} = a^{-1}(V_0), \quad (7.1)$$

where $V_0 = \{a \in \mathbb{C}^n : |y(a)| < 1, h(a) \neq 0\}$. The image of the multi-valued map

$$\underline{y}^{1/\tilde{q}} = (\underline{y}_1^{1/\tilde{q}}, \dots, \underline{y}_r^{1/\tilde{q}}, \underline{y}_{r+1}, \dots, \underline{y}_n) : \Omega_0 \rightsquigarrow \mathbb{C}^n,$$

where $\tilde{q} \in \mathbb{N}_+$, is contained in \mathbb{B} . Since $h = \text{unit} \cdot \prod_{i=1}^r y_i^{m_i}$ for $m_i > 0$ on \mathcal{U} (see (6.3)), we have $|h| \lesssim |y_i|$ for all $1 \leq i \leq r$. Let $\tilde{\psi} \in C^1(\overline{\mathbb{B}})$ and consider the multi-valued function (cf. (6.7))

$$\varphi := \underline{h}^{\tilde{\alpha}} \tilde{\psi}(\underline{y}^{1/\tilde{q}}) : \Omega_0 \rightsquigarrow \mathbb{C};$$

additionally we define $\varphi(x) := 0$ if $\underline{h}(x) = 0$.

$$\begin{array}{ccccccc} \mathbb{C}^n = M_1 & \longleftarrow & V_0 & \longrightarrow & \mathcal{U} & \hookrightarrow & M_\ell \\ & & \uparrow a & \searrow y_j = \frac{P_j}{h^{\tilde{q}}} & \downarrow \underline{y} & & \\ \mathbb{R}^m & \longleftarrow & \Omega_0 & \xrightarrow{\underline{y}} & \mathbb{B} & \hookrightarrow & \mathbb{C}^n \end{array}$$

Remark 7.1. The functions \underline{h} and P_i are defined on \mathbb{R}^m and have compact support, but their supports need not be contained in Ω_0 .

Proposition 7.2. *Let $k \in \mathbb{N}_+$ be a multiple of \tilde{q} satisfying $k \geq \max\{\frac{2s}{\tilde{\alpha}}, m\}$ for the numbers $\tilde{q}, s, \tilde{\alpha}$ associated with the chart \mathcal{U} and φ (see (6.7)). Suppose that $a \in C_c^{k-1,1}(\mathbb{R}^m, \overline{\mathbb{B}})$. There exists a finite collection of C^{k-1} -hypersurfaces $E_i \subseteq \mathbb{R}^m$ and a parameterization ϕ of φ on Ω_0 such that $\phi \in SBV(\Omega_0)$, ϕ is continuous on $\Omega_0 \setminus \bigcup_i E_i$, and*

$$\|\phi\|_{BV(\Omega_0)} \leq C (|\Omega_0| + 1) \left(\|\underline{h}\|_{C^{k-1,1}}^{1/k} + \sum_{i=1}^n \|P_i\|_{C^{k-1,1}}^{1/k} \right), \quad (7.2)$$

with $C > 0$ depending only on $m, k, \tilde{\alpha}, \tilde{q}, s, \|\underline{h}\|_{L^\infty}, \|\tilde{\psi}\|_{C^1}$, and $\|P_j\|_{L^\infty}$ for $1 \leq j \leq r$.

Proof. This follows from Proposition 8.3 which will be proved in the next section. \square

Convention 7.3. We will consistently write ϕ for a parameterization of the multi-valued function φ . Then ϕ is an ordered tuple of single-valued complex functions.

Proposition 7.2, Proposition 4.4, and Formula (6.4) give a parameterization of the roots by functions of bounded variation on the set corresponding to one chart. Our next step is to cut and paste such local roots. For this we use the following lemma.

Lemma 7.4. *Let $u \in C^1(\overline{\mathbb{B}})$ have values in the interval $[b, c]$. Suppose that $a \in C_c^{k-1,1}(\mathbb{R}^m, \overline{\mathbb{B}})$, where $k \geq \max\{\frac{s}{\tilde{\alpha}}, m\}$. Consider the function $u \circ \underline{y} : \Omega_0 \rightarrow \mathbb{R}$. There is a constant $C > 0$ which depends only on $n, m, k, \|\underline{h}\|_{L^\infty}$, and $\|\nabla u\|_{L^\infty}$ such that for all small $\epsilon > 0$ we have*

$$\left| \left\{ r \in [b, c] : \int_{(u \circ \underline{y})^{-1}(r)} |\underline{h}|^{\tilde{\alpha}} d\mathcal{H}^{m-1} \geq \epsilon^{-1} C \left(\|\underline{h}\|_{C^{k-1,1}}^{1/k} + \sum_{j=1}^n \|P_j\|_{C^{k-1,1}}^{1/k} \right) \right\} \right| \leq \epsilon.$$

Proof. We have, on Ω_0 ,

$$\begin{aligned} & |\underline{h}|^{\tilde{\alpha}} |\nabla(u \circ \underline{y})| \\ & \lesssim |\nabla u(\underline{y})| \max_j \left(|\underline{y}_j|^{1-1/k} |\underline{h}|^{\tilde{\alpha}-s/k} \frac{|\nabla \underline{P}_j|}{|\underline{P}_j|^{1-1/k}} + s |\underline{y}_j| |\underline{h}|^{\tilde{\alpha}-1/k} \frac{|\nabla \underline{h}|}{|\underline{h}|^{1-1/k}} \right). \end{aligned}$$

By Theorem 2.7,

$$\int_{\Omega_0} |\underline{h}|^{\tilde{\alpha}} |\nabla(u \circ \underline{y})| dx \leq C \left(\|\underline{h}\|_{C^{k-1,1}}^{1/k} + \sum_{j=1}^n \|\underline{P}_j\|_{C^{k-1,1}}^{1/k} \right), \quad (7.3)$$

for a constant C as specified in the assertion. By the coarea formula (Corollary 3.2),

$$\int_{\Omega_0} |\underline{h}|^{\tilde{\alpha}} |\nabla(u \circ \underline{y})| dx = \int_b^c \int_{(u \circ \underline{y})^{-1}(r)} |\underline{h}|^{\tilde{\alpha}} d\mathcal{H}^{m-1} dr,$$

and the assertion follows as at the end of the proof of Theorem 3.3. \square

7.2. Choice of neighborhoods. In what follows we fix a finite family of extended chains

$$\mathcal{CV} = \{\mathbf{E}_j\} = \{(p_{j,\ell}, f_{j,\ell}, h_{j,\ell}, P_{j,\ell,i}, s_{j,\ell}, r_{j,\ell}, \mathcal{U}_{j,\ell})\} \quad (7.4)$$

such that $\sigma_n^{-1}(\overline{\mathbb{B}}) \subseteq \bigcup_{j=1}^N \mathcal{U}_{j,n}$. Such a family exists by compactness of $\sigma_n^{-1}(\overline{\mathbb{B}})$. Since $\sigma_{\ell,\ell-1}(\mathcal{U}_{j,\ell}) \subseteq \mathcal{U}_{j,\ell-1}$ for all ℓ , we have

$$\sigma_\ell^{-1}(\overline{\mathbb{B}}) \subseteq \bigcup_j \mathcal{U}_{j,\ell} \quad (7.5)$$

for all levels ℓ . We will denote by $C(\mathcal{CV}, m)$ any constant which depends only on the family \mathcal{CV} and m .

7.3. Definition of $k(n, m)$. Let $\mathbf{E} = (p_\ell, f_\ell, h_\ell, P_{\ell,i}, s_\ell, r_\ell, \mathcal{U}_\ell)$ be an extended chain and let \tilde{q}_ℓ and $\tilde{\alpha}_\ell$ be as specified in (6.7). Let $k_{\mathbf{E}} \in \mathbb{N}_+$ be the smallest common multiple of the integers \tilde{q}_ℓ which satisfies

$$k_{\mathbf{E}} \geq \max_\ell \frac{2s_\ell}{\tilde{\alpha}_\ell}.$$

Let $\mathcal{CV} = \{\mathbf{E}_j\} = \{(p_{j,\ell}, f_{j,\ell}, h_{j,\ell}, P_{j,\ell,i}, s_{j,\ell}, r_{j,\ell}, \mathcal{U}_{j,\ell})\}$ be the finite family of extended chains fixed in Section 7.2. Then we set

$$k = k(n, m) := \max_j \{k_{\mathbf{E}_j}, m\}.$$

7.4. Cutting and pasting. Let ℓ be fixed. For each $j = 1, \dots, N$ (here N is the number of extended chains in the family \mathcal{CV}) we have a chart $(\mathcal{U}_{j,\ell}, \mathbf{y}_{j,\ell})$ with a privileged system of coordinates, as in Section 7.1, and we consider the multi-valued functions

$$\varphi_{j,\ell} = \underline{h}_{j,\ell}^{\tilde{\alpha}_{j,\ell}} \tilde{\psi}_{j,\ell} \left(\underline{y}_{j,\ell,1}^{1/\tilde{q}_{j,\ell}}, \dots, \underline{y}_{j,\ell,r_{j,\ell}}^{1/\tilde{q}_{j,\ell}}, \underline{y}_{j,\ell,r_{j,\ell}+1}, \dots, y_{j,\ell,n} \right) : \Omega_{j,\ell,0} \rightsquigarrow \mathbb{C},$$

where $\Omega_{j,\ell,0} \subseteq \mathbb{R}^m$ is the open set defined in (7.1). Then

$$\Omega_{\ell,0} := \mathbb{R}^m \setminus a^{-1}(V(\mathcal{D}_\ell)) = \bigcup_{j=1}^N \Omega_{j,\ell,0}. \quad (7.6)$$

which belongs to $SBV(\Omega_{\ell,0})$ and verifies

$$\|\phi_\ell\|_{BV(\Omega_{\ell,0})} \leq C(\mathcal{CV}, m) \max_{1 \leq j \leq N} \left(\|h_{j,\ell}\|_{C^{k-1,1}}^{1/k} + \sum_{i=1}^n \|P_{j,\ell,i}\|_{C^{k-1,1}}^{1/k} \right). \quad (7.8)$$

In the definition of ϕ_ℓ all $\varphi_{j,\ell}$, $j \in J$, are equal as multi-valued functions on $\Omega_{J,\ell,0}$ and hence their arithmetic mean should be interpreted as the same multi-valued function.

Let us extend ϕ_ℓ by 0 to the set $a^{-1}(V(\mathcal{D}_\ell)) = \mathbb{R}^m \setminus \Omega_{\ell,0}$. Then ϕ_ℓ is continuous at all points of $\mathbb{R}^m \setminus \Omega_{\ell,0}$; furthermore, ϕ_ℓ has compact support. By Proposition 4.4, we may conclude that ϕ_ℓ has bounded variation on \mathbb{R}^m and $\|\phi_\ell\|_{BV(\mathbb{R}^m)} = \|\phi_\ell\|_{BV(\Omega_{\ell,0})}$ is bounded by (7.8). We have $S_{\phi_\ell} = J_{\phi_\ell}$ and the Cantor part of $D\phi_\ell$ vanishes in $\Omega_{\ell,0}$. The Cantor part of $D\phi_\ell$ also vanishes in $\mathbb{R}^m \setminus \Omega_{\ell,0}$, by [3, Proposition 3.92], since ϕ_ℓ vanishes and is continuous on this set. Thus $\phi_\ell \in SBV(\mathbb{R}^m)$.

Now we can use Theorem 6.4, that gives a parameterization $\lambda : \mathbb{R}^m \rightarrow \mathbb{C}^n$ of the roots of P_a , where $a \in C_c^{k-1,1}(\mathbb{R}^m, \mathbb{B})$, as a finite sum of the above constructed ϕ_ℓ and therefore it belongs to $SBV(\mathbb{R}^m)$ and satisfies

$$\|\lambda\|_{BV(\mathbb{R}^m)} \leq C(\mathcal{CV}, m) \max_{1 \leq \ell \leq n} \max_{1 \leq j \leq N} \left(\|h_{j,\ell}\|_{C^{k-1,1}}^{1/k} + \sum_{i=1}^n \|P_{j,\ell,i}\|_{C^{k-1,1}}^{1/k} \right). \quad (7.9)$$

Remark 7.6. Actually, to get an everywhere defined parameterization λ of the roots of P_a a modification of the parameterizations ϕ_ℓ defined in (7.7) is necessary on their discontinuity sets which are \mathcal{L}^m -negligible.

Their values on these sets can be chosen arbitrarily, with the only condition that they should satisfy $P_a(Z) = \prod_{j=1}^n (Z - \lambda_j)$, since the membership in $SBV(\mathbb{R}^m)$ and the bound for $\|\phi_\ell\|_{BV(\mathbb{R}^m)}$ are not affected by this modification.

7.5. General Case. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and let $a \in C^{k-1,1}(\overline{\Omega}, \mathbb{C}^n)$. By Whitney's extension theorem, a admits a $C^{k-1,1}$ -extension \hat{a} to \mathbb{R}^m with compact support in the open 1-neighborhood Ω_1 of Ω such that

$$\|\hat{a}\|_{C^{k-1,1}(\overline{\Omega}_1)} \leq C(m, k, \Omega) \|a\|_{C^{k-1,1}(\overline{\Omega})}; \quad (7.10)$$

see Section 3.2 for more details. Let us first prove Theorem 1.1 for \hat{a} .

We may suppose that the discriminant of $P_{\hat{a}(x)}$ is not identically equal to zero.

In general, the image $\hat{a}(\mathbb{R}^m)$ is not necessarily contained in the closed unit ball $\overline{\mathbb{B}} \subseteq \mathbb{C}^n$. To reduce to this case we use an \mathbb{R}_+ action on the coefficient vector $\hat{a} \in \mathbb{C}^n$. For $\eta > 0$ and $\hat{a} \in \mathbb{C}^n$ we define $\eta * \hat{a} \in \mathbb{C}^n$ by $(\eta * \hat{a})_i = \eta^i \hat{a}_i$. Then λ is a root of $P_{\hat{a}}$ if and only if $\eta\lambda$ is a root of $P_{\eta * \hat{a}}$.

Fix $\rho \geq \max\{1, \|\hat{a}\|_{L^\infty}\}$. Then $\|\rho^{-1} * \hat{a}\|_{L^\infty} \leq 1$. By (7.9) applied to $P_{\rho^{-1} * \hat{a}}$ the roots of $P_{\hat{a}}$ admit a parameterization $\lambda : \mathbb{R}^m \rightarrow \mathbb{C}^n$ in $SBV(\mathbb{R}^m)$ such that

$$\|\lambda\|_{BV(\mathbb{R}^m)} \leq \rho C(n, m) \max_{1 \leq \ell \leq n} \max_{1 \leq j \leq N} \left(\|h_{j,\ell}\|_{C^{k-1,1}}^{1/k} + \sum_{i=1}^n \|P_{j,\ell,i}\|_{C^{k-1,1}}^{1/k} \right),$$

where for a polynomial $g \in \mathbb{C}[\hat{a}]$ we set $g(x) := g(\rho^{-1} * \hat{a}(x))$. (The dependence of the constant on the cover \mathcal{CV} is subsumed under its dependence on n .)

By Section 7.2 and by the regularity of the composition of $C^{k-1,1}$ -maps (cf. [15, Theorem 4.3]), we conclude that

$$\begin{aligned} \|\lambda\|_{BV(\mathbb{R}^m)} &\leq C(n, m) \max_{1 \leq \ell \leq n} \max_{1 \leq j \leq N} \left(\|h_{j,\ell}\|_{C^{k-1,1}(\overline{\mathbb{B}})}^{1/k} + \sum_{i=1}^n \|P_{j,\ell,i}\|_{C^{k-1,1}(\overline{\mathbb{B}})}^{1/k} \right) \\ &\quad \times \max\{1, \|\hat{a}\|_{L^\infty}\} (1 + \|\hat{a}\|_{C^{k-1,1}}) \\ &= \tilde{C}(n, m) \max\{1, \|\hat{a}\|_{L^\infty}\} (1 + \|\hat{a}\|_{C^{k-1,1}}) \end{aligned}$$

for a different constant $\tilde{C}(n, m)$.

The proof of Theorem 1.1 for coefficients $a \in C^{k-1,1}(\overline{\Omega}, \mathbb{C}^n)$ defined on an arbitrary bounded Lipschitz domain now follows in view of (7.10): we obtain that the roots of P_a admit a parameterization $\lambda : \Omega \rightarrow \mathbb{C}^n$ in $SBV(\Omega)$ such that

$$\|\lambda\|_{BV(\Omega)} \leq C(n, m, \Omega) \max\{1, \|a\|_{L^\infty(\Omega)}\} (1 + \|a\|_{C^{k-1,1}(\overline{\Omega})}).$$

The statement about the discontinuity set of λ follows from the construction. To finish the proof of Theorem 1.1 we just need to complete the proof of Proposition 7.2 which is done in the next section.

8. A LOCAL COMPUTATION

In this section we complete the proof of Proposition 7.2 by showing a slightly more general Proposition 8.3. First we fix the setup and the notation that we use throughout this section.

Setup 8.1. Let m and n be integers ≥ 2 . Let $q, s \in \mathbb{N}_+$ and $\alpha \in q^{-1}\mathbb{N}_+$. Let

$$k \in \mathbb{N}_+ \text{ be a multiple of } q \text{ satisfying } k \geq \max\left\{\frac{2s}{\alpha}, m\right\}. \quad (8.1)$$

Then $k/q \in \mathbb{N}_+$ and $k\alpha \in \mathbb{N}_+$. Let $h, P_j \in C_c^{k-1,1}(\mathbb{R}^m)$, $j = 1, \dots, n$, and set $y_j := P_j/h^s$. Put $y := (y_1, \dots, y_n)$ and consider the bounded open subset of \mathbb{R}^m ,

$$\Omega_0 := \{x \in \mathbb{R}^m : |y(x)| < 1, h(x) \neq 0\}.$$

We assume that, on Ω_0 ,

$$|h| \lesssim |y_j| \quad \text{for all } j = 1, \dots, n. \quad (8.2)$$

This implies that no P_j , $j = 1, \dots, n$, vanishes on Ω_0 . Let $y^{1/q}$ denote the multi-valued function

$$y^{1/q} := (y_1^{1/q}, \dots, y_r^{1/q}, y_{r+1}, \dots, y_n) : \Omega_0 \rightsquigarrow \mathbb{C}^n; \quad (8.3)$$

it takes values in the open unit ball $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ in \mathbb{C}^n . Let $\psi \in C^1(\overline{\mathbb{B}})$ and consider the multi-valued function

$$\varphi := h^\alpha \psi(y^{1/q}) = h^\alpha \psi(y_1^{1/q}, \dots, y_r^{1/q}, y_{r+1}, \dots, y_n) : \Omega_0 \rightsquigarrow \mathbb{C}; \quad (8.4)$$

additionally we define $\varphi(x) := 0$ if $h(x) = 0$.

8.1. Continuous parameterization of φ . The multi-valued function φ in (8.4) has a continuous parameterization in the complement of a finite union of C^{k-1} -hypersurfaces.

Lemma 8.2. *For each $1 \leq j \leq r$, there is a C^{k-1} -hypersurfaces $E_j \subseteq \Omega_0$ such that the multi-valued function $y_j^{1/q}$ has a continuous parameterization on $\Omega_0 \setminus E_j$. There exists a C^{k-1} -hypersurface $E_0 \subseteq \mathbb{R}^m$ such that h^α has a continuous parameterization on $\mathbb{R}^m \setminus E_0$. There is a continuous parameterization of φ on $\Omega_0 \setminus \bigcup_{j=0}^r E_j$. Every parameterization of φ extends continuously to the zero set of h . (Here (8.1) can be replaced by the weaker condition $k \geq m$.)*

Proof. Each $y_j = P_j/h^s$ is of class $C^{k-1,1}$ on Ω_0 . By Proposition 2.2, there exists a C^{k-1} -hypersurface $E_j \subseteq \Omega_0$ such that the multi-valued function $y_j^{1/q}$, where $1 \leq j \leq r$, has a continuous parameterization on $\Omega_0 \setminus E_j$. (If $r < j \leq n$ then y_j is a single-valued continuous function on Ω_0 .) Likewise the multi-valued function h^α has a continuous parameterization on $\mathbb{R}^m \setminus E_0$, where E_0 is a C^{k-1} -hypersurface in \mathbb{R}^m . Now any parameterization of φ extends continuously to $h^{-1}(0)$ because $\psi(y^{1/q})$ is bounded. \square

8.2. Parameterization of φ with bounded variation. Now we are ready to show that φ admits a parameterization of bounded variation.

Proposition 8.3. *There exists a finite collection of C^{k-1} -hypersurfaces $E_j \subseteq \mathbb{R}^m$ and a parameterization ϕ of φ on Ω_0 such that $\phi \in SBV(\Omega_0)$, ϕ is continuous on $\Omega_0 \setminus \bigcup_j E_j$,*

$$|D\phi|(\Omega_0) \leq C \left(\|h\|_{C^{k-1,1}}^{1/k} + \sum_{j=1}^n \|P_j\|_{C^{k-1,1}}^{1/k} \right),$$

and

$$\|\phi\|_{BV(\Omega_0)} \leq C (|\Omega_0| + 1) \left(\|h\|_{C^{k-1,1}}^{1/k} + \sum_{j=1}^n \|P_j\|_{C^{k-1,1}}^{1/k} \right), \quad (8.5)$$

with $C > 0$ depending only on $m, k, \alpha, q, s, \|h\|_{L^\infty}, \|\psi\|_{C^1}$, and $\|P_j\|_{L^\infty}$ for $1 \leq j \leq r$. We have $S_\phi = J_\phi \subseteq \bigcup_j E_j$ and the Cantor part $D\phi$ vanishes in Ω_0 .

Proof. First we choose a parameterization ϕ of φ and its discontinuity set: By Theorem 1.4, there is a closed C^{k-1} -hypersurface E_0 in $\mathbb{R}^m \setminus h^{-1}(0)$ and $h^{1/k}$ admits a parameterization in $SBV(\mathbb{R}^m)$ (recall that h has compact support), again denoted by $h^{1/k}$, which is continuous on $\mathbb{R}^m \setminus E_0$ and satisfies

$$S_{h^{1/k}} = J_{h^{1/k}} = E_0 \quad \text{and} \quad D^c(h^{1/k}) = 0$$

and

$$\int_{E_0} |h|^{1/k} d\mathcal{H}^{m-1} \leq C(m, k) \|h\|_{C^{k-1,1}}^{1/k}. \quad (8.6)$$

Analogously, for each $1 \leq j \leq r$, there is a closed C^{k-1} -hypersurface E_j in $\mathbb{R}^m \setminus P_j^{-1}(0)$ and $P_j^{1/k}$ admits a parameterization in $SBV(\mathbb{R}^m)$, again denoted by $P_j^{1/k}$, which is continuous on $\mathbb{R}^m \setminus E_j$ and satisfies

$$S_{P_j^{1/k}} = J_{P_j^{1/k}} = E_j \quad \text{and} \quad D^c(P_j^{1/k}) = 0$$

and

$$\int_{E_j} |P_j|^{1/k} d\mathcal{H}^{m-1} \leq C(m, k) \|P_j\|_{C^{k-1,1}}^{1/k}. \quad (8.7)$$

Since $k\alpha \in \mathbb{N}_+$ (cf. (8.1)),

$$(h^{1/k})^{k\alpha} = \exp\left(\frac{1}{k} \log h\right)^{k\alpha} = \exp\left(\frac{k\alpha}{k} \log h\right) = \exp(\alpha \log h) = h^\alpha$$

is a parameterization of the multi-valued function h^α in $SBV(\mathbb{R}^m)$ which is continuous on $\mathbb{R}^m \setminus E_0$ and satisfies

$$S_{h^\alpha} = J_{h^\alpha} = E_0 \quad \text{and} \quad D^c(h^\alpha) = 0.$$

Similarly, since $k/q \in \mathbb{N}_+$, $h^{1/q} = (h^{1/k})^{k/q}$ is a parameterization of the multi-valued function $h^{1/q}$ in $SBV(\mathbb{R}^m)$ which is continuous on $\mathbb{R}^m \setminus E_0$ and satisfies

$$S_{h^{1/q}} = J_{h^{1/q}} = E_0 \quad \text{and} \quad D^c(h^{1/q}) = 0.$$

By the same reason $P_j^{1/q} = (P_j^{1/k})^{k/q}$, for $1 \leq j \leq r$, is a parameterization of the multi-valued function $P_j^{1/q}$ in $SBV(\mathbb{R}^m)$ which is continuous on $\mathbb{R}^m \setminus E_j$ and satisfies

$$S_{P_j^{1/q}} = J_{P_j^{1/q}} = E_j \quad \text{and} \quad D^c(P_j^{1/q}) = 0.$$

For $1 \leq j \leq r$, the identity $y_j^{1/q} = P_j^{1/q}/h^{s/q}$ yields a parameterization in $SBV_{\text{loc}}(\Omega_0)$ of the multi-valued function $y_j^{1/q}$ which is continuous on $\Omega_0 \setminus (E_0 \cup E_j)$ and satisfies

$$S_{y_j^{1/q}} = J_{y_j^{1/q}} \subseteq \Omega_0 \cap (E_0 \cup E_j) \quad \text{and} \quad D^c(y_j^{1/q}) = 0.$$

Then, by Theorem 4.2 and (4.3) (note that ψ admits a Lipschitz C^1 -extension to \mathbb{R}^{2n}), there is a parameterization ϕ of φ in $SBV_{\text{loc}}(\Omega_0)$ which is continuous in $\Omega_0 \setminus E$, where $E := \bigcup_{j=0}^r E_j$, and satisfies

$$S_\phi = J_\phi \subseteq \Omega_0 \cap E \quad \text{and} \quad D^c\phi = 0.$$

The rest of the proof is devoted to showing that ϕ has bounded variation on Ω_0 (not just locally); for this we use the chain rule (4.2) and the product rule (4.3). Note that on $h^{1/k}$ coincides with its precise representative on $\mathbb{R}^m \setminus E_0$; similarly for $P_j^{1/k}$, etc. In the following computations we let $q = 1$ if $r+1 \leq j \leq n$. First of all,

$$\begin{aligned} \tilde{D}\phi &= h^\alpha \tilde{D}(\psi(y^{1/q})) + \psi(y^{1/q}) \tilde{D}(h^\alpha) \\ &= h^\alpha \nabla \psi(y^{1/q}) \tilde{D}(y^{1/q}) + \psi(y^{1/q}) \tilde{D}(h^\alpha). \end{aligned}$$

Since $k\alpha \in \mathbb{N}_+$, the map $z \mapsto z^{k\alpha}$ is C^1 and hence, by the chain rule (4.2),

$$\tilde{D}(h^\alpha) = \tilde{D}((h^{1/k})^{k\alpha}) = k\alpha (h^{1/k})^{k\alpha-1} \tilde{D}(h^{1/k}) = k\alpha h^{\alpha-1/k} \tilde{D}(h^{1/k}).$$

Analogously, we find

$$\tilde{D}(h^{1/q}) = \frac{k}{q} h^{1/q-1/k} \tilde{D}(h^{1/k}) \quad \text{and} \quad \tilde{D}(P_j^{1/q}) = \frac{k}{q} P_j^{1/q-1/k} \tilde{D}(P_j^{1/k}).$$

Altogether,

$$\begin{aligned}
\tilde{D}\phi &= h^\alpha \nabla \psi(y^{1/q}) \left(P_j^{1/q} \tilde{D}(h^{-s/q}) + h^{-s/q} \tilde{D}(P_j^{1/q}) \right)_j + \psi(y^{1/q}) \tilde{D}(h^\alpha) \\
&= h^\alpha \nabla \psi(y^{1/q}) \left(-\frac{ks}{q} P_j^{1/q} h^{-s/q-1/k} \tilde{D}(h^{1/k}) + \frac{k}{q} h^{-s/q} P_j^{1/q-1/k} \tilde{D}(P_j^{1/k}) \right)_j \\
&\quad + k\alpha \psi(y^{1/q}) h^{\alpha-1/k} \tilde{D}(h^{1/k}) \\
&= \nabla \psi(y^{1/q}) \left(-\frac{ks}{q} y_j^{1/q} h^{\alpha-1/k} \tilde{D}(h^{1/k}) + \frac{k}{q} y_j^{1/q-1/k} h^{\alpha-s/k} \tilde{D}(P_j^{1/k}) \right)_j \\
&\quad + k\alpha \psi(y^{1/q}) h^{\alpha-1/k} \tilde{D}(h^{1/k}).
\end{aligned}$$

Since $\psi \in C^1(\overline{\mathbb{B}})$ and k satisfies (8.1), we have on Ω_0 ,

$$|\tilde{D}\phi| \leq C \left(|\tilde{D}(h^{1/k})| + \sum_{j=1}^n |\tilde{D}(P_j^{1/k})| \right),$$

where C is a constant only depending on $k, \alpha, q, s, \|h\|_{L^\infty}$, and $\|\psi\|_{C^1}$ (recall that $|y_j| < 1$).

On the other hand for the jump part, (4.2) and (4.3) imply

$$D^j \phi = \Pi \otimes \nu_{(h^\alpha, \psi(y^{1/q}))} \mathcal{H}^{m-1} \llcorner J_{(h^\alpha, \psi(y^{1/q}))},$$

where

$$\Pi(x) = \begin{cases} ((h^\alpha)^+(x) - (h^\alpha)^-(x)) \psi(y^{1/q}(x)) & \text{if } x \in J_{h^\alpha} \setminus J_{\psi(y^{1/q})}, \\ h^\alpha(x) (\psi(y^{1/q})^+(x) - \psi(y^{1/q})^-(x)) & \text{if } x \in J_{\psi(y^{1/q})} \setminus J_{h^\alpha}, \\ (h^\alpha)^+(x) \psi(y^{1/q})^+(x) - (h^\alpha)^-(x) \psi(y^{1/q})^-(x) & \text{if } x \in J_{h^\alpha} \cap J_{\psi(y^{1/q})}. \end{cases}$$

Thus

$$|D^j \phi| \leq C \sum_{j=0}^r |h|^\alpha \mathcal{H}^{m-1} \llcorner (\Omega_0 \cap E)$$

for some constant C depending only on $\|\psi\|_{L^\infty}$. We may conclude that $\phi \in BV(\Omega_0)$ and

$$\begin{aligned}
|D\phi|(\Omega_0) &\leq |\tilde{D}\phi|(\Omega_0) + |D^j \phi|(\Omega_0) \\
&\leq C \left(|\tilde{D}(h^{1/k})|(\Omega_0) + \sum_{j=1}^n |\tilde{D}(P_j^{1/k})|(\Omega_0) + \sum_{j=0}^r \int_{\Omega_0 \cap E_j} |h|^\alpha d\mathcal{H}^{m-1} \right).
\end{aligned}$$

By Theorem 2.7 (or Theorem A.1), (8.6), and (8.7) (using $|h|^\alpha = |h|^{1/k} |h|^{\alpha-1/k}$ and $|h|^\alpha \lesssim |P_j|^{\alpha/(s+1)} = |P_j|^{1/k} |P_j|^{\alpha/(s+1)-1/k}$, by (8.2), where the second factors are bounded in both cases, by (8.1)), we obtain

$$|D\phi|(\Omega_0) \leq C \left(\|h\|_{C^{k-1,1}}^{1/k} + \sum_{j=1}^n \|P_j\|_{C^{k-1,1}}^{1/k} \right),$$

for a constant $C > 0$ which depends only on $m, k, \alpha, q, s, \|h\|_{L^\infty}, \|\psi\|_{C^1}$, and $\|P_j\|_{L^\infty}$ for $1 \leq j \leq r$. Then (8.5) follows, since $\|\phi\|_{L^1(\Omega_0)} \leq |\Omega_0| \|h\|_{L^\infty}^\alpha \|\psi\|_{L^\infty}$. \square

APPENDIX A. SOBOLEV REGULARITY OF CONTINUOUS ROOTS

The next theorem is a refinement of [33, Theorem 2] in the case that Ω is a Lipschitz domain.

Theorem A.1. *Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain. Let P_a be a monic polynomial (1.1) with coefficients $a_j \in C^{n-1,1}(\overline{\Omega})$, $j = 1, \dots, n$. Let $\lambda \in C^0(V)$ be a root of P_a on an open subset $V \subseteq \Omega$. Then λ belongs to the Sobolev space $W^{1,p}(V)$ for every $1 \leq p < \frac{n}{n-1}$. The distributional gradient $\nabla \lambda$ satisfies*

$$\|\nabla \lambda\|_{L^p(V)} \leq C(m, n, p, \Omega) \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}(\overline{\Omega})}^{1/j}. \quad (\text{A.1})$$

Proof. By [33, Theorem 1], λ is absolutely continuous along affine lines parallel to the coordinate axes (restricted to V). So λ possesses the partial derivatives $\partial_i \lambda$, $i = 1, \dots, m$, which are defined almost everywhere and are measurable.

Set $x = (t, y)$, where $t = x_1$, $y = (x_2, \dots, x_m)$, and let V_1 be the orthogonal projection of V on the hyperplane $\{x_1 = 0\}$. For each $y \in V_1$ we denote by $V^y := \{t \in \mathbb{R} : (t, y) \in V\}$ the corresponding section of V .

Let λ_j^y , $j = 1, \dots, n$, be a continuous system of the roots of $P_a(\cdot, y)$ on V^y such that $\lambda(\cdot, y) = \lambda_1^y$; it exists since $\lambda(\cdot, y)$ can be completed to a continuous system of the roots of $P_a(\cdot, y)$ on each connected component of V^y by [38, Lemma 6.17]. Our goal is to bound

$$\|\partial_t \lambda(\cdot, y)\|_{L^p(V^y)} = \|(\lambda_1^y)'\|_{L^p(V^y)}$$

uniformly with respect to $y \in V_1$.

Let $R = I_1 \times \dots \times I_m \subseteq \mathbb{R}^m$ be an open box containing Ω and such that $|I_i| \leq \text{diam}(\Omega)$ for all $i = 1, \dots, m$. By Whitney's extension theorem (cf. Section 3.2), the coefficients a_j of P_a admit a $C^{n-1,1}$ -extension \hat{a}_j to \mathbb{R}^m such that

$$\max_{1 \leq j \leq n} \|\hat{a}_j\|_{C^{n-1,1}(\overline{R})}^{1/j} \leq C(m, n, \Omega) \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}(\overline{\Omega})}^{1/j}. \quad (\text{A.2})$$

Let \mathcal{C}^y denote the set of connected components J of the open subset $V^y \subseteq \mathbb{R}$. For each $J \in \mathcal{C}^y$ we extend the system of roots $\lambda_j^y|_J$, $j = 1, \dots, n$, continuously to I_1 , i.e., we choose continuous functions $\lambda_j^{y,J}$, $j = 1, \dots, n$, on I_1 such that $\lambda_j^{y,J}|_J = \lambda_j^y|_J$ for all j and

$$P_a(t, y)(Z) = \prod_{j=1}^n (Z - \lambda_j^{y,J}(t)), \quad t \in I_1.$$

This is possible since $\lambda_j^y|_J$ has a continuous extension to the endpoints of the (bounded) interval J , by [26, Lemma 4.3], and can then be extended on the left and on the right of J by a continuous system of the roots of $P_a(\cdot, y)$ after suitable permutations.

By [33, Theorem 1], for each $y \in V_1$, $J \in \mathcal{C}^y$, and $j = 1, \dots, n$, the function $\lambda_j^{y,J}$ is absolutely continuous on I_1 and $(\lambda_j^{y,J})' \in L^p(I_1)$, for $1 \leq p < n/(n-1)$, with

$$\|(\lambda_j^{y,J})'\|_{L^p(I_1)} \leq C(n, p, |I_1|) \max_{1 \leq i \leq n} \|\hat{a}_i\|_{C^{n-1,1}(\overline{R})}^{1/i}. \quad (\text{A.3})$$

Let $J, J_0 \in \mathcal{C}^y$ be arbitrary. By [32, Lemma 3.6], $(\lambda_j^y)'$ as well as $(\lambda_j^{y, J_0})'$ belong to $L^p(J)$ and we have

$$\sum_{j=1}^n \|(\lambda_j^y)'\|_{L^p(J)}^p = \sum_{j=1}^n \|(\lambda_j^{y, J})'\|_{L^p(J)}^p = \sum_{j=1}^n \|(\lambda_j^{y, J_0})'\|_{L^p(J)}^p.$$

Thus,

$$\begin{aligned} \sum_{j=1}^n \|(\lambda_j^y)'\|_{L^p(V^y)}^p &= \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \|(\lambda_j^y)'\|_{L^p(J)}^p = \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \|(\lambda_j^{y, J_0})'\|_{L^p(J)}^p \\ &= \sum_{j=1}^n \|(\lambda_j^{y, J_0})'\|_{L^p(V^y)}^p \leq \sum_{j=1}^n \|(\lambda_j^{y, J_0})'\|_{L^p(I_1)}^p. \end{aligned}$$

In particular, by (A.3),

$$\|\partial_t \lambda(\cdot, y)\|_{L^p(V^y)} = \|(\lambda_1^y)'\|_{L^p(V^y)} \leq C(n, p, |I_1|) \max_{1 \leq i \leq n} \|\hat{a}_i\|_{C^{n-1,1}(\bar{R})}^{1/i},$$

and so, by Fubini's theorem,

$$\begin{aligned} \int_V |\partial_1 \lambda(x)|^p dx &= \int_{V_1} \int_{V^y} |\partial_1 \lambda(t, y)|^p dt dy \\ &\leq \left(C(n, p, |I_1|) \max_{1 \leq i \leq n} \|\hat{a}_i\|_{C^{n-1,1}(\bar{R})}^{1/i} \right)^p \int_{V_1} dy. \end{aligned}$$

Thus, thanks to $|I_1| \leq \text{diam}(\Omega)$,

$$\|\partial_1 \lambda\|_{L^p(V)} \leq C(n, p, \text{diam}(\Omega)) \max_{1 \leq i \leq n} \|\hat{a}_i\|_{C^{n-1,1}(\bar{R})}^{1/i}.$$

In view of (A.2) this implies (A.1), since the other partial derivatives $\partial_i \lambda$, $i \geq 2$, are treated analogously. \square

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