ON THE BOREL MAPPING IN THE QUASIANALYTIC SETTING

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Abstract. The Borel mapping takes germs at 0 of smooth functions to the sequence of iterated partial derivatives at 0. We prove that the Borel mapping restricted to the germs of any quasianalytic ultradifferentiable class strictly larger than the real analytic class is never onto the corresponding sequence space.

1. Introduction

It is a classical result due to Carleman [10], [11] that the Borel mapping that takes germs at 0 of functions in a quasianalytic Denjoy–Carleman class $\mathcal{E}^{\{M\}}$ to the sequence of iterated partial derivatives at 0 is never *onto* the corresponding sequence space $\Lambda^{\{M\}}$ unless $\mathcal{E}^{\{M\}}$ is contained in the real analytic class. Here $M = (M_k)$ is a weight sequence that dominates the growth of the iterated partial derivatives of the functions in $\mathcal{E}^{\{M\}}$, and *quasianalytic* means that the Borel mapping is injective on $\mathcal{E}^{\{M\}}$ (precise definitions will be given below). Carleman's proof is based on his formula for reconstructing the function $f \in \mathcal{E}^{\{M\}}$ from the sequence of its iterated derivatives at 0 (due to quasianalyticity f is unique); see also [19] for a modern account of the proof.

In the recent paper [6] Bonet and Meise prove this result (non-surjectivity of the Borel mapping) for proper quasianalytic classes $\mathcal{E}^{\{\omega\}}$ (and $\mathcal{E}^{(\omega)}$); the brackets { } and ( ) refer to classes of Roumieu and Beurling type, respectively (see definitions below). These classes were introduced by Beurling [3] and Björck [4], by imposing decay conditions at infinity for the Fourier transform in terms of a weight function ω, and they were equivalently described by Braun, Meise, and Taylor [8]. We shall refer to these classes as Braun–Meise–Taylor classes. In [6] the problem is transferred to weighted spaces of entire functions via the Fourier–Laplace transform and functional-analytic methods are applied.

Sometimes $\mathcal{E}^{\{\omega\}} = \mathcal{E}^{\{M\}}$ (and $\mathcal{E}^{(\omega)} = \mathcal{E}^{(M)}$) for a suitable sequence M, but in general the families of classes described by weight functions ω and those described by weight sequences M are mutually distinct; see [7]. However, the method developed in [14] allows us to describe the classes $\mathcal{E}^{\{\omega\}}$ (and $\mathcal{E}^{(\omega)}$) as unions (or intersections) of associated one parameter families of Denjoy–Carleman classes $\mathcal{E}^{\{W^x\}}$ (or $\mathcal{E}^{(W^x)}$), where W^x are weight sequences associated with ω in a precise way depending on a real parameter x. More generally, this construction can be turned into a definition, and in this manner one obtains ultradifferentiable classes $\mathcal{E}^{\{\mathfrak{M}\}}$ (and

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$\mathcal{E}^{(\mathfrak{M})}$) defined in terms of *weight matrices* $\mathfrak{M} = \{M^x\}_x$. These comprise the classical Denjoy–Carleman classes $\mathcal{E}^{\{M\}}$, $\mathcal{E}^{(M)}$, the Braun–Meise–Taylor classes $\mathcal{E}^{\{\omega\}}$, $\mathcal{E}^{(\omega)}$, and many more; cf. [14, Theorem 5.22]. This new technique often makes it possible to treat all these classes uniformly, while previously every setting required a special proof.

In the present paper we will show in an elementary way that the Borel mappings

$$j^\infty : \mathcal{E}^{\{\mathfrak{M}\}}_{0,n} \to \Lambda^{\{\mathfrak{M}\}}_n \quad \text{and} \quad j^\infty : \mathcal{E}^{(\mathfrak{M})}_{0,n} \to \Lambda^{(\mathfrak{M})}_n, \quad j^\infty f = (\partial^\alpha f(0))_\alpha,$$

are never surjective in the proper quasianalytic setting (*proper* means not contained in the real analytic class). Here $\mathcal{E}^{\{\mathfrak{M}\}}_{0,n}$ (resp. $\mathcal{E}^{(\mathfrak{M})}_{0,n}$) denotes the ring of germs at $0 \in \mathbb{R}^n$ of functions in $\mathcal{E}^{\{\mathfrak{M}\}}$ (resp. $\mathcal{E}^{(\mathfrak{M})}$), and $\Lambda^{\{\mathfrak{M}\}}_n$ (resp. $\Lambda^{(\mathfrak{M})}_n$) is the corresponding sequence space. As a corollary we recover the results of Bonet and Meise [6].

We actually show more: if $\mathcal{E}^{\{\mathfrak{M}\}}$ (resp. $\mathcal{E}^{(\mathfrak{M})}$) is a proper quasianalytic class then there exist elements in $\Lambda^{\{\mathfrak{M}\}}_n$ (resp. $\Lambda^{(\mathfrak{M})}_n$) that are not contained in

$$j^\infty\Big(\bigcup\Big\{\mathcal{E}^{\{\mathfrak{N}\}}_{0,n} : \mathcal{E}^{\{\mathfrak{N}\}}_{0,n} \text{ is quasianalytic}\Big\}\Big); \tag{1.1}$$

see Theorems 5 and 6. Note that, since trivially $\mathcal{E}^{(\mathfrak{N})}_{0,n} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}_{0,n}$, this result implies all statements above. In particular, $\Lambda^{\{\omega\}}_n$ (resp. $\Lambda^{(\omega)}_n$) is not contained in

$$j^\infty\Big(\bigcup\Big\{\mathcal{E}^{\{\sigma\}}_{0,n} : \sigma \text{ is a quasianalytic weight function}\Big\}\Big).$$

Our proof is based on Bernstein's theorem on absolutely monotone functions [2] and on a theorem due to Bang [1] (Theorem 1 below), which we recall with full proof for the sake of completeness.

Let us emphasize that our proof also provides some partial information on the image (1.1). If $n = 1$ (for simplicity) then (1.1) cannot contain any strictly positive sequence $a = (a_k)$ unless a defines a real analytic germ. Even for a single quasianalytic weight sequence M it is generally not known how to identify the elements of $j^\infty \mathcal{E}^{\{M\}}_{0,1}$ among those of $\Lambda^{\{M\}}_1$.

We wish to mention the recent paper by Sfouli [18] in which Carleman's result is obtained for quasianalytic local rings defined in an abstract way. This abstract definition includes stability under composition and differentiation. These rather restrictive properties (see e.g. [15] for a characterization of the former) are not required in our setting. Moreover, the approach of Sfouli yields Carleman's result only in dimension $n \ge 2$.

2. Weight sequences and Denjoy–Carleman classes

Let us recall some basic facts on weight sequences and define Denjoy–Carleman classes and its germs.

2.1. **Denjoy–Carleman classes and its germs.** Let $M = (M_k)_{k\in\mathbb{N}}$ be a positive sequence and let $U \subseteq \mathbb{R}^n$ be an open non-empty set. Then the set $\mathcal{E}^{\{M\}}(U)$ of all $f \in C^\infty(U)$ such that for all compact $K \subseteq U$ there exists $\rho > 0$ with

$$\|f\|^M_{K,\rho} := \sup_{x\in K,\ \alpha\in\mathbb{N}^n} \frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|} M_{|\alpha|}} < \infty,$$

is called the *Denjoy–Carleman class of Roumieu type* associated with M. It is endowed with the natural projective topology over K and the inductive topology over $\rho \in \mathbb{N}$. Analogously, we define the *Denjoy–Carleman class of Beurling type* $\mathcal{E}^{(M)}(U)$ consisting of all $f \in C^\infty(U)$ such that for all compact $K \subseteq U$ and all $\rho > 0$, $\|f\|^M_{K,\rho} < \infty$, and endow it with its natural Fréchet topology ($1/\rho \in \mathbb{N}$).

Let us define the spaces of germs at $0 \in \mathbb{R}^n$,

$$\mathcal{E}^{\{M\}}_{0,n} := \operatorname{ind}_{k\in\mathbb{N}} \mathcal{E}^{\{M\}}((-\tfrac{1}{k},\tfrac{1}{k})^n)$$
$$\mathcal{E}^{(M)}_{0,n} := \operatorname{ind}_{k\in\mathbb{N}} \mathcal{E}^{(M)}((-\tfrac{1}{k},\tfrac{1}{k})^n).$$

Finally we consider the sequence spaces

$$\Lambda^{\{M\}}_n := \{a = (a_\alpha) \in \mathbb{C}^{\mathbb{N}^n} : \exists \rho > 0 : |a|^M_\rho < \infty\},$$
$$\Lambda^{(M)}_n := \{a = (a_\alpha) \in \mathbb{C}^{\mathbb{N}^n} : \forall \rho > 0 : |a|^M_\rho < \infty\},$$

where

$$|a|^M_\rho := \sup_{\alpha\in\mathbb{N}^n} \frac{|a_\alpha|}{\rho^{|\alpha|} M_{|\alpha|}}.$$

Then $\Lambda^{\{M\}}_n$ is an (LB)-space and $\Lambda^{(M)}_n$ is a Fréchet space.

With the sequence $M_k = k!$ we recover the real analytic functions $\mathcal{E}^{\{k!\}}(U) = C^\omega(U)$ and restrictions of the entire functions $\mathcal{E}^{(k!)}(U) = \mathcal{H}(\mathbb{C}^n)$ if U is connected. We denote by $\mathcal{O}_{0,n}$ the ring of germs of real analytic functions at $0 \in \mathbb{R}^n$.

By convention we write $\mathcal{E}^{[M]}$ if we mean either $\mathcal{E}^{\{M\}}$ or $\mathcal{E}^{(M)}$, similarly $\Lambda^{[M]}_n$ stands for $\Lambda^{\{M\}}_n$ and $\Lambda^{(M)}_n$, etc.

2.2. **Weight sequences and properties of Denjoy–Carleman classes.** We shall impose some mild regularity properties on the sequence $M = (M_k)$ that guarantee, in particular, that $\mathcal{E}^{[M]}_{0,n}$ is a ring.

By definition, a *weight sequence* is a sequence of positive real numbers $M = (M_k)_{k\in\mathbb{N}}$ such that:

$$1 = M_0 \le M_1, \tag{2.1}$$

$$k \mapsto M_k \text{ is logarithmically convex (log-convex for short),} \tag{2.2}$$

$$\liminf_k m_k^{1/k} > 0. \tag{2.3}$$

Given a sequence $M = (M_k)$ we associate the sequences $m = (m_k)$ and $\mu = (\mu_k)$ given by

$$m_k := \frac{M_k}{k!}, \quad \mu_k := \frac{M_k}{M_{k-1}}.$$

Note that (2.1) and (2.2) imply that M_k and $M_k^{1/k}$ are non-decreasing.

Remark 1. Under the assumption that $C^\omega \subseteq \mathcal{E}^{\{M\}}$ (resp. $C^\omega \subseteq \mathcal{E}^{(M)}$) which we shall always make, (2.1), (2.2), and (2.3) are no restriction of generality for our problem, because one can change to the log-convex minorant $\underline{M}$ of M which describes the same function space: $\mathcal{E}^{[\underline{M}]} = \mathcal{E}^{[M]}$, see [14, Theorem 2.15], whereas $\Lambda^{[\underline{M}]} \subseteq \Lambda^{[M]}$.

In [14] and [15] we denoted by $M = (M_k)$ the sequence which here is denoted by $m = (m_k)$. We deviate from our former convention for notational simplicity.

For arbitrary positive sequences $M = (M_k)$ and $N = (N_k)$ we define

$$M \preceq N \quad :\Leftrightarrow \quad \exists C, \rho > 0 \, \forall k : M_k \le C\rho^k N_k \quad \Leftrightarrow \quad \sup_k \Big(\frac{M_k}{N_k}\Big)^{1/k} < \infty$$

and

$$M \lhd N \quad :\Leftrightarrow \quad \forall \rho > 0 \, \exists C > 0 \, \forall k : M_k \le C\rho^k N_k \quad \Leftrightarrow \quad \lim_k \Big(\frac{M_k}{N_k}\Big)^{1/k} = 0.$$

Then $M \preceq N$ implies $\mathcal{E}^{[M]} \subseteq \mathcal{E}^{[N]}$ and $\Lambda^{[M]} \subseteq \Lambda^{[N]}$, and $M \lhd N$ implies $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)}$ and $\Lambda^{\{M\}} \subseteq \Lambda^{(N)}$. The converse implications hold if M is a weight sequence; cf. [14, Proposition 2.12] and [12, Lemma 2.2], that $\Lambda^{(M)} \subseteq \Lambda^{(N)}$ implies $M \preceq N$ follows from the argument in [9]. In particular, (2.3) holds if and only if the real analytic class is contained in $\mathcal{E}^{\{M\}}$, and furthermore, if and only if the restrictions of all entire functions are contained in $\mathcal{E}^{(M)}$. The inclusion of the real analytic class in $\mathcal{E}^{(M)}$ is equivalent to the condition

$$m_k^{1/k} \to \infty. \tag{2.4}$$

A weight sequence $M = (M_k)$ is called *quasianalytic* if

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} = \infty, \quad \text{or equivalently,} \quad \sum_{k=1}^{\infty} \frac{1}{M_k^{1/k}} = \infty.$$

The famous Denjoy–Carleman theorem (cf. [12, Theorem 2.1]) holds that $M = (M_k)$ is quasianalytic if and only if $\mathcal{E}^{[M]}$ is *quasianalytic*, i.e., for open connected $U \subseteq \mathbb{R}^n$ and each $a \in U$ the Borel mapping $f \mapsto (\partial^\alpha f(a))_\alpha$ is injective on $\mathcal{E}^{[M]}(U)$.

Remark 2. A class $\mathcal{E}^{[M]}$ is called non-quasianalytic if it is not quasianalytic. This is equivalent to the fact that there exist non-trivial $\mathcal{E}^{[M]}$-functions with compact support.

3. A proof of Carleman's theorem

We have the Borel mapping

$$j^\infty : \mathcal{E}^{[M]}_{0,n} \to \Lambda^{[M]}_n, \quad f \mapsto (\partial^\alpha f(0))_{\alpha \in \mathbb{N}^n}. \tag{3.1}$$

If $M = (M_k)$ is a quasianalytic weight sequence, then this mapping is injective. In this section we will show that it is never surjective if $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{[M]}_{0,n}$.

3.1. **The Roumieu case.** Let us first concentrate on the Roumieu case. Due to a theorem of Carleman, the mapping $j^\infty : \mathcal{E}^{\{M\}}_{0,n} \to \Lambda^{\{M\}}_n$ is never surjective if $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{M\}}_{0,n}$, or equivalently,

$$\sup_k m_k^{1/k} = \infty.$$

A concise proof (following the main ideas of Carleman) may be found in [19].

We shall give another proof based on Bernstein's theorem (cf. [20, p. 146] for a proof and [5] for a survey of related results) and the following elementary theorem due to Bang [1]. We reproduce the proof of the latter for the convenience of the reader and for the sake of completeness (cf. also [13]).

Theorem 1 (Bang [1])**.** *Let $M = (M_k)$ be a quasianalytic weight sequence and let $f \in C^\infty([0,1])$ satisfy*

$$\sup_{x\in[0,1]} |f^{(j)}(x)| \le M_j, \quad j \in \mathbb{N}. \tag{3.2}$$

If f is not identically 0 and for all $j \in \mathbb{N}$ there exists $x_j \in [0,1]$ such that $f^{(j)}(x_j) = 0$, then the series $\sum_{j=0}^\infty |x_j - x_{j+1}|$ is divergent.

Proof. For $N \in \mathbb{N}$ and $x \in [0,1]$ set

$$B_{f,N}(x) := \max_{j\ge N} \frac{|f^{(j)}(x)|}{e^j M_j}.$$

Let us collect some properties of $B_{f,N}$:

(1) $B_{f,N}(x) \le e^{-N}$,
(2) $B_{f,N}(x) \ge B_{f,N+1}(x)$, and $B_{f,N}(x) = B_{f,N+1}(x)$ if $f^{(N)}(x) = 0$,
(3) for all $k > N$ and all $x, x+h \in [0,1]$ $(h \ne 0)$,

$$B_{f,N}(x+h) < \max\{B_{f,N}(x), e^{-k}\}\, e^{e|h|\mu_k}.$$

(1) and (2) follow easily from the definition and from (3.2). To see (3) let $k > N$, $N \le j < k$, and $x, x+h \in [0,1]$. Then, by Taylor's formula, for some ξ between x and $x+h$,

$$\begin{aligned}
\frac{|f^{(j)}(x+h)|}{e^j M_j} &\le \sum_{i=0}^{k-j-1} \frac{|f^{(j+i)}(x)|\,|h|^i}{e^j M_j\, i!} + \frac{|f^{(k)}(\xi)|\,|h|^{k-j}}{e^j M_j\,(k-j)!} \\
&= \sum_{i=0}^{k-j-1} \frac{M_{j+i}}{M_j} \frac{|f^{(j+i)}(x)|}{e^{j+i} M_{j+i}} \frac{(e|h|)^i}{i!} + e^{-k} \frac{M_k}{M_j} \frac{|f^{(k)}(\xi)|}{M_k} \frac{(e|h|)^{k-j}}{(k-j)!} \\
&\le B_{f,N}(x) \sum_{i=0}^{k-j-1} \Big(\frac{M_k}{M_{k-1}}\Big)^i \frac{(e|h|)^i}{i!} + e^{-k} \Big(\frac{M_k}{M_{k-1}}\Big)^{k-j} \frac{(e|h|)^{k-j}}{(k-j)!} \\
&< \max\{B_{f,N}(x), e^{-k}\}\, e^{e|h|\mu_k},
\end{aligned}$$

where we used that $M = (M_k)$ is log-convex. If $j \ge k$ then, by (3.2),

$$\frac{|f^{(j)}(x+h)|}{e^j M_j} \le e^{-j} < \max\{B_{f,N}(x), e^{-k}\}\, e^{e|h|\mu_k}.$$

This implies (3).

Let f and x_j be as in the theorem. Set $\tau_k := \sum_{j=0}^{k-1} |x_j - x_{j+1}|$, $k \ge 1$, $\tau_0 := 0$, and define for $t \in [\tau_{N-1}, \tau_N]$,

$$\tilde B_{f,N}(t) := \begin{cases} B_{f,N}(x_{N-1} + \tau_{N-1} - t) & \text{if } x_N < x_{N-1}, \\ B_{f,N}(x_{N-1} - \tau_{N-1} + t) & \text{if } x_N \ge x_{N-1}. \end{cases}$$

By (3), the function $\tilde B_{f,N}$ is continuous and, by (2), $\tilde B_{f,N}(\tau_N) = B_{f,N}(x_N) = B_{f,N+1}(x_N) = \tilde B_{f,N+1}(\tau_N)$. So we obtain a continuous function $\tilde B_f$ on the interval $[0,\tau)$, where $\tau := \sup_k \tau_k$, by setting

$$\tilde B_f(t) := \tilde B_{f,N}(t) \quad \text{if } t \in [\tau_{N-1}, \tau_N],\ N \ge 1.$$

By (1) and (2) we find that $\tilde B_f(t) \le e^{-N}$ for all $t \ge \tau_{N-1}$ and hence $\tilde B_f(t) \to 0$ as $t \to \tau$. Since f and thus also $\tilde B_f$ does not vanish identically, the range of $\tilde B_f$

contains all numbers e^{-k} for sufficiently large k, say $k \ge k_0$. So we may choose a strictly increasing sequence t_k such that $\tilde B_f(t_k) = e^{-k}$ and $\tilde B_f(t) > e^{-k}$ for all $t \in (t_{k-1}, t_k)$ (recursively, take for t_k the smallest $t \in \tilde B_f^{-1}(e^{-k})$ with $t > t_{k-1}$). By (3) (applied to each interval in the subdivision of (t_{k-1}, t_k) induced by the points τ_N between t_{k-1} and t_k) we may conclude that

$$\tilde B_f(t_{k-1}) \le \tilde B_f(t_k)\, e^{e(t_k - t_{k-1})\mu_k},$$

or equivalently,

$$t_k - t_{k-1} \ge \frac{1}{e\mu_k},$$

and therefore

$$t_k \ge t_{k_0} + \frac{1}{e} \sum_{j=k_0+1}^{k} \frac{1}{\mu_j}.$$

By the choice of the sequence t_k we find that $\tau_k \ge t_k$, which implies the assertion as $M = (M_k)$ is quasianalytic. □

Corollary 1 (Bang [1]). *Let $M = (M_k)$ be a quasianalytic weight sequence and let $f \in C^\infty([0,1])$ satisfy* (3.2). *If $f^{(j)}(0) > 0$ for all $j \in \mathbb{N}$, then $f^{(j)}(x) > 0$ for all $x \in [0,1]$ and $j \in \mathbb{N}$.*

Proof. Suppose that $f^{(j)}(0) > 0$ for all $j \in \mathbb{N}$ and that some derivative $f^{(j)}$ has a zero $x_j \in (0,1]$. By Rolle's theorem, we find a strictly decreasing sequence $x_j > x_{j+1} > \cdots > 0$, where x_k is a zero of $f^{(k)}$ for all $k \ge j$. This contradicts Theorem 1. □

We may deduce not only that $j^\infty : \mathcal{E}^{\{M\}}_{0,n} \to \Lambda^{\{M\}}_n$ is not surjective if $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{M\}}_{0,n}$, but that there exist elements in $\Lambda^{\{M\}}_n$ that are not contained in $j^\infty \mathcal{E}^{\{N\}}_{0,n}$ for *any* quasianalytic weight sequence $N = (N_k)$.

Theorem 2. *Let $M = (M_k)$ be a quasianalytic weight sequence such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{M\}}_{0,n}$. Then there exist elements in $\Lambda^{\{M\}}_n$ that are not contained in $j^\infty \mathcal{E}^{\{N\}}_{0,n}$ for any quasianalytic weight sequence $N = (N_k)$.*

Proof. Without loss of generality we may assume that $n = 1$. Let $a = (a_j) \in \Lambda^{\{M\}}_1$ be positive, i.e., $a_j > 0$ for all j. Let $N = (N_k)$ be any quasianalytic weight sequence. We claim that if there exists $f \in \mathcal{E}^{\{N\}}_{0,1}$ such that $j^\infty f = a$ then $f \in \mathcal{O}_{0,1}$. There is $r > 0$ such that $f \in \mathcal{E}^{\{N\}}((-r,r))$ and $0 < r_1 < r$ and $\rho > 0$ such that

$$\sup_{x \in [0,r_1]} |f^{(j)}(x)| \le \rho^{j+1} N_j, \quad j \in \mathbb{N};$$

abusing notation we denote germs and its representatives by the same symbol. Let us define $\tilde f(x) := \rho^{-1} f(r_1 x)$ and $\tilde N_j := (\rho\, r_1)^j N_j$. Then

$$\sup_{x \in [0,1]} |\tilde f^{(j)}(x)| \le \tilde N_j, \quad j \in \mathbb{N},$$

and hence Corollary 1 implies that $\tilde f^{(j)}(x) > 0$ for all $x \in [0,1]$ and all $j \in \mathbb{N}$, that is $f^{(j)}(x) > 0$ for all $x \in [0, r_1]$ and all $j \in \mathbb{N}$. By Bernstein's theorem (e.g. [20, p. 146]), $f \in \mathcal{O}_{0,1}$.

Thus if $a = (a_j)$ is chosen such that it does not define a real analytic germ, which is possible by the assumption $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{M\}}_{0,n}$, then it cannot belong to $j^\infty \mathcal{E}^{\{N\}}_{0,1}$ for any quasianalytic weight sequence $N = (N_k)$. □

3.2. **The Beurling case.** Here we assume that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{(M)}_{0,n}$ which is equivalent to the condition (2.4), i.e., $m_k^{1/k} \to \infty$. We shall use the following representation result which is a special case of Proposition 3 below.

Proposition 1 ([14, Proposition 2.12])**.** *If $M = (M_k)$ is a positive sequence such that $m_k^{1/k} \to \infty$, then*

$$\Lambda_n^{(M)} = \bigcup \big\{ \Lambda_n^{\{L\}} : L \lhd M,\ \ell_k^{1/k} \to \infty \big\}.$$

This proposition allows us to reduce the Beurling to the Roumieu case.

Theorem 3. *Let $M = (M_k)$ be a quasianalytic weight sequence such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{(M)}_{0,n}$. Then there exist elements in $\Lambda_n^{(M)}$ that are not contained in $j^\infty \mathcal{E}^{\{N\}}_{0,n}$ for any quasianalytic weight sequence $N = (N_k)$.*

In particular, there are elements in $\Lambda_n^{(M)}$ not contained in $j^\infty \mathcal{E}^{(N)}_{0,n}$ for any quasianalytic weight sequence $N = (N_k)$, since always $\mathcal{E}^{(N)} \subseteq \mathcal{E}^{\{N\}}$.

Proof. Let $L = (L_k)$ be a positive sequence satisfying $L \lhd M$ and $\ell_k^{1/k} \to \infty$. Let $\underline{L} = (\underline{L}_k)$ denote the log-convex minorant of L. We still have $\underline{L} \lhd M$ and $\underline{\ell}_k^{1/k} \to \infty$; cf. [14, Lemma 2.6 and Theorem 2.15]. Thus $\underline{L}$ is a quasianalytic weight sequence (since so is M). The condition $\underline{\ell}_k^{1/k} \to \infty$ implies that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{\underline{L}\}}_{0,n}$.

By Theorem 2, there exist elements in $\Lambda_n^{\{\underline{L}\}}$ that are not in $j^\infty \mathcal{E}^{\{N\}}_{0,n}$ for any quasianalytic weight sequence $N = (N_k)$. This implies the statement by Proposition 1. □

4. Weight functions, weight matrices, and Braun–Meise–Taylor classes

4.1. **Weight functions.** A *weight function* is a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega|_{[0,1]} = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$ that satisfies

$$\omega(2t) = O(\omega(t)) \quad \text{as } t \to \infty, \tag{4.1}$$

$$\omega(t) = O(t) \quad \text{as } t \to \infty, \tag{4.2}$$

$$\log t = o(\omega(t)) \quad \text{as } t \to \infty, \tag{4.3}$$

$$\varphi(t) := \omega(e^t) \text{ is convex.} \tag{4.4}$$

For a weight function ω we consider the *Young conjugate* φ^* of φ,

$$\varphi^*(x) := \sup_{y\ge 0} xy - \varphi(y), \quad x \ge 0,$$

which is a convex increasing function satisfying $\varphi^*(0) = 0$, $\varphi^{**} = \varphi$, and $x/\varphi^*(x) \to 0$ as $x \to \infty$; see [8].

4.2. **Braun–Meise–Taylor classes and its germs.** Let ω be a weight function and let $U \subseteq \mathbb{R}^n$ be an open non-empty set. Then the set $\mathcal{E}^{\{\omega\}}(U)$ of all $f \in C^\infty(U)$ such that for all compact $K \subseteq U$ there exists $\rho > 0$ with

$$\|f\|^\omega_{K,\rho} := \sup_{x \in K,\ \alpha \in \mathbb{N}^n} \frac{|\partial^\alpha f(x)|}{\exp(\frac{1}{\rho}\varphi^*(\rho|\alpha|))} < \infty,$$

is called the *Braun–Meise–Taylor class of Roumieu type* associated with ω. It is endowed with the natural projective topology over K and the inductive topology over $\rho \in \mathbb{N}$. Analogously, we define the *Braun–Meise–Taylor class of Beurling type* $\mathcal{E}^{(\omega)}(U)$ consisting of all $f \in C^\infty(U)$ such that for all compact $K \subseteq U$ and all $\rho > 0$, $\|f\|^\omega_{K,\rho} < \infty$, and endow it with its natural Fréchet topology ($1/\rho \in \mathbb{N}$).

Let us define the rings of germs at $0 \in \mathbb{R}^n$,

$$\mathcal{E}^{\{\omega\}}_{0,n} := \operatorname{ind}_{k\in\mathbb{N}} \mathcal{E}^{\{\omega\}}((-\tfrac{1}{k}, \tfrac{1}{k})^n)$$
$$\mathcal{E}^{(\omega)}_{0,n} := \operatorname{ind}_{k\in\mathbb{N}} \mathcal{E}^{(\omega)}((-\tfrac{1}{k}, \tfrac{1}{k})^n).$$

and consider the sequence spaces

$$\Lambda^{\{\omega\}}_n := \{a = (a_\alpha) \in \mathbb{C}^{\mathbb{N}^n} : \exists \rho > 0 : |a|^\omega_\rho < \infty\},$$
$$\Lambda^{(\omega)}_n := \{a = (a_\alpha) \in \mathbb{C}^{\mathbb{N}^n} : \forall \rho > 0 : |a|^\omega_\rho < \infty\},$$

where

$$|a|^\omega_\rho := \sup_{\alpha \in \mathbb{N}^n} \frac{|a_\alpha|}{\exp(\frac{1}{\rho}\varphi^*(\rho|\alpha|))}.$$

Then $\Lambda^{\{\omega\}}_n$ is an (LB)-space and $\Lambda^{(\omega)}_n$ is a Fréchet space.

With $\omega(t) = t$ we recover the real analytic functions $\mathcal{E}^{\{t\}}(U) = C^\omega(U)$ and restrictions of the entire functions $\mathcal{E}^{(t)}(U) = \mathcal{H}(\mathbb{C}^n)$ if U is connected.

Again $\mathcal{E}^{[\omega]}$ stands for either $\mathcal{E}^{\{\omega\}}$ or $\mathcal{E}^{(\omega)}$, $\Lambda^{[\omega]}_n$ for $\Lambda^{\{\omega\}}_n$ or $\Lambda^{(\omega)}_n$, etc.

For weight functions ω and σ we define

$$\omega \preceq \sigma \quad :\Leftrightarrow \quad \sigma(t) = O(\omega(t)) \text{ as } t \to \infty$$

and

$$\omega \lhd \sigma \quad :\Leftrightarrow \quad \sigma(t) = o(\omega(t)) \text{ as } t \to \infty.$$

Then $\omega \preceq \sigma$ if and only if $\mathcal{E}^{[\omega]} \subseteq \mathcal{E}^{[\sigma]}$ if and only if $\Lambda^{[\omega]} \subseteq \Lambda^{[\sigma]}$, and $\omega \lhd \sigma$ if and only if $\mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{(\sigma)}$ if and only if $\Lambda^{\{\omega\}} \subseteq \Lambda^{(\sigma)}$; cf. [14, Corollary 5.17]. In particular, (4.2) holds if and only if the real analytic class is contained in $\mathcal{E}^{\{\omega\}}$, and furthermore, if and only if the restrictions of all entire functions are contained in $\mathcal{E}^{(\omega)}$. The inclusion of the real analytic class in $\mathcal{E}^{(\omega)}$ is equivalent to the condition

$$\omega(t) = o(t) \quad \text{as } t \to \infty.$$

A weight function ω is called *quasianalytic* if

$$\int_1^\infty \frac{\omega(t)}{t^2}\, dt = \infty.$$

This condition is equivalent to quasianalyticity of $\mathcal{E}^{[\omega]}$.

4.3. **The weight matrix associated with a weight function.** Given a weight function ω we may associate a *weight matrix* $\mathfrak{W} = \{W^x\}_{x>0}$ by setting

$$W^x_k := \exp(\tfrac{1}{x}\varphi^*(xk)), \quad k \in \mathbb{N}.$$

By the properties of φ^*, each W^x is a weight sequence (in the sense of Section 2.2) and $W^x \le W^y$ if $x \le y$.

The weight function ω is quasianalytic if and only if each (equivalently, some) W^x is quasianalytic; see [14, Corollary 5.8].

The associated weight matrix $\mathfrak{W}$ allows us to describe any Braun–Meise–Taylor class $\mathcal{E}^{[\omega]}$ as a union or an intersection of Denjoy–Carleman classes.

Theorem 4 ([14, Corollary 5.15])**.** *Let ω be a weight function and let $\mathfrak{W} = \{W^x\}_{x>0}$ be the associated weight matrix. Let $U \subseteq \mathbb{R}^n$ be any open non-empty set. Then*

$$\mathcal{E}^{\{\omega\}}(U) = \operatorname{proj}_{K\subseteq U} \operatorname{ind}_{x>0} \operatorname{ind}_{\rho>0} \mathcal{E}^{W^x}_\rho(K),$$
$$\mathcal{E}^{(\omega)}(U) = \operatorname{proj}_{K\subseteq U} \operatorname{proj}_{x>0} \operatorname{proj}_{\rho>0} \mathcal{E}^{W^x}_\rho(K)$$

as locally convex spaces (K runs through a compact exhaustion of U). Here $\mathcal{E}^{W^x}_\rho(K)$ denotes the Banach space

$$\mathcal{E}^{W^x}_\rho(K) := \{f \in C^\infty(K) : \|f\|^{W^x}_{K,\rho} < \infty\}.$$

4.4. **Weight matrices and associated ultradifferentiable classes.** More abstractly, we define a *weight matrix* to be a family of weight sequences $\mathfrak{M} = \{M^x\}_{x\in X}$ indexed by a subset $X \subseteq \mathbb{R}$ such that

$$M^x \le M^y \quad \text{if } x \le y. \tag{4.5}$$

For a weight matrix $\mathfrak{M} = \{M^x\}_{x\in X}$ and an open non-empty set $U \subseteq \mathbb{R}^n$, we define the locally convex spaces

$$\mathcal{E}^{\{\mathfrak{M}\}}(U) = \operatorname{proj}_{K\subseteq U} \operatorname{ind}_{x>0} \operatorname{ind}_{\rho>0} \mathcal{E}^{M^x}_\rho(K),$$
$$\mathcal{E}^{(\mathfrak{M})}(U) = \operatorname{proj}_{K\subseteq U} \operatorname{proj}_{x>0} \operatorname{proj}_{\rho>0} \mathcal{E}^{M^x}_\rho(K),$$

its rings of germs at $0 \in \mathbb{R}^n$,

$$\mathcal{E}^{\{\mathfrak{M}\}}_{0,n} := \operatorname{ind}_{k\in\mathbb{N}} \mathcal{E}^{\{\mathfrak{M}\}}((-\tfrac{1}{k}, \tfrac{1}{k})^n),$$
$$\mathcal{E}^{(\mathfrak{M})}_{0,n} := \operatorname{ind}_{k\in\mathbb{N}} \mathcal{E}^{(\mathfrak{M})}((-\tfrac{1}{k}, \tfrac{1}{k})^n),$$

and the sequence spaces

$$\Lambda^{\{\mathfrak{M}\}}_n := \{a = (a_\alpha) \in \mathbb{C}^{\mathbb{N}^n} : \exists x \in X\ \exists \rho > 0 : |a|^{M^x}_\rho < \infty\},$$
$$\Lambda^{(\mathfrak{M})}_n := \{a = (a_\alpha) \in \mathbb{C}^{\mathbb{N}^n} : \forall x \in X\ \forall \rho > 0 : |a|^{M^x}_\rho < \infty\}$$

with the natural (LB)- and Fréchet topology. As usual $\mathcal{E}^{[\mathfrak{M}]}$ means either $\mathcal{E}^{\{\mathfrak{M}\}}$ or $\mathcal{E}^{(\mathfrak{M})}$, etc.

For weight matrices $\mathfrak{M} = \{M^x\}_{x\in X}$ and $\mathfrak{N} = \{N^y\}_{y\in Y}$ we define

$$\mathfrak{M}\{\preceq\}\mathfrak{N} \quad :\Leftrightarrow \quad \forall x \in X\ \exists y \in Y : M^x \preceq N^y$$
$$\mathfrak{M}(\preceq)\mathfrak{N} \quad :\Leftrightarrow \quad \forall y \in Y\ \exists x \in X : M^x \preceq N^y$$

and

$$\mathfrak{M}\{\lhd)\mathfrak{N} \quad :\Leftrightarrow \quad \forall x \in X\ \forall y \in Y : M^x \lhd N^y.$$

Then $\mathfrak{M}[\preceq]\mathfrak{N}$ if and only if $\mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ if and only if $\Lambda^{[\mathfrak{M}]} \subseteq \Lambda^{[\mathfrak{N}]}$, and $\mathfrak{M}\{\lhd)\mathfrak{N}$ if and only if $\mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{(\mathfrak{N})}$ if and only if $\Lambda^{\{\mathfrak{M}\}} \subseteq \Lambda^{(\mathfrak{N})}$; see [14, Proposition 4.6].

Analogously to Theorem 4 we get:

Proposition 2. *Let ω be a weight function and let $\mathfrak{W} = \{W^x\}_{x>0}$ be the associated weight matrix. Then*

$$\Lambda_n^{\{\omega\}} = \Lambda_n^{\{\mathfrak{W}\}} \quad \text{and} \quad \Lambda_n^{(\omega)} = \Lambda_n^{(\mathfrak{W})}$$

as locally convex spaces.

Proof. This follows from the proof of Theorem 4 in [14, Theorem 5.14]. The argument is based on the following two facts: by definition,

$$|a|_x^\omega = |a|_1^{W^x}, \quad a \in \mathbb{C}^{\mathbb{N}^n}, \tag{4.6}$$

and, by [14, Lemma 5.9],

$$\forall \sigma > 0 \; \exists H \geq 1 \; \forall x > 0 \; \exists C \geq 1 \; \forall k \in \mathbb{N} : \sigma^k W_k^x \leq C W_k^{Hx}. \tag{4.7}$$

The (continuous) inclusions $\Lambda_n^{\{\omega\}} \subseteq \Lambda_n^{\{\mathfrak{W}\}}$ and $\Lambda_n^{(\omega)} \supseteq \Lambda_n^{(\mathfrak{W})}$ follow easily from (4.6).

If we combine (4.6) and (4.7) we obtain

$$\forall \sigma > 0 \; \exists H \geq 1 \; \forall x > 0 \; \exists C \geq 1 :$$
$$|a|_{Hx}^\omega \leq C \, |a|_\sigma^{W^x} \quad \text{and} \quad |a|_{1/\sigma}^{W^x} \leq C \, |a|_{x/H}^\omega, \quad a \in \mathbb{C}^{\mathbb{N}^n},$$

which implies the continuous inclusions $\Lambda_n^{\{W^x\}} \subseteq \Lambda_n^{\{\omega\}}$ and $\Lambda_n^{(\omega)} \subseteq \Lambda_n^{(W^x)}$, for all $x > 0$. □

5. Non-surjectivity of the Borel mapping for proper quasianalytic classes

We shall show in this section that the Borel mapping is never surjective in the proper quasianalytic setting. We will work in the framework of ultradifferentiable classes $\mathcal{E}^{[\mathfrak{M}]}$ defined in terms of a weight matrix $\mathfrak{M}$. In view of Theorem 4 this includes all Braun–Meise–Taylor classes and thus we recover the result of Bonet and Meise [6]. The approach via weight matrices allows us to apply the results on Denjoy–Carleman classes in Section 2 in a direct way.

Let $\mathfrak{M} = \{M^x\}_{x \in X}$ be a weight matrix. Let us consider the Borel mapping

$$j^\infty : \mathcal{E}_{0,n}^{[\mathfrak{M}]} \to \Lambda_n^{[\mathfrak{M}]}, \quad f \mapsto (\partial^\alpha f(0))_{\alpha \in \mathbb{N}^n}. \tag{5.1}$$

The mapping (5.1) specializes to the mapping (3.1) if $\mathfrak{M}$ consists of a single weight sequence M, and it specializes to the mapping

$$j^\infty : \mathcal{E}_{0,n}^{[\omega]} \to \Lambda_n^{[\omega]}, \quad f \mapsto (\partial^\alpha f(0))_{\alpha \in \mathbb{N}^n},$$

if ω is a weight function, thanks to Theorem 4 and Proposition 2.

5.1. **Quasianalytic weight matrices.** It is easy to see that the ring $\mathcal{E}_{0,n}^{\{\mathfrak{M}\}}$ is quasianalytic (i.e., the Borel mapping $j^\infty : \mathcal{E}_{0,n}^{\{\mathfrak{M}\}} \to \Lambda_n^{\{\mathfrak{M}\}}$ is injective) if and only if each weight sequence M^x in the weight matrix $\mathfrak{M} = \{M^x\}_{x \in X}$ is quasianalytic.

In the Beurling case, $\mathcal{E}_{0,n}^{(\mathfrak{M})}$ is quasianalytic if and only if at least one weight sequence M^x in the weight matrix $\mathfrak{M} = \{M^x\}_{x \in X}$ is quasianalytic; this follows from [17, Proposition 4.7]. In that case we can assume that all weight sequences in

$\mathfrak{M}$ are quasianalytic by removing all non-quasianalytic ones; by the property (4.5) this leaves the spaces $\mathcal{E}^{(\mathfrak{M})}(U)$, $\mathcal{E}^{(\mathfrak{M})}_{0,n}$, and $\Lambda^{(\mathfrak{M})}_n$ unchanged.

In light of this remark we call a weight matrix $\mathfrak{M} = \{M^x\}_{x\in X}$ *quasianalytic* if each weight sequence M^x is quasianalytic. (We warn the reader that the formal negation of this notion, i.e., $\mathfrak{M}$ *is not quasianalytic*, means that $\mathcal{E}^{\{\mathfrak{M}\}}_{0,n}$ is non-quasianalytic, but not necessarily $\mathcal{E}^{(\mathfrak{M})}_{0,n}$.)

5.2. **The Roumieu case.** We shall assume that $\mathfrak{M} = \{M^x\}_{x\in X}$ is a quasianalytic weight matrix such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{\mathfrak{M}\}}_{0,n}$. The latter condition holds if and only if $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{M^x\}}_{0,n}$ for some $x \in X$, or equivalently

$$\exists x \in X : \sup_k (m^x_k)^{1/k} = \infty,$$

where $m^x_k := M^x_k/k!$.

Theorem 5. *Let $\mathfrak{M} = \{M^x\}_{x\in X}$ be a quasianalytic weight matrix such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{\mathfrak{M}\}}_{0,n}$. Then there exist elements in $\Lambda^{\{\mathfrak{M}\}}_n$ that are not contained in $j^\infty\mathcal{E}^{\{\mathfrak{N}\}}_{0,n}$ for any quasianalytic weight matrix $\mathfrak{N} = \{N^y\}_{y\in Y}$.*

Proof. By assumption there exists $x \in X$ such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{M^x\}}_{0,n}$. Then Theorem 2 implies that there is an element $a = (a_\alpha) \in \Lambda^{\{M^x\}}_n \subseteq \Lambda^{\{\mathfrak{M}\}}_n$ such that $a \notin j^\infty\mathcal{E}^{\{N\}}_{0,n}$ for all quasianalytic weight sequences $N = (N_k)$.

In particular, $a \notin j^\infty\mathcal{E}^{\{\mathfrak{N}\}}_{0,n}$ for every quasianalytic weight matrix $\mathfrak{N} = \{N^y\}_{y\in Y}$. In fact, suppose that $a \in j^\infty\mathcal{E}^{\{\mathfrak{N}\}}_{0,n}$ for some quasianalytic weight matrix $\mathfrak{N}$. Then there exist $r > 0$ and $f \in \mathcal{E}^{\{\mathfrak{N}\}}((-2r, 2r)^n)$ such that $j^\infty f = a$. By restriction, we can assume that $f \in \mathcal{E}^{\{\mathfrak{N}\}}([-r, r]^n)$ and in turn that there exists $y \in Y$ such that $f \in \mathcal{E}^{\{N^y\}}([-r, r]^n)$. But this contradicts the first paragraph. □

In view of Theorem 4 and Proposition 2 we immediately obtain the following corollary.

Corollary 2. *Let ω be a quasianalytic weight function such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{\omega\}}_{0,n}$. Then there exist elements in $\Lambda^{\{\omega\}}_n$ that are not contained in $j^\infty\mathcal{E}^{\{\sigma\}}_{0,n}$ for any quasianalytic weight function σ.*

Note that the strict inclusion $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{\{\omega\}}_{0,n}$ holds if and only if

$$\liminf_{t\to\infty} \frac{\omega(t)}{t} = 0$$

which is immediate from the inclusion relations recalled in Section 4.2.

5.3. **The Beurling case.** Here we assume that $\mathfrak{M} = \{M^x\}_{x\in X}$ is a quasianalytic weight matrix such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{(\mathfrak{M})}_{0,n}$. As we will see below this strict inclusion holds if and only if $\mathcal{O}_{0,n} \subsetneq \mathcal{E}^{(M^x)}_{0,n}$ for all $x \in X$, or equivalently

$$\forall x \in X : (m^x_k)^{1/k} \to \infty, \tag{5.2}$$

where $m^x_k := M^x_k/k!$.

We will reduce the Beurling to the Roumieu case. The key to this reduction is the following lemma.

Lemma 1. *Let $\mathfrak{M} = \{M^x\}_{x\in X}$ be a weight matrix. Let $L = (L_k)$ be any positive sequence satisfying $L\{\lhd)\mathfrak{M}$. Then there exists a positive sequence $N = (N_k)$ satisfying $L \lhd N\{\lhd)\mathfrak{M}$.*

Proof. Without loss of generality we can assume that $X = (0,\infty)$. The assumption $L\{\lhd)\mathfrak{M}$ precisely means that

$$\forall x \in (0,\infty)\ \forall \rho > 0\ \exists C \geq 1\ \forall k \in \mathbb{N} : L_k \leq C\,\rho^k M_k^x.$$

In particular, (taking $x = \rho = 1/p$)

$$\forall p \in \mathbb{N}_{\geq 1}\ \exists C \geq 1\ \forall k \in \mathbb{N} : L_k \leq C\,p^{-k} M_k^{1/p}. \tag{5.3}$$

Let C_p denote the minimal constant C such that (5.3) holds. This defines a non-decreasing sequence $(C_p)_p$ (by (4.5)). Fix a real number $A > 1$. Choose a strictly increasing sequence $(j_p)_{p\geq 1}$ of positive integers such that $C_p \leq A^{j_p}$.

We define

$$N_j := \sqrt{L_j M_j^{1/p}} \quad \text{for} \quad j_p \leq j < j_{p+1};$$

for $0 \leq j < j_1$ any choice of N_j works. Then $L \lhd N$ since, by (5.3), for $j_p \leq j < j_{p+1}$,

$$\Big(\frac{L_j}{N_j}\Big)^{1/j} = \Big(\sqrt{\frac{L_j}{M_j^{1/p}}}\Big)^{1/j} \leq \sqrt{\frac{C_p^{1/j}}{p}} \leq \sqrt{\frac{A}{p}}$$

which tends to 0 as $j \to \infty$. Let $x > 0$ be fixed. If $j_p \leq j < j_{p+1}$ where $p \geq 1/x$, then by (4.5),

$$\Big(\frac{N_j}{M_j^x}\Big)^{1/j} = \Big(\frac{\sqrt{L_j M_j^{1/p}}}{M_j^x}\Big)^{1/j} \leq \Big(\sqrt{\frac{L_j}{M_j^x}}\Big)^{1/j}$$

which tends to 0 as $j \to \infty$ because $L \lhd M^x$. That is $N\{\lhd)\mathfrak{M}$ and the proof is complete. $\square$

Corollary 3. *If $\mathfrak{M} = \{M^x\}_{x\in X}$ is a weight matrix satisfying (5.2), then there exists a positive sequence $N = (N_k) = (k!\,n_k)$ satisfying $n_k^{1/k} \to 0$ and $N\{\lhd)\mathfrak{M}$.*

Proof. If L denotes the sequence $L_k = k!$, then (5.2) means exactly $L\{\lhd)\mathfrak{M}$ and the above lemma implies the assertion. $\square$

Thus, if a weight matrix $\mathfrak{M} = \{M^x\}_{x\in X}$ satisfies (5.2), then there is a positive sequence $N = (N_k)$ such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}_{0,n}^{(N)} \subsetneq \mathcal{E}_{0,n}^{(\mathfrak{M})}$, and the assertion at the beginning of the section is proved.

Proposition 3. *If $\mathfrak{M} = \{M^x\}_{x\in X}$ is a weight matrix satisfying (5.2), then*

$$\Lambda_n^{(\mathfrak{M})} = \bigcup \big\{\Lambda_n^{\{L\}} : L\{\lhd)\mathfrak{M},\ \ell_k^{1/k} \to \infty\big\}. \tag{5.4}$$

Proof. Let us show the nontrivial inclusion $\subseteq$. Let $a = (a_\alpha) \in \Lambda_n^{(\mathfrak{M})}$ and set $L_k := \max\{\max_{|\alpha|=k} |a_\alpha|, k!\}$. Then $L\{\lhd)\mathfrak{M}$ as $a \in \Lambda_n^{(\mathfrak{M})}$ and by (5.2). Lemma 1 provides a positive sequence $N = (N_k)$ such that $L \lhd N\{\lhd)\mathfrak{M}$, and thus $a \in \Lambda_n^{(N)} \subseteq \Lambda_n^{\{N\}}$. In particular, $(k!)_k \lhd N$, that is $n_k^{1/k} \to \infty$. The proof is complete. $\square$

Remark 3. (1) The proof of Proposition 3 actually shows that

$$\Lambda_n^{(\mathfrak{M})} = \bigcup \big\{\Lambda_n^{(L)} : L\{\lhd)\mathfrak{M},\ \ell_k^{1/k} \to \infty\big\}. \tag{5.5}$$

(2) It is easy to see that (5.4) and (5.5) hold with $\Lambda_n^{(\mathfrak{M})}$ and $\Lambda_n^{[L]}$ replaced by $\mathcal{E}^{(\mathfrak{M})}(K)$ and $\mathcal{E}^{[L]}(K)$ for any compact $K \subseteq \mathbb{R}^n$, where

$$\mathcal{E}^{\{\mathfrak{M}\}}(K) := \operatorname{ind}_{x>0} \operatorname{ind}_{\rho>0} \mathcal{E}_\rho^{M^x}(K),$$
$$\mathcal{E}^{(\mathfrak{M})}(K) := \operatorname{proj}_{x>0} \operatorname{proj}_{\rho>0} \mathcal{E}_\rho^{M^x}(K).$$

(3) In the situation of Lemma 1 it is sometimes possible to transfer properties of $\mathfrak{M}$ to N, just as in Corollary 3. Another instance is the following: if $L\{\lhd)\mathfrak{M}$, where L satisfies (2.3) and $\mathcal{E}^{(\mathfrak{M})}$ is non-quasianalytic, then there is a log-convex non-quasianalytic N satisfying $L \lhd N\{\lhd)\mathfrak{M}$.

(4) In analogy to the inductive representations in (5.4) and (5.5), there are projective representations of the form

$$\Lambda_n^{\{\mathfrak{M}\}} = \bigcap \Lambda_n^{(L)} = \bigcap \Lambda_n^{\{L\}},$$

where the intersections are taken over all weight sequences L with $\mathfrak{M}\{\lhd)L$; similarly, for $\mathcal{E}^{\{\mathfrak{M}\}}(U)$ and $\mathcal{E}^{[L]}(U)$ and any open $U \subseteq \mathbb{R}^n$. See [16, Proposition 9.4.4].

Now we are ready to show our main result in the Beurling case.

Theorem 6. *Let $\mathfrak{M} = \{M^x\}_{x\in X}$ be a quasianalytic weight matrix such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}_{0,n}^{(\mathfrak{M})}$. Then there exist elements in $\Lambda_n^{(\mathfrak{M})}$ that are not contained in $j^\infty \mathcal{E}_{0,n}^{\{\mathfrak{N}\}}$ for any quasianalytic weight matrix $\mathfrak{N} = \{N^y\}_{y\in Y}$.*

In particular, there are elements in $\Lambda_n^{(\mathfrak{M})}$ not contained in $j^\infty \mathcal{E}_{0,n}^{(\mathfrak{N})}$ for any quasianalytic weight matrix $\mathfrak{N} = \{N^y\}_{y\in Y}$, since always $\mathcal{E}^{(\mathfrak{N})} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$.

Proof. Let $L = (L_k)$ be a positive sequence satisfying $L\{\lhd)\mathfrak{M}$ and $\ell_k^{1/k} \to \infty$ which exists by Proposition 3. Let $\underline{L} = (\underline{L}_k)$ denote the log-convex minorant of L. Then still $\underline{L}\{\lhd)\mathfrak{M}$ and $\underline{\ell}_k^{1/k} \to \infty$, by [14, Theorem 2.15]. It follows that $\underline{L}$ is a quasianalytic weight sequence, since $\mathcal{E}^{(\mathfrak{M})}$ is quasianalytic. Thanks to $\underline{\ell}_k^{1/k} \to \infty$ we have $\mathcal{O}_{0,n} \subsetneq \mathcal{E}_{0,n}^{\{\underline{L}\}}$.

Now the assertion is a direct consequence of Theorem 2 (or Theorem 5) applied to $\underline{L}$. □

By Theorem 4 and Proposition 2, we immediately get the following corollary.

Corollary 4. *Let ω be a quasianalytic weight function such that $\mathcal{O}_{0,n} \subsetneq \mathcal{E}_{0,n}^{(\omega)}$. Then there exist elements in $\Lambda_n^{(\omega)}$ that are not contained in $j^\infty \mathcal{E}_{0,n}^{\{\sigma\}}$ for any quasianalytic weight function σ.*

The strict inclusion $\mathcal{O}_{0,n} \subsetneq \mathcal{E}_{0,n}^{(\omega)}$ holds if and only if $\omega(t) = o(t)$ as $t \to \infty$; cf. [14, Corollary 5.17(3)].

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