THE BOREL MAP IN THE MIXED BEURLING SETTING

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Abstract. The Borel map takes a smooth function to its infinite jet of derivatives (at zero). We study the restriction of this map to ultradifferentiable classes of Beurling type in a very general setting which encompasses the classical Denjoy–Carleman and Braun–Meise–Taylor classes. More precisely, we characterize when the Borel image of one class covers the sequence space of another class in terms of the two weights that define the classes. We present two independent solutions to this problem, one by reduction to the Roumieu case and the other by dualization of the involved Fréchet spaces, a Phragmén–Lindelöf theorem, and Hörmander's solution of the $\overline{\partial}$-problem.

1. Introduction

The Borel map $j_0^\infty : C^\infty(\mathbb{R}) \to \mathbb{C}^\mathbb{N}$ at 0 is defined by $j_0^\infty f := (f^{(n)}(0))_{n\in\mathbb{N}}$. We will study the restriction of j_0^∞ to ultradifferentiable classes in a general setting which allows us to treat the classical Denjoy–Carleman and Braun–Meise–Taylor classes at the same time. Our classes are defined in terms of one-parameter families $\mathfrak{M} = (M^{(x)})_{x>0}$, $\mathfrak{N} = (N^{(x)})_{x>0}$, etc., of weight sequences; we call them weight matrices. In this article, we are mostly interested in classes of Beurling type

$$\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) :\ \forall j,k,l \in \mathbb{N}_{\ge 1} : \sup_{x\in[-j,j]} \sup_{p\in\mathbb{N}} \frac{|f^{(p)}(x)|}{(\frac{1}{l})^p N_p^{(\frac{1}{k})}} < \infty \Big\},$$

but we shall have to use also results on the Roumieu counterpart

$$\mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) :\ \forall j \in \mathbb{N}_{\ge 1}\ \exists k,l \in \mathbb{N}_{\ge 1} : \sup_{x\in[-j,j]} \sup_{p\in\mathbb{N}} \frac{|f^{(p)}(x)|}{l^p N_p^{(k)}} < \infty \Big\}.$$

By definition, the image $j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ is contained in the sequence space

$$\Lambda^{(\mathfrak{N})} := \Big\{ a = (a_p) \in \mathbb{C}^\mathbb{N} :\ \forall k,l \in \mathbb{N}_{\ge 1} : \sup_{p\in\mathbb{N}} \frac{|a_p|}{(\frac{1}{l})^p N_p^{(\frac{1}{k})}} < \infty \Big\};$$

and likewise $j_0^\infty \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R}) \subseteq \Lambda^{\{\mathfrak{N}\}}$, where $\Lambda^{\{\mathfrak{N}\}}$ is defined analogously. The goal of this paper is to find necessary and sufficient conditions for

$$\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R}), \tag{1.1}$$

in terms of $\mathfrak{M}$ and $\mathfrak{N}$.

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The Roumieu case is well understood, see our recent article [24]: under some mild assumptions on $\mathfrak{M}$ and $\mathfrak{N}$, we have $\Lambda^{\{\mathfrak{M}\}} \subseteq j_0^\infty \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R})$ if and only if

$$\forall x > 0 \; \exists y > 0 : \; M^{(x)} \prec_{SV} N^{(y)},$$

where $M^{(x)} \prec_{SV} N^{(y)}$ means

$$\exists s \in \mathbb{N} : \sup_{j \ge 1} \sup_{0 \le i < j} \Big(\frac{M_j^{(x)}}{s^j N_i^{(y)}} \Big)^{\frac{1}{j-i}} \frac{1}{j} \sum_{k=j}^\infty \frac{N_{k-1}^{(y)}}{N_k^{(y)}} < \infty,$$

a condition introduced by Schmets and Valdivia in [37].

The characterization of (1.1) is considerably more difficult (partly, because the image of an intersection is, in general, smaller than the intersection of the images). We solve this problem in two independent ways:

(1) The first method reduces the Beurling to the Roumieu problem, and uses the solution of the latter. This is a well-known approach which has been used, in various disguises, in several settings; see e.g. [9], [37], [17], and [26]. The additional parameter (i.e., x in the weight matrix) makes this delicate reduction quite involved. As a result we prove in Theorem 3.1 that (again under mild assumptions) (1.1) is equivalent to

$$\forall y > 0 \; \exists x > 0 : \; M^{(x)} \prec_{SV} N^{(y)}.$$

(2) The second approach is based on dualization of (1.1) and identification of the strong duals $(\Lambda^{(\mathfrak{M})})'$ and $\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})'$ with suitable spaces of entire functions. This strategy has been implemented by [6] (for Braun–Meise–Taylor classes, following [8] and [10]). In fact, our analysis is based on the abstract functional-analytic result [6, Corollary 2.3] (which we restate in Proposition 5.9). It translates the problem to a question about bounded sets in the mentioned spaces of entire functions, where a Phragmén–Lindelöf theorem and Hörmander's solution of the $\overline{\partial}$-problem can be brought to bear. We find in Theorem 5.1 that (under other mild assumptions) (1.1) is equivalent to

$$\forall y > 0 \; \exists x > 0 : \; M^{(x)} \prec_L N^{(y)}.$$

The condition $M^{(x)} \prec_L N^{(y)}$ means

$$\exists C > 0 \; \forall s \ge 0 : \; \frac{s}{\pi} \int_{-\infty}^\infty \frac{\omega_{N^{(y)}}(t)}{t^2 + s^2} \, dt \le \omega_{M^{(x)}}(Cs) + C,$$

where $\omega_M(t) := \sup_{k \in \mathbb{N}} \log(\frac{t^k}{M_k})$ is the pre-weight function associated with a weight sequence M. It appears as (2.14') in Langenbruch's paper [20] and it is closely related to the condition appearing in [6], but with a little twist; see Remark 5.2 and Section 6.

Let us briefly describe the structure of the paper. In Section 2, we gather all relevant notation and conditions concerning weight sequences, functions, and matrices and we introduce the corresponding ultradifferentiable function and sequence spaces. The solution by reduction (1) is obtained in Section 3. In Section 4, we identify the duals $(\Lambda^{(\mathfrak{M})})'$ and $\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})'$ with certain weighted spaces of entire functions. This allows us to carry out the solution by dualization (2) in Section 5. In the final Section 6, we show that our theorems specialize to the known results for Denjoy–Carleman and Braun–Meise–Taylor classes; see Theorem 6.2, Supplement 6.3, and Theorem 6.4. In the short appendix, we prove a technical statement needed in Section 4, namely that the entire functions are dense in an auxiliary function space. Since the inclusion of the entire functions is continuous, also the polynomials are dense.

2. Ultradifferentiable classes and weights

Ultradifferentiable classes are weighted classes of smooth functions.

2.1. **Weight sequences.** We call a sequence of positive real numbers $M = (M_k)$ a *weight sequence*, if $M_0 = 1$ and $M_k = \mu_1 \cdots \mu_k$, $k \ge 1$, for an increasing sequence $0 < \mu_1 \le \mu_2 \le \cdots$ tending to ∞. We call a weight sequence *normalized* if $\mu_1 \ge 1$ and put $\mu_0 := 1$. Let us also set $m_k := \frac{M_k}{k!}$.

That μ_k is increasing means that M_k is log-convex. Here are some easy consequences of the definition: $M_j M_k \le M_{j+k}$, $(M_k)^{1/k} \le \mu_k$, and $(M_k)^{1/k} \to \infty$ if and only if $\mu_k \to \infty$ (cf. [27, Lemma 2.3]).

For a weight sequence M, we define the *Denjoy–Carleman class of Beurling type*

$$\mathcal{E}^{(M)}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) : \ \forall K \subset\subset \mathbb{R} \ \ \forall r > 0 : \|f\|^M_{K,r} := \sup_{x \in K, k \in \mathbb{N}} \frac{|f^{(k)}(x)|}{r^k M_k} < \infty \Big\}.$$

It is endowed with the natural projective topology and thus has the structure of a Fréchet space. If the universal quantifier in front of r is replaced by an existential quantifier one gets the Denjoy–Carleman class $\mathcal{E}^{\{M\}}(\mathbb{R})$ of *Roumieu type.*

It is immediate that the restriction of the Borel map j^∞_0 to $\mathcal{E}^{(M)}(\mathbb{R})$ takes values in the corresponding sequence space

$$\Lambda^{(M)} := \Big\{ \lambda = (\lambda_k)_k \in \mathbb{C}^\mathbb{N} : \ \forall r > 0 : \|\lambda\|^M_r := \sup_{k \in \mathbb{N}} \frac{|\lambda_k|}{r^k M_k} < \infty \Big\},$$

which again is endowed with its natural Fréchet topology. By the Denjoy-Carleman theorem, $j^\infty_0|_{\mathcal{E}^{(M)}(\mathbb{R})}$ is injective if and only if

$$\sum_{k \ge 1} \frac{1}{\mu_k} = \infty;$$

see e.g. [19, Theorem 4.2], [14, Theorem 1.3.8], or [27, Theorem 3.6]. In that case, the class (and the weight sequence) is called *quasianalytic*, and *non-quasianalytic* otherwise.

We say that M has *moderate growth*, if

$$\exists C > 0 \ \ \forall j, k \in \mathbb{N} : \quad M_{j+k} \le C^{j+k} M_j M_k,$$

and M is *derivation closedness*, if

$$\exists C > 0 \ \ \forall j \in \mathbb{N} : \quad M_{j+1} \le C^{j+1} M_j.$$

All these conditions are frequently used in the theory of ultradifferentiable classes; in [19], non-quasianalyticity is denoted by $(M.3)'$, derivation closedness by $(M.2)'$ and moderate growth by $(M.2)$.

Given two weight sequences M and N, we write $M \le N$ if $M_k \le N_k$ for all k, and $M \preccurlyeq N$ if $\sup_{k>0} \big(\frac{M_k}{N_k}\big)^{1/k} < \infty$. We say that M and N are *equivalent* if $M \preccurlyeq N$ and $N \preccurlyeq M$. Note that both moderate growth and derivation closedness are preserved under equivalence. Two weight sequences are equivalent if and only if they generate the same class. In fact,

$$\mathcal{E}^{(M)}(\mathbb{R}) \subseteq \mathcal{E}^{(N)}(\mathbb{R}) \quad \Longleftrightarrow \quad M \preccurlyeq N \quad \Longleftrightarrow \quad \Lambda^{(M)} \subseteq \Lambda^{(N)}.$$

We shall also need the relation $M \lhd N$, defined by $\lim_{k\to\infty} \big(\frac{M_k}{N_k}\big)^{1/k} = 0$, which is equivalent to $\mathcal{E}^{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}^{(N)}(\mathbb{R})$ as well as $\Lambda^{\{M\}} \subseteq \Lambda^{(N)}$. All this can be found in [28, Proposition 2.12].

2.2. **Weight functions.** The second approach to ultradifferentiable classes is based on weight functions, i.e., increasing continuous functions $\omega : [0,\infty) \to [0,\infty)$ satisfying some additional properties which will be specified shortly. Originally, ω was used, by Beurling [1] and Björck [2], to impose growth restrictions at infinity on the Fourier transform of the functions in question. In the modern approach due to Braun, Meise, and Taylor [7], the derivatives of the functions are controlled by the *Young conjugate* of $y \mapsto \varphi_\omega(y) := \omega(e^y)$, that is

$$\varphi^*_\omega(x) := \sup\{xy - \varphi_\omega(y) : \ y \ge 0\}, \quad x \ge 0.$$

Assuming that $\log(t) = o(\omega(t))$ as $t \to \infty$, which ensures that $\varphi^*_\omega(x)$ is finite for all $x > 0$, one defines the *Braun–Meise–Taylor class of Beurling type*

$$\mathcal{E}^{(\omega)}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) : \ \forall K \subset\subset \mathbb{R} \ \forall r > 0 : \|f\|^\omega_{K,r} := \sup_{x\in K, k\in\mathbb{N}} \frac{|f^{(k)}(x)|}{e^{\varphi^*_\omega(rk)/r}} < \infty \Big\}$$

and endows it with the natural Fréchet topology. Similarly, we have the Fréchet space

$$\Lambda^{(\omega)} := \Big\{ \lambda = (\lambda_k)_k \in \mathbb{C}^\mathbb{N} : \ \forall r > 0 : \|\lambda\|^\omega_r := \sup_{k\in\mathbb{N}} \frac{|\lambda_k|}{e^{\varphi^*_\omega(rk)/r}} < \infty \Big\}$$

and the map $j^\infty_0|_{\mathcal{E}^{(\omega)}(\mathbb{R})} : \mathcal{E}^{(\omega)}(\mathbb{R}) \to \Lambda^{(\omega)}$. Again there is a Roumieu version of these classes of functions and sequences, where r is subjected to an existential quantifier; we refer to [24] for details.

Let us now make precise the relevant regularity properties for ω. We say that an increasing continuous function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ is a *pre-weight function*, if $\log(t) = o(\omega(t))$ as $t \to \infty$ (in particular, $\omega(t) \to \infty$), and φ_ω is convex. We call a pre-weight ω a *weight function* if it also fulfills

$$\omega(2t) = O(\omega(t)) \text{ as } t \to \infty. \qquad (\omega_1)$$

The map $j^\infty_0|_{\mathcal{E}^{(\omega)}(\mathbb{R})}$ is injective if and only if

$$\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt = \infty;$$

see e.g. [7], [35, Section 4], or [27, Theorem 11.17]. Then the class and the weight function ω are called *quasianalytic*, and *non-quasianalytic* otherwise. It is straightforward to see that non-quasianalyticity implies $\omega(t) = o(t)$ as $t \to \infty$.

Two pre-weight functions are called *equivalent*, written $\omega \sim \sigma$, if $\omega(t) = O(\sigma(t))$ and $\sigma(t) = O(\omega(t))$ as $t \to \infty$. This is precisely the case if they generate the same classes. Indeed,

$$\mathcal{E}^{(\omega)}(\mathbb{R}) \subseteq \mathcal{E}^{(\sigma)}(\mathbb{R}) \Longleftrightarrow \Lambda^{(\omega)} \subseteq \Lambda^{(\sigma)} \Longleftrightarrow \sigma(t) = O(\omega(t)) \text{ as } t \to \infty,$$

see [28, Corollary 5.17]. For every pre-weight function there is an equivalent pre-weight function which vanishes on $[0,1]$.

Remark 2.1. We will frequently consider the radially symmetric extension $\mathbb{C} \ni z \mapsto \omega(|z|)$ of a pre-weight function ω. By abuse of notation, we will still write $\omega(z)$ instead of $\omega(|z|)$.

2.3. **The associated weight function.** Let M be a weight sequence. Then

$$\omega_M(t) := \sup_{k\in\mathbb{N}} \log\Big(\frac{t^k}{M_k}\Big),$$

is a pre-weight function; cf. [21, Chapitre I] and [19, Section 3.1]. See [33, Theorem 3.1] for necessary and sufficient conditions for ω_M being a weight function. For $\lambda > 0$, we set $\mu_M(\lambda) := |\{p \in \mathbb{N}_{\geq 1} :\ \mu_p \leq \lambda\}|$. Then we have the following integral representation of ω_M, cf. e.g. [19, (3.11)] and references therein,

$$\omega_M(t) = \int_0^t \frac{\mu_M(\lambda)}{\lambda}\, d\lambda. \tag{2.1}$$

If M is normalized, then $\omega_M|_{[0,1]} = 0$. And ω_M is non-quasianalytic if and only if M is non-quasianalytic; see [19, Lemma 4.1]. Note that a weight sequence M can be recovered from ω_M by

$$M_k = \sup_{t>0} \frac{t^k}{\exp(\omega_M(t))}, \quad k \in \mathbb{N}. \tag{2.2}$$

In general, $\mathcal{E}^{(M)}(\mathbb{R})$ and $\mathcal{E}^{(\omega_M)}(\mathbb{R})$ may differ, unless M has moderate growth; see [5] and [28, Section 5].

2.4. **Weight matrices.** In [28] and [34], Denjoy–Carleman and Braun–Meise–Taylor classes were understood as special cases of ultradifferentiable classes defined by weight matrices. A *weight matrix* is a one-parameter family of weight sequences $\mathfrak{M} = (M^{(x)})_{x>0}$ such that $M^{(x)} \leq M^{(y)}$ if $x \leq y$ and, for all $x > 0$,

$$(m_j^{(x)})^{1/j} \to \infty \text{ as } j \to \infty. \tag{2.3}$$

We call $\mathfrak{M}$ *normalized* if all $M^{(x)} \in \mathfrak{M}$ are normalized.

We define the classes of *Beurling type*

$$\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) :\ \forall K \subset\subset \mathbb{R}\ \ \forall r, x > 0 :\ \|f\|_{K,r}^{M^{(x)}} < \infty \Big\},$$

and

$$\Lambda^{(\mathfrak{M})} := \Big\{ \lambda = (\lambda_k)_k \in \mathbb{C}^{\mathbb{N}} :\ \forall r, x > 0 :\ \|\lambda\|_r^{M^{(x)}} < \infty \Big\},$$

and endow both spaces with their natural Fréchet topology. Note that, by our assumption (2.3), each class $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ contains all real analytic functions on $\mathbb{R}$ (cf. [28, Section 4.1]).

If *all* $M^{(x)} \in \mathfrak{M}$ are non-quasianalytic, we call $\mathfrak{M}$ *non-quasianalytic*. Non-quasianalyticity of $\mathfrak{M}$ is equivalent to the existence of bump functions in $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$; see [35, Proposition 4.7] or [27, Theorem 11.16].

Any weight sequence M induces a weight matrix $\mathfrak{M} = (M^{(x)})_{x>0}$ with $M^{(x)} = M$ for all $x > 0$. Then, obviously, $\mathcal{E}^{(M)}(\mathbb{R}) = \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ and $\Lambda^{(M)}(\mathbb{R}) = \Lambda^{(\mathfrak{M})}(\mathbb{R})$.

2.5. **Weight matrices associated with pre-weight functions.** To a pre-weight function ω (vanishing on $[0,1]$) such that $\omega(t) = o(t)$ as $t \to \infty$, we assign the normalized weight matrix $\Omega = (W^{(x)})_{x>0}$ defined by

$$W_k^{(x)} := \exp\Big(\frac{1}{x}\varphi_\omega^*(xk)\Big). \tag{2.4}$$

If ω is actually a weight function, then

$$\mathcal{E}^{(\omega)}(\mathbb{R}) \cong \mathcal{E}^{(\Omega)}(\mathbb{R}) \quad \text{and} \quad \Lambda^{(\omega)} \cong \Lambda^{(\Omega)} \tag{2.5}$$

as locally convex spaces; see [28] and [34]. Let us remark that here $\omega(t) = o(t)$ as $t \to \infty$ is assumed so that Ω satisfies our standard assumption (2.3).

Let us collect some useful properties of Ω.

Lemma 2.2. *The weight matrix $\Omega = (W^{(x)})_{x>0}$ satisfies:*

1. $\vartheta^{(x)} \le \vartheta^{(y)}$ *if* $x \le y$, *where* $\vartheta_k^{(x)} := \frac{W_k^{(x)}}{W_{k-1}^{(x)}}$.
2. $W_{j+k}^{(x)} \le W_j^{(2x)} W_k^{(2x)}$ *for all* $x > 0$ *and* $j,k \in \mathbb{N}$.
3. $\omega \sim \omega_{W^{(x)}}$ *for each* $x > 0$. *More precisely,*
$$\forall x > 0\ \exists D_x > 0 : x\omega_{W^{(x)}} \le \omega \le 2x\omega_{W^{(x)}} + D_x. \tag{2.6}$$
4. $(w_k^{(x)})^{1/k} \to \infty$ *for all* $x > 0$ *if and only if* $\omega(t) = o(t)$ *as* $t \to \infty$.
5. ω *is non-quasianalytic if and only if each* $W^{(x)}$ *is non-quasianalytic, i.e., if and only if* Ω *is non-quasianalytic.*
6. *If* ω *is a weight function, then*
$$\forall h \ge 1\ \exists A \ge 1\ \forall x > 0\ \exists D \ge 1\ \forall j \in \mathbb{N} : h^j W_j^{(x)} \le D W_j^{(Ax)}, \tag{2.7}$$
which is crucial to have (2.5).

Proof. Cf. [28, Section 5] and [29, Section 2.5]. For (3) see also [34, Theorem 4.0.3, Lemma 5.1.3] and [16, Lemma 2.5]. □

2.6. **Order relations of weight matrices.** For two weight matrices $\mathfrak{M}$ and $\mathfrak{N}$, we write $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ if for all y there exists x such that $M^{(x)} \preccurlyeq N^{(y)}$. By [28, Proposition 4.6(1)],

$$\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) \quad \Longleftrightarrow \quad \Lambda^{(\mathfrak{M})} \subseteq \Lambda^{(\mathfrak{N})} \quad \Longleftrightarrow \quad \mathfrak{M}(\preccurlyeq)\mathfrak{N}.$$

If $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ and $\mathfrak{N}(\preccurlyeq)\mathfrak{M}$ hold simultaneously, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ are *equivalent*. This is the case if and only if $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) = \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ as well as $\Lambda^{(\mathfrak{M})} = \Lambda^{(\mathfrak{N})}$ (as sets and, in turn, also as locally convex vector spaces).

Remark 2.3. Typically, for each notion of Beurling type there is a related version of Roumieu type. Since in this paper we are principally concerned with the Beurling case, we will only mention the former without emphasizing every time that it is the Beurling version.

If M is a weight sequence and $\mathfrak{N}$ a weight matrix, then $M \lhd N^{(x)}$ for all $x > 0$ if and only if $\mathcal{E}^{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$; see [28, Proposition 4.6(2)].

2.7. **Moderate growth and derivation closedness.** For weight sequences M, N, consider

$$mg(M,N) := \sup_{j+k\ge 1}\Big(\frac{M_{j+k}}{N_j N_k}\Big)^{\frac{1}{j+k}} \in (0,\infty]$$

and

$$dc(M,N) := \sup_{j\in\mathbb{N}}\left(\frac{M_{j+1}}{N_j}\right)^{\frac{1}{j+1}} \in (0,\infty].$$

A weight matrix $\mathfrak{M} = (M^{(x)})_{x>0}$ is said to have *moderate growth* if

$$\forall y>0\ \exists x>0:\ mg(M^{(x)},M^{(y)})<\infty, \tag{$\mathfrak{M}_{(mg)}$}$$

and to be *derivation closed* if

$$\forall y>0\ \exists x>0:\ dc(M^{(x)},M^{(y)})<\infty. \tag{$\mathfrak{M}_{(dc)}$}$$

Note that moderate growth, derivation closedness, and non-quasianalyticity are preserved under equivalence.

Derivation closedness allows for absorption of log-terms in associated weight functions:

Lemma 2.4. *Let $M^{(k)}$, for $1 \le k \le l+1$, be weight sequences such that $dc(M^{(k)}, M^{(k+1)}) < \infty$ for all $1 \le k \le l$. Then there exists $C > 0$ such that*

$$\omega_{M^{(l+1)}}(t) + \log(1+t^l) \le \omega_{M^{(1)}}(Ct) + C, \quad t \ge 0.$$

Proof. An iterated application of [4, Lemma 2] yields the result. □

2.8. **Absorbing exponential growth.** Inspired by (2.7) (cf. [28, Section 4.1]), we say that a weight matrix $\mathfrak{M}$ *absorbs exponential growth* if

$$\forall y,h>0\ \exists x,A>0\ \forall k\in\mathbb{N}: \quad h^k M^{(x)}_k \le A M^{(y)}_k. \tag{$\mathfrak{M}_{(L)}$}$$

The weight matrix Ω associated with a weight function always has this property, by Lemma 2.2.

The following lemma states that for any weight matrix $\mathfrak{M}$ we may find an equivalent weight matrix with the property $(\mathfrak{M}_{(L)})$. For the sake of completeness, we mention that an analogous statement holds true in the Roumieu setting as well.

Lemma 2.5. *Let $\mathfrak{M}$ be a (normalized) weight matrix. Then there exists an equivalent (normalized) weight matrix $\mathfrak{N}$ that satisfies $(\mathfrak{M}_{(L)})$. Actually, we can choose $\mathfrak{N}$ such that for all $k \in \mathbb{N}_{\ge 1}$ there exist A_k and B_k such that for all $j \in \mathbb{N}$*

$$A_k\Big(\frac{1}{2^k}\Big)^j M_j^{(\frac{1}{k})} \le N_j^{(\frac{1}{k})} \le B_k\Big(\frac{1}{2^k}\Big)^j M_j^{(\frac{1}{k})}. \tag{2.8}$$

Consequently, for all $t \ge 0$,

$$\omega_{M^{(1/k)}}(2^k t) - \log(B_k) \le \omega_{N^{(1/k)}}(t) \le \omega_{M^{(1/k)}}(2^k t) - \log(A_k). \tag{2.9}$$

Proof. We will construct normalized weight sequences $N^{(\frac{1}{k})}$, indexed by $k \in \mathbb{N}_{\ge 1}$, satisfying (2.8) and $N^{(\frac{1}{k+1})} \le N^{(\frac{1}{k})}$ for all k. If we set $N^{(x)} := N^{(\frac{1}{k})}$ for $\frac{1}{k+1} < x \le \frac{1}{k}$, then (2.8) implies that $\mathfrak{M}$ and $\mathfrak{N}$ are equivalent. Moreover, (2.9) follows from (2.8) and the definition of the associated weight function. To see that $\mathfrak{N}$ fulfills $(\mathfrak{M}_{(L)})$, fix y and h and choose $k, n \in \mathbb{N}$ such that $h \le 2^k$ and $\frac{1}{n} \le y$. By (2.8),

$$h^j N_j^{(\frac{1}{k+n})} \le B_{k+n}\Big(\frac{1}{2^n}\Big)^j M_j^{(\frac{1}{k+n})} \le B_{k+n}\Big(\frac{1}{2^n}\Big)^j M_j^{(\frac{1}{n})} \le \frac{B_{k+n}}{A_n} N_j^{(\frac{1}{n})} \le \frac{B_{k+n}}{A_n} N_j^{(y)},$$

for all j.

Let us now construct the sequences $N^{(\frac{1}{k})}$. In the following, we work with $\mu_j^{(x)} = \frac{M_j^{(x)}}{M_{j-1}^{(x)}}$ and $\nu_j^{(x)} = \frac{N_j^{(x)}}{N_{j-1}^{(x)}}$. Choose $j_0 \in \mathbb{N}$ minimal such that $\mu_j^{(1)} \ge 2$ for all $j \ge j_0$. Set

$$\nu_j^{(1)} := 1 \text{ for } j \le j_0, \quad \nu_j^{(1)} := \frac{1}{2}\mu_j^{(1)} \text{ for } j > j_0,$$

and $N_j^{(1)} := \nu_0^{(1)}\nu_1^{(1)} \cdots \nu_j^{(1)}$, for $j \in \mathbb{N}$. Thus $N^{(1)}$ is clearly log-convex and satisfies (2.8) for $k = 1$.

Now assume we have found sequences $N^{(\frac{1}{l})}$ such that (2.8) and $N^{(\frac{1}{l})} \le N^{(\frac{1}{l-1})}$ is satisfied for $l \le k$. Then we construct $N^{(\frac{1}{k+1})}$ as follows. Choose j_0 such that $\mu_j^{(\frac{1}{k+1})} \ge 2^{k+1}$ for all $j \ge j_0$. By (2.8) and the pointwise order of $\mathfrak{M}$, for $j \ge j_0$,

$$\begin{aligned}
\Big(\frac{1}{2^{k+1}}\Big)^{j-j_0} \mu_{j_0+1}^{(\frac{1}{k+1})} \cdots \mu_j^{(\frac{1}{k+1})} &= \Big(\frac{1}{2^{k+1}}\Big)^{j-j_0} \frac{M_j^{(\frac{1}{k+1})}}{M_{j_0}^{(\frac{1}{k+1})}} \\
&\le \frac{2^{(k+1)j_0}}{M_{j_0}^{(\frac{1}{k+1})}} \Big(\frac{1}{2^k}\Big)^j M_j^{(\frac{1}{k})} \\
&\le \frac{2^{(k+1)j_0}}{A_k M_{j_0}^{(\frac{1}{k+1})}} N_j^{(\frac{1}{k})} \\
&= \frac{2^{(k+1)j_0} N_{j_0}^{(\frac{1}{k})}}{A_k M_{j_0}^{(\frac{1}{k+1})}} \nu_{j_0+1}^{(\frac{1}{k})} \cdots \nu_j^{(\frac{1}{k})} =: B_k \nu_{j_0+1}^{(\frac{1}{k})} \cdots \nu_j^{(\frac{1}{k})}.
\end{aligned}$$

Since $\mu_j^{(\frac{1}{k+1})} \to \infty$ as $j \to \infty$, there exists $j_1 > j_0$ such that

$$\Big(\frac{1}{2^{k+1}}\Big)^{j_1-j_0} \mu_{j_0+1}^{(\frac{1}{k+1})} \cdots \mu_{j_1}^{(\frac{1}{k+1})} \ge B_k.$$

Now set

$$\nu_j^{(\frac{1}{k+1})} = 1 \text{ for } j \le j_1, \quad \nu_j^{(\frac{1}{k+1})} = \frac{1}{2^{k+1}}\mu_j^{(\frac{1}{k+1})} \text{ for } j > j_1.$$

Then (2.8) is immediate. Combining the last two estimates, we also get $N^{(\frac{1}{k+1})} \le N^{(\frac{1}{k})}$. This ends the proof. □

Corollary 2.6. *For any weight matrix $\mathfrak{M}$ there is an equivalent weight matrix $\mathfrak{N}$ such that $\{\| \cdot \|^{N^{(1/k)}}_{[-k,k],1} : k \in \mathbb{N}_{\ge 1}\}$ (resp. $\{\| \cdot \|^{N^{(1/k)}}_1 : k \in \mathbb{N}_{\ge 1}\}$) is a fundamental system of seminorms for $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ (resp. $\Lambda^{(\mathfrak{M})}$).*

2.9. **Strong (ω_1) condition.** Let us write $M \prec_{s\omega_1} N$ if and only if

$$\exists C > 0\ \forall t \ge 0 : \quad \omega_M(2t) \le \omega_N(t) + C, \qquad (s\omega_1)$$

and say that M and N satisfy the *strong (ω_1) condition*.

Then another immediate consequence of Lemma 2.5 is the following.

Corollary 2.7. *Up to equivalence, we can assume that a weight matrix $\mathfrak{M}$ satisfies*

$$\forall x > 0\ \exists y > 0 : \ M^{(x)} \prec_{s\omega_1} M^{(y)}. \qquad (2.10)$$

Remark 2.8. In analogy to (ω_1), one is led to the following condition:

$$\forall x > 0\ \exists y > 0 : \quad \omega_{M^{(x)}}(2t) = O(\omega_{M^{(y)}}(t)) \text{ as } t \to \infty; \tag{2.11}$$

see [35] and [18]. But (2.10) is stronger than (2.11). Cf. the results from [18, Section 3] and the citations therein as well as Remark 5.2.

3. Reduction to the Roumieu case

The goal of this section is to prove the following theorem.

Theorem 3.1. *Let $\mathfrak{M}, \mathfrak{N}$ be weight matrices that are ordered with respect to their quotient sequences, i.e., $\mu^{(x)} \le \mu^{(y)}$ and $\nu^{(x)} \le \nu^{(y)}$ if $x \le y$. Then*

$$\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) \quad \Longleftrightarrow \quad \forall y > 0\ \exists x > 0 : \ M^{(x)} \prec_{SV} N^{(y)}. \tag{SV}$$

We shall see in Lemma 3.3 that both sides of the equivalence (SV) imply $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ and non-quasianalyticity of $\mathfrak{N}$. Recall that $M \prec_{SV} N$ means

$$\exists C, s \in \mathbb{N}_{\ge 1} : \sup_{j \ge 1} \sup_{0 \le i < j} \Big(\frac{M_j}{s^j N_i}\Big)^{\frac{1}{j-i}} \frac{1}{j} \sum_{k=j}^{\infty} \frac{N_{k-1}}{N_k} \le C. \tag{3.1}$$

We will deduce Theorem 3.1 from the following result for Denjoy–Carleman classes of Roumieu type. It is due to [37] under slightly stronger conditions; the version stated here is a special case of [17, Theorem 3.2].

Theorem 3.2. *Let $M \preccurlyeq N$ be weight sequences with $\liminf_{p\to\infty} (m_p)^{1/p} > 0$. Then $\Lambda^{\{M\}} \subseteq j_0^\infty \mathcal{D}^{\{N\}}([-1,1])$ if and only if $M \prec_{SV} N$.*

Here $\mathcal{D}^{(N)}([-1,1])$ (resp. $\mathcal{D}^{\{N\}}([-1,1])$) denotes the space of $\mathcal{E}^{(N)}$ (resp. $\mathcal{E}^{\{N\}}$) functions supported in $[-1,1]$.

3.1. **Auxiliary results.** We show first that both sides of the equivalence (SV) imply $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ and non-quasianalyticity of $\mathfrak{N}$. Similar results hold in the Roumieu case; cf. [24].

Lemma 3.3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be weight matrices. Both sides of the equivalence* (SV) *imply $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ and non-quasianalyticity of $\mathfrak{N}$.*

Proof. By [36, Lemma 3.2], $M^{(x)} \prec_{SV} N^{(y)}$ implies $M^{(x)} \preccurlyeq N^{(y)}$ so that $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ is clearly a consequence of the right-hand side of (SV).

To see that it also follows from the left-hand side, suppose that $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ is violated which means that there is $y > 0$ such that $(M_k^{(x)}/N_k^{(y)})^{1/k}$ is unbounded for all $x > 0$. Thus, for all $j \in \mathbb{N}_{\ge 1}$ we find $k_j \ge j$ such that

$$\Big(\frac{M_{k_j}^{(1/j)}}{N_{k_j}^{(y)}}\Big)^{1/k_j} \ge j.$$

Consider the sequence $a = (a_\ell)$ with $a_{k_j} = (\frac{1}{j})^{k_j} M_{k_j}^{(1/j)}$ and $a_\ell = 0$ otherwise. Then $a \in \Lambda^{(\mathfrak{M})}$, because for given $h, z > 0$ and j so large that $\frac{1}{j} \le \min\{h, z\}$, we have $|a_{k_j}| = (\frac{1}{j})^{k_j} M_{k_j}^{(1/j)} \le h^{k_j} M_{k_j}^{(z)}$. On the other hand, we claim that $a \notin j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$. Indeed, if there is $f \in \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ with $j_0^\infty f = a$, then $N_{k_j}^{(y)} \le a_{k_j} = f^{(k_j)}(0) \le A_{h,z} h^{k_j} N_{k_j}^{(z)}$ for all $h, z > 0$ and j; a contradiction for $z = y$ and $h = 1/2$.

To infer non-quasianalyticity of $\mathfrak{N}$, we distinguish two cases. If $\mathfrak{M}$ is non-quasianalytic so is $\mathfrak{N}$, since we already know that $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$. If $\mathfrak{M}$ is quasianalytic, then the assertion follows either from [30, Theorem 6], which shows that no (proper) quasianalytic class is contained in the image of the Borel map of any other quasi-analytic class, or from the observation that $M^{(x)} \prec_{SV} N^{(y)}$ cannot hold if $N^{(y)}$ is quasianalytic (since then (3.1) is infinite). □

We restate [9, Lemme 16] which is crucial for the reduction.

Lemma 3.4. *Let (α_j) be a sequence of nonnegative real numbers such that $\sum_{j=1}^\infty \alpha_j < \infty$. Let (β_j) and (γ_j) be sequences of positive real numbers such that $\lim_{j\to\infty} \beta_j = 0 = \lim_{j\to\infty} \gamma_j$, and assume that (γ_j) is decreasing. Then there exists an increasing sequence (θ_j) tending to ∞ such that*

(1) *$\theta_j\gamma_j$ is decreasing,*
(2) *$\theta_j\beta_j \to 0$,*
(3) *$\sum_{k=j}^\infty \theta_k\alpha_k \le 8\theta_j \sum_{k=j}^\infty \alpha_k$ for all $j \ge 1$.*

3.2. **Scheme of proof.** The direction $\Rightarrow$ in (SV) follows from a rather direct generalization of the proof of [17, Theorem 4.7] which we sketch in Section 3.4.

The more delicate part is the converse implication. Our aim is to reduce its proof to the Roumieu case, i.e., Theorem 3.2. More specifically, we show that for any given $\lambda \in \Lambda^{(\mathfrak{M})}$ we find weight sequences R, S such that

(i) $\lambda \in \Lambda^{\{R\}}$,
(ii) $R \prec_{SV} S$,
(iii) $\mathcal{E}^{\{S\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$.

Then Theorem 3.2 (together with Lemma 3.3) gives the desired conclusion.

3.3. **Proof of Theorem 3.1($\Leftarrow$).** We construct the sequences R, S in several steps.

Step (I)**.** *Up to equivalence, we may assume that $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the following conditions:*

(a) *For all $\alpha \in \mathbb{N}_{\ge 1}$,*

$$N_j^{(\frac{1}{\alpha})} \ge 2^j N_j^{(\frac{1}{\alpha+1})}, \quad \text{for large enough } j. \tag{3.2}$$

(b) *For all $y > 0$ we have $M^{(y)} \prec_{SV} N^{(y)}$ with $C = s = 1$ in (3.1).*
(c) *For all $y > 0$ we have $M^{(y)} \le N^{(y)}$.*

Proof. (a) follows from Lemma 2.5.

(b), (c) Fix $y > 0$. By assumption, there is $x = x(y)$ such that $M^{(x)} \prec_{SV} N^{(y)}$ and thus $M^{(x)} \preccurlyeq N^{(y)}$. We may assume that $y \mapsto x(y)$ is increasing and $x(y) \le y$ (by the order of the weight matrices). Then $(M^{(x(y))})_{y>0}$ is equivalent to $\mathfrak{M}$. Finally, there exists an increasing function r with $r(0) = 0$ such that the family $\mathfrak{M}'$ with $M_j'^{(y)} := r(y)^j M_j^{(x(y))}$ satisfies the additional assumption of (b) and (c). This matrix is not normalized, but we can use an analogous technique as in the proof of Lemma 2.5 to force this as well.

Note that all constructions yield matrices that are still ordered with respect to their quotients. □

We assume from now on that $\mathfrak{M}$ and $\mathfrak{N}$ satisfy (a),(b), and (c). Fix $\lambda \in \Lambda^{(\mathfrak{M})}$.

Step (II)**.** *There exist a decreasing 0-sequence* (ε_j) *and a strictly increasing sequence of positive integers* (a_α) *such that*

$$|\lambda_j| \le \varepsilon_1 \cdots \varepsilon_j M_j^{(\frac{1}{\alpha+1})}, \quad \text{if } a_\alpha \le j < a_{\alpha+1}. \tag{3.3}$$

Proof. By definition of $\Lambda^{(\mathfrak{M})}$, the sequence $\varepsilon^{(\alpha)} := (\varepsilon_j^{(\alpha)})$ defined by

$$\varepsilon_j^{(\alpha)} := \sup_{k \ge j} \Big(\frac{|\lambda_k|}{M_k^{(\frac{1}{\alpha+1})}} \Big)^{1/k}$$

is decreasing and tending to 0 for each α. By the order of $\mathfrak{M}$, we also have $\varepsilon^{(\alpha)} \le \varepsilon^{(\alpha+1)}$. We define sequences (a_α) and (a'_α) of positive integers as follows:

- Set $a_1 := 1$.
- For given a_α, we choose a'_α and in turn $a_{\alpha+1}$ such that

$$\varepsilon^{(\alpha+1)}_{a_{\alpha+1}} < \varepsilon^{(\alpha)}_{a'_\alpha} \le \frac{1}{1+\alpha} \varepsilon^{(\alpha)}_{a_\alpha}.$$

It is clear that the sequences (a_α) and (a'_α) are strictly increasing and interlacing. Finally, define $\varepsilon = (\varepsilon_j)$ by

$$\varepsilon_j := \varepsilon_j^{(\alpha)} \quad \text{for } a_\alpha \le j \le a'_\alpha, \qquad \varepsilon_j := \varepsilon^{(\alpha)}_{a'_\alpha} \quad \text{for } a'_\alpha < j < a_{\alpha+1}.$$

Then ε is decreasing, tending to 0, and, by construction

$$\Big(\frac{|\lambda_j|}{M_j^{(\frac{1}{\alpha+1})}} \Big)^{1/j} \le \varepsilon_j, \quad \text{if } a_\alpha \le j < a_{\alpha+1},$$

which gives (3.3). □

Step (III)**.** *There exist an increasing sequence* $(\underline{\mu}_j)$ *with* $\underline{\mu}_j/j \to \infty$*, and strictly increasing sequences of integers* (b_α) *and* (C_α) *such that* $\underline{M}_j := \underline{\mu}_0 \underline{\mu}_1 \cdots \underline{\mu}_j$ *satisfies* $\underline{M} \le C_\alpha M^{(1/\alpha)}$*, for all* α*, and*

$$M_j^{(1/\alpha)} \le \underline{M}_j, \quad \text{for all } \alpha \text{ and } j \le b_\alpha. \tag{3.4}$$

Proof. Let (a_α) be the sequence from Step (II). We define sequences of positive integers (b_α) and (b'_α) as follows:

- Set $b'_1 := 1$.
- For given b'_α, we choose b_α such that

$$b_\alpha > \max\{a_\alpha, b'_\alpha\} \quad \text{and} \quad \mu_j^{(\frac{1}{\alpha+1})} \ge \alpha j, \quad \text{for } j \ge b_\alpha. \tag{3.5}$$

- For given b_α, we choose $b'_{\alpha+1} > b_\alpha$ minimal to ensure

$$\mu^{(\frac{1}{\alpha+1})}_{b'_{\alpha+1}} > \mu^{(\frac{1}{\alpha})}_{b_\alpha}.$$

Note that (b_α) and (b'_α) are strictly increasing, interlacing, and $\mu_j^{(\frac{1}{\alpha+1})} \le \mu_{b_\alpha}^{(\frac{1}{\alpha})}$ for all $j \le b'_{\alpha+1} - 1$. Finally, set

$$\underline{\mu}_j := \mu_j^{(\frac{1}{\alpha})}, \quad \text{for } b'_\alpha \le j \le b_\alpha, \qquad \underline{\mu}_j := \mu_{b_\alpha}^{(\frac{1}{\alpha})}, \quad \text{for } b_\alpha < j < b'_{\alpha+1}, \tag{3.6}$$

and $\underline{\mu}_0 := 1$. By construction, $\underline{\mu}_j$ is increasing, $\underline{\mu}_j/j \to \infty$, and (3.4) holds. For fixed α, one has $\underline{\mu}_j \le \mu_j^{(1/\alpha)}$ for all $j \ge b'_\alpha$ which yields $\underline{M} \le C_\alpha M^{(1/\alpha)}$ for some positive constant C_α. Clearly, we may assume that C_α are integers, strictly increasing in α. □

Step (IV)**.** *There exist an increasing sequence $(\underline{\nu}_j)$ tending to ∞, strictly increasing sequences of positive integers (c_α) and (d_α), and an increasing sequence (D_α) tending to ∞ such that $\underline{N}_j := \underline{\nu}_0\underline{\nu}_1\cdots\underline{\nu}_j$ satisfies $\underline{N} \le D_\alpha N^{(1/\alpha)}$, for all α,*

$$N_j^{(1/\alpha)} \le \underline{N}_j, \quad \text{for all } \alpha \text{ and } j \le d_\alpha, \tag{3.7}$$

and there is a constant $D \ge 1$ such that, for all α and $c_\alpha \le j < c_{\alpha+1}$,

$$\sum_{k\ge j} \frac{1}{\underline{\nu}_k} \le 2 \sum_{k \ge j} \frac{1}{\nu_k^{(\frac{1}{\alpha+2})}}, \tag{3.8}$$

$$C_{\alpha+3} N_i^{(\frac{1}{\alpha+3})} \le D 2^{j-i} \underline{N}_i, \quad \text{for all } \ 0 \le i < j, \tag{3.9}$$

where C_α are the constants from Step (III).

Proof. We define sequences (c_α) and (d_α) of positive integers as follows:

- Set $c_1 := 1$ and $d_0 := 0$.
- For given c_α, we choose $d_\alpha \ge C_{\alpha+4} + d_{\alpha-1}$ such that

$$\sum_{k>d_\alpha} \frac{1}{\nu_k^{(\frac{1}{\alpha+1})}} \le \frac{1}{2} \sum_{k>c_\alpha} \frac{1}{\nu_k^{(\frac{1}{\alpha})}}, \tag{3.10}$$

$$N_j^{(\frac{1}{\alpha+2})} \ge 2^j N_j^{(\frac{1}{\alpha+3})}, \quad \text{for } j \ge d_\alpha. \tag{3.11}$$

- For given d_α, we choose $c_{\alpha+1} > d_\alpha$ minimal such that

$$\nu_{c_{\alpha+1}}^{(\frac{1}{\alpha+1})} > \nu_{d_\alpha}^{(\frac{1}{\alpha})}. \tag{3.12}$$

Then (c_α) and (d_α) are strictly increasing and interlacing. Set

$$\underline{\nu}_j := \nu_j^{(\frac{1}{\alpha})}, \quad c_\alpha \le j \le d_\alpha, \qquad \underline{\nu}_j := \nu_{d_\alpha}^{(\frac{1}{\alpha})}, \quad d_\alpha < j < c_{\alpha+1}, \tag{3.13}$$

and $\underline{\nu}_0 := 1$. Completely analogous to Step (III), we may conclude that $\underline{N} \le D_\alpha N^{(1/\alpha)}$ and (3.7).

Let us show (3.9). It clearly suffices to show the claim for $\alpha \ge 3$ (the finitely many remaining values can be controlled by possibly enlarging D). So let $\alpha \ge 3$, $c_\alpha \le j < c_{\alpha+1}$, and $0 \le i < j$. Our construction yields $j \ge c_\alpha > d_{\alpha-1} \ge C_{\alpha+3}$, and therefore

$$C_{\alpha+3} N_0^{(\frac{1}{\alpha+3})} = C_{\alpha+3} \le 2^{d_{\alpha-1}} \le 2^j = 2^j \underline{N}_0$$

which finishes the case $i = 0$. So let $1 \le i < j$. There is $\beta \le \alpha$ such that $c_\beta \le i < c_{\beta+1}$. If $\beta \ge 2$, then $d_{\beta-1} \le i \le d_{\beta+1}$. By (3.7) and (3.11),

$$\frac{\underline{N}_i}{N_i^{(\frac{1}{\alpha+3})}} \ge \frac{N_i^{(\frac{1}{\beta+1})}}{N_i^{(\frac{1}{\alpha+3})}} \ge \frac{N_i^{(\frac{1}{\beta+1})}}{N_i^{(\frac{1}{\beta+2})}} \ge 2^i.$$

Since $2^j \ge d_{\alpha-1} \ge C_{\alpha+3}$, (3.9) follows. It remains to consider $1 \le i \le d_1$, in which case $N_i^{(\frac{1}{\alpha+3})} \le N_i^{(1)} = \underline{N}_i$ is clear and $2^{j-i} \ge 2^{d_{\alpha-1}-d_1} \ge C_{\alpha+3}$. Thus (3.9) is proved.

Let us now prove (3.8). First assume $c_\alpha \le j \le d_\alpha$. Then

$$\sum_{k\ge j}\frac{1}{\underline{\nu}_k} = \sum_{k=j}^{d_\alpha}\frac{1}{\nu_k^{(\frac{1}{\alpha})}} + \sum_{i\ge 1}\Big(\frac{c_{\alpha+i}-d_{\alpha+i-1}-1}{\nu_{d_{\alpha+i-1}}^{(\frac{1}{\alpha+i-1})}} + \sum_{k=c_{\alpha+i}}^{d_{\alpha+i}}\frac{1}{\nu_k^{(\frac{1}{\alpha+i})}}\Big).$$

By the minimal choice of $c_{\alpha+i}$ (see (3.12)),

$$\frac{c_{\alpha+i}-d_{\alpha+i-1}-1}{\nu_{d_{\alpha+i-1}}^{(\frac{1}{\alpha+i-1})}} \le \sum_{k=d_{\alpha+i-1}+1}^{c_{\alpha+i}-1}\frac{1}{\nu_k^{(\frac{1}{\alpha+i})}}, \tag{3.14}$$

whence

$$\begin{aligned}\sum_{k\ge j}\frac{1}{\underline{\nu}_k} &\le \sum_{k=j}^{d_\alpha}\frac{1}{\nu_k^{(\frac{1}{\alpha+1})}} + \sum_{i\ge 1}\sum_{k=d_{\alpha+i-1}+1}^{d_{\alpha+i}}\frac{1}{\nu_k^{(\frac{1}{\alpha+i})}} \\ &= \sum_{k=j}^{d_{\alpha+1}}\frac{1}{\nu_k^{(\frac{1}{\alpha+1})}} + \sum_{i\ge 2}\sum_{k=d_{\alpha+i-1}+1}^{d_{\alpha+i}}\frac{1}{\nu_k^{(\frac{1}{\alpha+i})}}.\end{aligned}$$

Using (3.10), we find

$$\sum_{k\ge d_{\alpha+i-1}+1}\frac{1}{\nu_k^{(\frac{1}{\alpha+i})}} \le \frac{1}{2^{i-1}}\sum_{k\ge d_\alpha+1}\frac{1}{\nu_k^{(\frac{1}{\alpha+1})}}$$

from which it is easy to conclude

$$\sum_{k\ge j}\frac{1}{\underline{\nu}_k} \le 2\sum_{k\ge j}\frac{1}{\nu_k^{(\frac{1}{\alpha+1})}}, \tag{3.15}$$

in particular, (3.8). If $d_\alpha < j < c_{\alpha+1}$, then, using (3.15) for $j = c_{\alpha+1}$, we find

$$\sum_{k\ge j}\frac{1}{\underline{\nu}_k} = \sum_{k=j}^{c_{\alpha+1}-1}\frac{1}{\nu_{d_\alpha}^{(\frac{1}{\alpha})}} + \sum_{k\ge c_{\alpha+1}}\frac{1}{\underline{\nu}_k} \le \sum_{k=j}^{c_{\alpha+1}-1}\frac{1}{\nu_k^{(\frac{1}{\alpha+2})}} + 2\sum_{k\ge c_{\alpha+1}}\frac{1}{\nu_k^{(\frac{1}{\alpha+2})}} \le 2\sum_{k\ge j}\frac{1}{\nu_k^{(\frac{1}{\alpha+2})}}.$$

Thus (3.8) is proved. □

Step (V)**.** *There exist weight sequences R, S such that $(r_j)^{1/j} \to \infty$ and*

(i) $\lambda \in \Lambda^{\{R\}}$,
(ii) $R \prec_{SV} S$,
(iii) $\mathcal{E}^{\{S\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$.

Proof. For the construction of R, we apply Lemma 3.4 to

$$\alpha_j := 0, \quad \beta_j := \max\Big\{\varepsilon_j, \frac{j}{(\underline{M}_j)^{1/j}}\Big\}, \quad \gamma_j := \frac{1}{\underline{\mu}_j}.$$

This yields an increasing sequence (θ_j) tending to ∞ such that $\theta_j\gamma_j$ is decreasing and $\theta_j\beta_j \to 0$. We can assume $\theta_0 = 1$. Since $\theta_j\gamma_j \le \theta_j\beta_j$ (as $(\underline{M}_j)^{1/j} \le \underline{\mu}_j$), also $\theta_j\gamma_j \to 0$. Then

$$R_j := \prod_{i=0}^{j}\frac{\underline{\mu}_j}{\theta_j} = \frac{\underline{M}_j}{\theta_0\theta_1\cdots\theta_j}$$

is a weight sequence (not necessarily normalized). We have $(r_j)^{1/j} \to \infty$, since

$$\frac{j}{(R_j)^{1/j}} = \frac{j(\theta_1 \cdots \theta_j)^{1/j}}{(\underline{M}_j)^{1/j}} \le \frac{j\theta_j}{(\underline{M}_j)^{1/j}} \le \theta_j \beta_j.$$

By (3.3), (3.4), and (3.5),

$$|\lambda_j| \le \varepsilon_1 \cdots \varepsilon_j M_j^{(\frac{1}{\alpha+1})} \le \varepsilon_1 \theta_1 \cdots \varepsilon_j \theta_j R_j.$$

Since $\varepsilon_j \theta_j \le \beta_j \theta_j \to 0$, we get $\lambda \in \Lambda^{\{R\}}$. This finishes the proof of (i).

To obtain S we apply Lemma 3.4 to

$$\alpha'_j = \gamma'_j := \frac{1}{\underline{\nu}_j}, \quad \beta'_j := \max\Big\{\frac{1}{\sqrt{\theta_{\lfloor j/2 \rfloor}}}, \frac{1}{\underline{\nu}_j}\Big\},$$

where $\lfloor j/2 \rfloor$ denotes the integer part of $j/2$. We obtain an increasing sequence (θ'_j) tending to ∞ such that $\theta'_j \gamma'_j$ is decreasing, $\theta'_j \beta'_j \to 0$, and

$$\sum_{k=j}^{\infty} \frac{\theta'_k}{\underline{\nu}_k} \le 8\theta'_j \sum_{k=j}^{\infty} \frac{1}{\underline{\nu}_k}, \quad \text{for all } j. \tag{3.16}$$

Let $\theta'_0 := 1$. Then

$$S_j := A^j \prod_{i=0}^{j} \frac{\underline{\nu}_j}{\theta'_j} = A^j \frac{\underline{N}_j}{\theta'_0 \theta'_1 \cdots \theta'_j}$$

is a weight sequence. Here A is a constant chosen such that $A \ge \max\{1, \frac{\theta'_1}{\underline{\nu}_1}\}$ and

$$\frac{\theta'_i}{\theta_i} \le A, \tag{3.17}$$

$$\frac{\theta'_j}{(\theta_{i+1} \cdots \theta_j)^{\frac{1}{j-i}}} \le A, \quad \text{if } 0 \le i < j. \tag{3.18}$$

That (3.17) and (3.18) are possible is seen as follows. It is easy to see that the choice of β'_j enables (3.17). For $0 \le i < \lfloor j/2 \rfloor$, we have

$$(\theta_{i+1} \cdots \theta_j)^{\frac{1}{j-i}} \ge (\theta_{\lfloor j/2 \rfloor} \cdots \theta_j)^{\frac{1}{j-i}} \ge (\theta_{\lfloor j/2 \rfloor})^{\frac{j-\lfloor j/2 \rfloor}{j-i}} \ge \sqrt{\theta_{\lfloor j/2 \rfloor}},$$

since θ_j is increasing. If $\lfloor j/2 \rfloor \le i \le j-1$, then $(\theta_{i+1} \cdots \theta_j)^{1/(j-i)} \ge \theta_{i+1} \ge \theta_{\lfloor j/2 \rfloor}$. The choice of β'_j shows that the left-hand side of (3.18) is bounded.

Let us now show that $R \prec_{SV} S$. Fix $0 \le i < j$. There is α such that $c_\alpha \le j < c_{\alpha+1}$. We have (with $\sigma_k = S_k/S_{k-1}$)

$$\begin{aligned}
\sum_{k=j}^{\infty} \frac{1}{\sigma_k} &\overset{(3.16)}{\le} \frac{8}{A}\theta'_j \sum_{k=j}^{\infty} \frac{1}{\underline{\nu}_k} \overset{(3.8)}{\le} \frac{16}{A}\theta'_j \sum_{k=j}^{\infty} \frac{1}{\nu_k^{(\frac{1}{\alpha+2})}} \le \frac{16}{A}\theta'_j \sum_{k=j}^{\infty} \frac{1}{\nu_k^{(\frac{1}{\alpha+3})}} \\
&\overset{(\mathrm{I})_b}{\le} \frac{16}{A} j\theta'_j \Big(\frac{N_i^{(\frac{1}{\alpha+3})}}{M_j^{(\frac{1}{\alpha+3})}}\Big)^{\frac{1}{j-i}} \overset{(\mathrm{III})}{\le} \frac{16}{A} j\theta'_j \Big(\frac{C_{\alpha+3} N_i^{(\frac{1}{\alpha+3})}}{\underline{M}_j}\Big)^{\frac{1}{j-i}} \\
&\overset{(3.9)}{\le} \frac{32D}{A} j\theta'_j \Big(\frac{\underline{N}_i}{\underline{M}_j}\Big)^{\frac{1}{j-i}} = \frac{32D}{A} j\theta'_j \Big(\frac{\theta'_1 \cdots \theta'_i S_i}{A^i \theta_1 \cdots \theta_j R_j}\Big)^{\frac{1}{j-i}} \\
&\overset{(3.17)}{\le} \frac{32D}{A} j\theta'_j \Big(\frac{S_i}{\theta_{i+1} \cdots \theta_j R_j}\Big)^{\frac{1}{j-i}} \overset{(3.18)}{\le} 32Dj\Big(\frac{S_i}{R_j}\Big)^{\frac{1}{j-i}},
\end{aligned}$$

which finishes the proof of (ii).

For (iii) observe that, by Step (IV),

$$\frac{S_j}{N_j^{(1/\alpha)}} = \frac{A^j}{\theta'_1 \cdots \theta'_j} \frac{N_j}{N_j^{(1/\alpha)}} \le D_\alpha \frac{A^j}{\theta'_1 \cdots \theta'_j}.$$

We conclude that $S \lhd N^{(1/\alpha)}$ for all α, since $\theta'_j \to \infty$. □

Step (VI)**.** *There exists $f \in \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ such that $j_0^\infty f = \lambda$.*

Proof. By Step (V) and [36, Lemma 3.2], Theorem 3.2 can be applied to R and S. Thus there exists $f \in \mathcal{E}^{\{S\}}(\mathbb{R})$ with $j_0^\infty f = \lambda$. By $(\mathrm{V})_{iii}$, we know that $f \in \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$. □

3.4. **Proof of Theorem 3.1(⇒).** Since $\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ implies that $\mathfrak{N}$ is non-quasianalytic, by Lemma 3.3, and so there exist $\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$-cutoff functions, e.g. by [27, Corollary 3.2 and Theorem 11.16], we have $\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{D}^{(\mathfrak{N})}([-1,1])$.

Now we follow the ideas of [37, Proposition 4.3 and Theorem 4.4]. Let $E_{m,k}$ be $\mathcal{D}^{(\mathfrak{N})}([-1,1])$ endowed with the norm $\|f\|_{m,k} := \|f\|^{N^{(\frac{1}{k})}}_{[-1,1],\frac{1}{m}}$ and let $F_{m,k}$ be its completion. As in [37, Proposition 4.3], one sees that, for all m, k, there exists a continuous linear right inverse $T_{m,k} : \Lambda^{(\mathfrak{M})} \to F_{m,k}$ of $j_0^\infty|_{F_{m,k}}$. Then for every $m \in \mathbb{N}_{\ge 1}$ we can find $s \in \mathbb{N}_{\ge 1}$ and $C > 0$ such that

$$\|T_{m,1}(a)\|_{m,1} \le C\|a\|_s, \quad a \in \Lambda^{(\mathfrak{M})},$$

where $\|\cdot\|_s := \|\cdot\|^{M^{(\frac{1}{s})}}_{\frac{1}{s}}$. The proof of [37, Theorem 4.4], applied to the sequences $M^{(\frac{1}{s})}$ and $N^{(\frac{1}{m})}$, yields $M^{(\frac{1}{s})} \prec_{SV} N^{(\frac{1}{m})}$. This ends the proof of Theorem 3.1.

4. The duals of $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ and $\Lambda^{(\mathfrak{M})}$

In this section, we identify the duals of $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ and $\Lambda^{(\mathfrak{M})}$ with weighted spaces of entire functions. This will be of crucial importance in the proof of the second main result, i.e., Theorem 5.1.

4.1. **Weighted spaces of entire functions.** Let $g : \mathbb{C} \to [0,\infty)$ be a continuous function with $\lim_{|z|\to\infty} g(z) = \infty$. We define the Banach space

$$A_g := \Big\{ f \in \mathcal{H}(\mathbb{C}) : \ \|f\|_{A_g} := \sup_{z\in\mathbb{C}} \frac{|f(z)|}{e^{g(z)}} < \infty \Big\}.$$

Given an increasing sequence of continuous functions $\mathcal{G} = (g_k)_k$ of the mentioned type, we define

$$\mathcal{A}_{\mathcal{G}} := \bigcup_{k\in\mathbb{N}} A_{g_k},$$

and endow it with the natural inductive limit topology. Sometimes we need to work with L^2 weights. To this end, we set

$$A^2_g := \Big\{ f \in \mathcal{H}(\mathbb{C}) : \ \|f\|_{A^2_g} := \Big(\int_{\mathbb{C}} |f(z)|^2 e^{-g(z)}\, d\lambda(z) \Big)^{1/2} < \infty \Big\},$$

where λ denotes the Lebesgue measure in $\mathbb{C}$, and define the corresponding inductive limit $\mathcal{A}^2_{\mathcal{G}}$ analogously.

Lemma 4.1. *Let $g : \mathbb{C} \to [0,\infty)$ be a continuous function with $\lim_{|z|\to\infty} g(z) = \infty$. Then*

$$\|f\|_{A^2_{2g+\log(1+|z|^4)}} \le 3\pi \|f\|_{A_g}, \quad f \in A_g,$$

in particular, $A_g \hookrightarrow A^2_{2g+\log(1+|z|^4)}$.

Let $h : \mathbb{C} \to [0,\infty)$ be another continuous function and assume there exists $K > 0$ such that

$$g(z+u) \le h(z) + K, \quad z,u \in \mathbb{C},\ |u| \le 1. \tag{4.1}$$

Then

$$\|f\|_{A_{\frac{h}{2}}} \le e^K \|f\|_{A^2_g}, \quad f \in A^2_g,$$

in particular, $A^2_g \hookrightarrow A_{\frac{h}{2}}$.

Proof. For $f \in A_g$,

$$\|f\|^2_{A^2_{2g+\log(1+|z|^4)}} = \int_{\mathbb{C}} |f(z)|^2 e^{-2g(z)-\log(1+|z|^4)}\, d\lambda(z) \le \|f\|^2_{A_g} \int_{\mathbb{C}} \frac{d\lambda(z)}{1+|z|^4}$$

implies the first statement. For the second claim, we observe that an entire function f fulfills $f(z) = \frac{1}{\pi}\int_{|u|\le 1} f(z+u)\, d\lambda(u)$ for each $z \in \mathbb{C}$, which follows from Cauchy's integral formula and switching to polar coordinates. Thus,

$$f(z)^2 = \frac{1}{\pi}\int_{|u|\le 1} f(z+u)^2 e^{g(z+u)-g(z+u)}\, d\lambda(u),$$

and therefore

$$|f(z)|^2 \le \frac{1}{\pi} e^{h(z)+K} \int_{\mathbb{C}} |f(z+u)|^2 e^{-g(z+u)}\, d\lambda(u)$$

which gives the desired result. □

Remark 4.2. The proof shows that $\log(1+|z|^4)$ can be replaced by any function ρ such that $e^{-\rho} \in L^1(\mathbb{C})$; of course, the constant has to be adjusted accordingly.

Let us now show that, under some mild constraints on the family $\mathcal{G}$, the corresponding inductive limit is regular.

Proposition 4.3. *Let $\mathcal{G} = (g_k)_k$ be an increasing family of continuous functions $g_k : \mathbb{C} \to [0,\infty)$ tending to infinity as $|z| \to \infty$ such that for all k*

$$\lim_{|z|\to\infty} g_{k+1}(z) - g_k(z) = \infty. \tag{4.2}$$

Then $\mathcal{A}_{\mathcal{G}}$ is regular, complete, ultrabornological, reflexive, and webbed.

Proof. We will show that the connecting mappings are compact. Then the statements follow from [23, Satz 25.19, 25.20, 24.23, and Bemerkung 24.36].

Take $p := g_k$ and $q := g_{k+1}$. We show that the inclusion $A_p \hookrightarrow A_q$ is compact. Let (f_j) be a bounded sequence in A_p, i.e., there exists $D > 0$ such that

$$|f_j(z)| \le D e^{p(z)}, \quad z \in \mathbb{C}, \text{ for all } j.$$

Then this family of entire functions is locally uniformly bounded. Thus, by Montel's theorem, there exists a subsequence (f_{j_k}) that converges uniformly on compact subsets to $f \in \mathcal{H}(\mathbb{C})$. Clearly, $|f(z)| \le D e^{p(z)}$ for all $z \in \mathbb{C}$. Let us show that (f_{j_k}) converges to f in A_q. Let $\varepsilon > 0$. Choose $R > 0$ such that

$$e^{p(z)-q(z)} < \frac{\varepsilon}{2D}, \quad |z| > R,$$

where we use (4.2), and let k_0 be such that

$$|f_{j_k}(z) - f(z)| < \varepsilon, \quad k \geq k_0, \ |z| \leq R.$$

It follows that $\|f_{j_k} - f\|_{A_q} < \varepsilon$ for $k \geq k_0$. □

4.2. **The spaces $\mathcal{A}_{\Omega^+_{\mathfrak{M}}}$ and $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.** Given a weight matrix $\mathfrak{M} = (M^{(x)})_{x>0}$, we consider the sequences of functions

$$\Omega^+_{\mathfrak{M}} := \big(z \mapsto k|\operatorname{Im} z| + \omega_{M^{(1/k)}}(kz)\big)_k,$$
$$\Omega_{\mathfrak{M}} := \big(z \mapsto \omega_{M^{(1/k)}}(kz)\big)_k,$$

and the associated spaces $\mathcal{A}_{\Omega^+_{\mathfrak{M}}}$ and $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.

If the weight matrix is clear from the context, we also write $\omega^{(k)}(z) := \omega_{M^{(1/k)}}(z)$. Note that $\omega^{(k)} \leq \omega^{(l)}$ if $k \leq l$ by the definition of associated weight functions. Let us now see that Proposition 4.3 is applicable to $\mathcal{A}_{\Omega^+_{\mathfrak{M}}}$ and $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.

Let $l > k > 0$. Then (2.1) implies for all $t \geq 0$

$$\omega^{(l)}(lt) - \omega^{(k)}(kt) \geq \omega^{(k)}(lt) - \omega^{(k)}(kt) = \int_{kt}^{lt} \frac{\mu_{M^{(1/k)}}(\lambda)}{\lambda}\, d\lambda \geq \mu_{M^{(1/k)}}(kt)\log(l/k).$$

So for all $l > k > 0$ we get that $\omega^{(l)}(lz) - \omega^{(k)}(kz) \to \infty$ as $|z| \to \infty$ and thus we are able to infer the following corollary from Proposition 4.3.

Corollary 4.4. *$\mathcal{A}_{\Omega^+_{\mathfrak{M}}}$ and $\mathcal{A}_{\Omega_{\mathfrak{M}}}$ are regular, complete, ultrabornological, reflexive, and webbed.*

In what follows, unless mentioned otherwise, we assume that all weight sequences and matrices are normalized.

4.3. **The dual of $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$.** Let us recall a result of [38]. For this we need the Fourier transform of a distribution $T \in \mathcal{E}(\mathbb{R})'$,

$$\widehat{T}(z) := T(x \mapsto e^{ixz}).$$

For a weight sequence M, set

$$\lambda_M(t) := \sum_{j \geq 0} \frac{t^j}{M_j}, \quad t \geq 0.$$

One immediately infers (cf. [19, Proposition 4.5(a) ⇒ (b)])

$$e^{\omega_M(t)} \leq \lambda_M(t) \leq 2e^{\omega_M(2t)}, \quad t \geq 0. \tag{4.3}$$

Theorem 4.5 ([38, Theorem 2.8])**.** *Let M be a weight sequence. Then, for*

$$\mathfrak{E}^{(M)}(\mathbb{R}) := \big\{f \in C^\infty(\mathbb{R}) : \ \forall K \subset\subset \mathbb{R}, \ \forall m \in \mathbb{N}, \ \forall r > 0 : \|f\|^M_{K,m,r} < \infty\big\},$$

where

$$\|f\|^M_{K,m,r} := \sup_{j \in \mathbb{N},\, 0 \leq k \leq m,\, x \in K} \frac{|f^{(j+k)}(x)|}{r^j M_j}, \tag{4.4}$$

endowed with its natural Fréchet topology, we have

$$\mathfrak{E}^{(M)}(\mathbb{R})' \cong \mathcal{A}_{\Gamma_M},$$

where

$$\Gamma_M := \big(z \mapsto k\log(1 + |z|) + \log(\lambda_M(k|z|)) + k|\operatorname{Im} z|\big)_k,$$

and the isomorphism (of locally convex spaces) is realized by the Fourier transform.

For a weight matrix $\mathfrak{M}$, we set

$$\mathfrak{E}^{(\mathfrak{M})}(\mathbb{R}) := \bigcap_{x>0} \mathfrak{E}^{(M^{(x)})}(\mathbb{R}).$$

Note that $\mathfrak{E}^{(M)}(\mathbb{R}) = \mathcal{E}^{(M)}(\mathbb{R})$ (resp., $\mathfrak{E}^{(\mathfrak{M})}(\mathbb{R}) = \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$), if M (resp., $\mathfrak{M}$) is derivation closed.

Proposition 4.6. *Let $\mathfrak{M}$ be derivation closed. Then, as locally convex spaces,*

$$\mathcal{A}_{\Gamma_{\mathfrak{M}}} \cong \mathcal{A}_{\Omega^+_{\mathfrak{M}}},$$

where

$$\Gamma_{\mathfrak{M}} := \big(z \mapsto k \log(1+|z|) + \log(\lambda_{M^{(1/k)}}(k|z|)) + k|\operatorname{Im} z|\big)_k.$$

Proof. Using Lemma 2.4 together with (4.3), one easily checks that the respective inductive systems are equivalent and thus the topologies coincide. □

We are ready to identify the dual of $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$.

Theorem 4.7. *Let $\mathfrak{M}$ be derivation closed. Then $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})' \cong \mathcal{A}_{\Omega^+_{\mathfrak{M}}}$.*

Proof. We have $\mathfrak{E}^{(\mathfrak{M})}(\mathbb{R})' \cong \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})'$, since $\mathfrak{M}$ is derivation closed. First we show

$$\bigcup_{k\in\mathbb{N}_{\ge 1}} \mathfrak{E}^{(M^{(1/k)})}(\mathbb{R})' \cong \mathfrak{E}^{(\mathfrak{M})}(\mathbb{R})',$$

where the union on the left carries the locally convex inductive limit topology and the isomorphism is given by the restriction map which we denote by R. Observe that, for $k \le l$, we have a continuous inclusion $\mathfrak{E}^{(M^{(1/k)})}(\mathbb{R})' \hookrightarrow \mathfrak{E}^{(M^{(1/l)})}(\mathbb{R})'$, and the locally convex inductive limit exists.

The map R is surjective, since we can extend each continuous functional on $\mathfrak{E}^{(\mathfrak{M})}(\mathbb{R})$ to some $\mathfrak{E}^{(M^{(1/k)})}(\mathbb{R})$, by the Hahn–Banach theorem. By Lemma A.1, this extension is unique and thus R is also injective.

Let us now show continuity in both directions. Continuity of R follows from continuity of its restriction to any fixed $\mathfrak{E}^{(M^{(1/k)})}(\mathbb{R})'$ which is clear. Since $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ is a Fréchet–Schwartz space (see the proof of Proposition 5.10), the dual $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})'$ is bornological (cf. [23, Satz 24.23]). Thus it suffices to show that R^{-1} maps bounded sets to bounded sets. Now

$$U_n := \big\{f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) : p_n(f) := \|f\|^{M^{(1/n)}}_{[-n,n],1/n} \le \tfrac{1}{n}\big\}, \quad n \in \mathbb{N}_{\ge 1},$$

is a fundamental system of 0-neighborhoods in $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$, and the polars U_n° form a fundamental system of bounded sets in $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})'$ (cf. [23, Lemma 25.5]). If $T \in U_n^\circ$, then

$$|T(f)| \le n p_n(f), \quad f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}),$$

and, by the Hahn–Banach theorem, T extends to $\mathfrak{E}^{(M^{(1/n)})}(\mathbb{R})$ and satisfies this estimate for all $f \in \mathfrak{E}^{(M^{(1/n)})}(\mathbb{R})$. So $R^{-1}(U_n^\circ)$ is bounded in $\mathfrak{E}^{(M^{(1/n)})}(\mathbb{R})'$, therefore R^{-1} is continuous.

By Theorem 4.5, $\mathfrak{E}^{(M^{(1/k)})}(\mathbb{R})' \cong \mathcal{A}_{\Gamma_{M^{(1/k)}}}$ so that Proposition 4.6 yields the desired result. □

4.4. The dual of $\Lambda^{(\mathfrak{M})}$.

Theorem 4.8. *Let $\mathfrak{M}$ be a weight matrix. Then $(\Lambda^{(\mathfrak{M})})' \cong A_{\Omega_{\mathfrak{M}}}$, and this isomorphism is realized by*

$$\widetilde{S} : (\Lambda^{(\mathfrak{M})})' \to A_{\Omega_{\mathfrak{M}}}, \quad T \mapsto \widetilde{S}(T) := \Big(z \mapsto \sum_{j\geq 0} T(e_j) z^j\Big),$$

with e_j denoting the j-th unit vector.

Since $z \mapsto iz$ is an automorphism of $\mathbb{C}$, the map $S(T) := (z \mapsto \sum_{j\geq 0} T(e_j) i^j z^j)$ realizes the isomorphism $(\Lambda^{(\mathfrak{M})})' \cong A_{\Omega_{\mathfrak{M}}}$, too.

Proof. First observe that for any sequence $b = (b_j)_j$ satisfying

$$\exists A, B, k > 0\ \forall j \in \mathbb{N} : \quad |b_j| \leq A B^j \frac{1}{M_j^{(1/k)}}, \tag{4.5}$$

the map

$$T_b(a) := \sum_{j\geq 0} a_j b_j, \quad a = (a_j)_j \in \Lambda^{(\mathfrak{M})}, \tag{4.6}$$

is an element of $(\Lambda^{(\mathfrak{M})})'$. Actually, every $T \in (\Lambda^{(\mathfrak{M})})'$ has the form (4.6). Indeed,

$$T(a_1 e_1 + \cdots + a_n e_n) = a_1 T(e_1) + \cdots + a_n T(e_n)$$

and since $a_1 e_1 + \cdots + a_n e_n \to a$ in $\Lambda^{(\mathfrak{M})}$ as $n \to \infty$, the statement follows with $b_j = T(e_j)$.

If b satisfies (4.5), then

$$f_b(z) := \widetilde{S}(T_b)(z) = \sum_{j\geq 0} b_j z^j$$

defines an element in $A_{\Omega_{\mathfrak{M}}}$. Indeed,

$$|f_b(z)| \leq A \sum_{j\geq 0} \frac{(B|z|)^j}{M_j^{(1/k)}} \leq A \sup_{k\in\mathbb{N}} \frac{(2B|z|)^k}{M_k^{(1/k)}} \sum_{j\geq 0} \frac{1}{2^j} = 2A e^{\omega^{(k)}(2Bz)}.$$

Conversely, if $f \in A_{\Omega_{\mathfrak{M}}}$, then, by the Cauchy estimates and (2.2),

$$\frac{|f^{(j)}(0)|}{j!} \leq A \inf_{r>0} \frac{e^{\omega^{(k)}(kr)}}{r^j} = A k^j \frac{1}{M_j^{(1/k)}}.$$

So $\widetilde{S} : (\Lambda^{(\mathfrak{M})})' \to A_{\Omega_{\mathfrak{M}}}$ is a linear isomorphism.

Next we show continuity of $\widetilde{S}^{-1}$. To this end, it is enough to show that $\widetilde{S}^{-1}|_{A_k}$ is continuous, where $A_k := A_{\omega^{(k)}(kz)}$. A typical 0-neighborhood in $(\Lambda^{(\mathfrak{M})})'$ is of the form $U = \{T : T(C) \leq r\}$ for some bounded set $C \subseteq \Lambda^{(\mathfrak{M})}$ and $r > 0$. Let $D > 0$ be such that $|a_j| \leq D \frac{1}{(2k)^j} M_j^{(\frac{1}{k})}$ for all $j \in \mathbb{N}$ and all $a = (a_j)_j \in C$. Then $\widetilde{S}^{-1}$ maps the A_k-ball of radius $\frac{r}{2D}$ into U.

For the continuity of $\widetilde{S}$ we observe that $(\Lambda^{(\mathfrak{M})})'$ is ultrabornological, since $\Lambda^{(\mathfrak{M})}$ is a Fréchet–Schwartz space (cf. [23, Satz 24.23]). Indeed, the Fréchet space $\Lambda^{(\mathfrak{M})}$ is nuclear (by the Grothendieck-Pietsch criterion, cf. [23, 28.15]) and hence a Schwartz space (cf. [23, Corollary 28.5]). On the other hand, $A_{\Omega_{\mathfrak{M}}}$ is webbed. So the assertion follows from the open mapping theorem (cf. [23, Satz 24.30]). □

5. Proof by dualization

This section builds on the techniques developed in [6] for Braun–Meise–Taylor classes. Let us first introduce some notation. For a (normalized) non-quasianalytic pre-weight function ω, we set

$$\begin{aligned} P_\omega(x+iy) &:= \frac{|y|}{\pi}\int_{-\infty}^{\infty} \frac{\omega(t)}{(t-x)^2+y^2}\,dt, \quad x,y\in\mathbb{R},\ y\neq 0 \\ P_\omega(x) &:= \omega(x), \quad x\in\mathbb{R}, \end{aligned} \tag{5.1}$$

the *harmonic extension* of ω to the open upper and lower half plane (and subharmonic extension to $\mathbb{C}$). A detailed exposition of its main features is presented in Section 5.2. P_ω is closely related to the concave weight function (cf. [6, Definition 3.1(b)])

$$\kappa_\omega(r) := \int_1^\infty \frac{\omega(rt)}{t^2}dt = r\int_r^\infty \frac{\omega(t)}{t^2}dt. \tag{5.2}$$

In fact,

$$\frac{1}{\pi}\kappa_\omega(r) \le P_\omega(ir) \le \frac{4}{\pi}\kappa_\omega(r), \quad r>0, \tag{5.3}$$

by [6, Lemma 3.3]. It was proved in [6] that, for a non-quasianalytic weight function ω and another weight function σ, the inclusion

$$\Lambda^{(\sigma)} \subseteq j_0^\infty \mathcal{E}^{(\omega)}(\mathbb{R}),$$

holds if and only if

$$\kappa_\omega(r) = O(\sigma(r)) \quad \text{as } r\to\infty. \tag{5.4}$$

Note that (5.4) is also equivalent to $\Lambda^{\{\sigma\}} \subseteq j_0^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$; cf. Theorem 6.4, [6], and [24] as well as [31], [32], and [26] for the more general mixed Whitney extension problem.

When M is a non-quasianalytic weight sequence, we also write P_M instead of P_{ω_M} and κ_M instead of κ_{ω_M}. The crucial mixed condition for weight sequences M,N in this section is $M \prec_L N$ defined by

$$\exists C>0\ \forall s\ge 0:\ P_N(is) \le \omega_M(Cs)+C. \tag{5.5}$$

This condition appears in [20, (2.14')].

Let us now formulate the main result of this section.

Theorem 5.1. *Let $\mathfrak{M},\mathfrak{N}$ be weight matrices, $\mathfrak{N}$ derivation closed. Then*

$$\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) \quad \Longleftrightarrow \quad \forall y>0\ \exists x>0:\ M^{(x)} \prec_L N^{(y)}. \tag{L}$$

Remark 5.2. Similarly to Remark 2.8, note that in (5.5) on the right-hand side the constant C appears in the argument of ω_M (not in front). This subtle difference will become important later on; it stems from the fact that we aim for results for classes defined by (a family of) weight sequences instead of (associated) weight functions. Even though (5.3) implies $\kappa_N \sim P_N$, generally we cannot replace P_N by κ_N in (5.5).

5.1. **Auxiliary results.** We saw in Lemma 3.3 that the left-hand side of (L) entails $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ and non-quasianalyticity of $\mathfrak{N}$. This is also true for the right-hand side.

Lemma 5.3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be weight matrices. The right-hand side of* (L) *entails $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ and non-quasianalyticity of $\mathfrak{N}$.*

Proof. Non-quasianalyticity is immediate, since $\omega_{N^{(y)}}$ is non-quasianalytic if and only if so is $N^{(y)}$; see [19, Lemma 4.1].

Fix $y > 0$. There is $x > 0$ such that $M^{(x)} \prec_L N^{(y)}$, i.e., there is $C > 0$ such that

$$\omega_{N^{(y)}}(s) \le P_{N^{(y)}}(is) \le \omega_{M^{(x)}}(Cs) + C, \quad s > 0,$$

where the first inequality will be justified later in (5.6). By (2.2), we conclude

$$N_k^{(y)} = \sup_{t>0} \frac{t^k}{e^{\omega_{N^{(y)}}(t)}} \ge \sup_{t>0} \frac{t^k}{e^{\omega_{M^{(x)}}(Ct)+C}} = e^{-C} C^{-k} M_k^{(x)}.$$

This shows $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$. □

5.2. **Properties of P_ω, the (sub-)harmonic extension of ω.** In this section we assume, without further mentioning, that ω has the following properties:

- $\omega : [0,\infty) \to [0,\infty)$ is increasing and continuous.
- $\log(t) = O(\omega(t))$ as $t \to \infty$.
- $\varphi_\omega = \omega \circ \exp$ is convex.
- $\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$.

So ω may be any non-quasianalytic pre-weight function, in particular, ω_M for a non-quasianalytic weight sequence M. Recall our convention $\omega(z) := \omega(|z|)$ for $z \in \mathbb{C}$.

The harmonic extension P_ω, defined in (5.1), will play a crucial role as a weight for a weighted space of entire functions. Let us list some obvious properties:

(1) $P_\omega(z) \ge 0$ for all $z \in \mathbb{C}$,
(2) P_ω is symmetric relative to the real and imaginary axis,
(3) $\sigma \le \omega$ implies $P_\sigma \le P_\omega$,
(4) $P_{\sigma+\omega} = P_\sigma + P_\omega$,
(5) $P_{t\mapsto\omega(nt)}(z) = P_\omega(nz)$.

Remark 5.4. For an increasing sequence of functions ω_j converging to ω uniformly on compact subsets of $\mathbb{R}$, we get directly from the definition that $P_{\omega_j} \to P_\omega$ uniformly on compact subsets of $\mathbb{C}$.

The following proposition is well-known.

Proposition 5.5. *P_ω is continuous on $\mathbb{C}$, harmonic in the open upper and lower half plane, and subharmonic on $\mathbb{C}$.*

Proof. That P_ω is continuous on $\mathbb{C}$ and harmonic in the open upper and lower half plane is clear. For the subharmonicity, note first that ω is subharmonic on $\mathbb{C}$; indeed, $\omega(z) = \varphi_\omega(\log|z|)$ and φ_ω is increasing and convex.

Next we show that

$$P_\omega(z) \ge \omega(z), \quad z \in \mathbb{C}. \tag{5.6}$$

In fact, $\xi + i\eta \mapsto P_\omega(e^{\xi+i\eta})$ is harmonic on the horizontal strip $\{0 < \eta < \pi\}$, convex in ξ, and thus concave and symmetric relative to $\frac{\pi}{2}$ in η (cf. the arguments in [8, p. 198]). So for any fixed ξ the map $\eta \mapsto P_\omega(e^{\xi+i\eta})$ takes its minimum at $\eta = 0$ (and $\eta = \pi$). Since P_ω extends ω, this proves (5.6).

Now, for $x \in \mathbb{R}$ and $\delta > 0$,

$$P_\omega(x) = \omega(x) \le \frac{1}{2\pi} \int_0^{2\pi} \omega(x + \delta e^{i\theta})\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} P_\omega(x + \delta e^{i\theta})\, d\theta,$$

which implies that P_ω is subharmonic on $\mathbb{C}$. □

5.3. **Consequences of properties of weight sequences for P_M.** If M, N are weight sequences such that $M \prec_{s\omega_1} N$, then one easily infers the existence of $K \ge 1$ such that

$$\omega_M(z + w) \le \omega_N(z) + \omega_N(w) + K, \quad z, w \in \mathbb{C}. \tag{5.7}$$

Moreover, if M, N are non-quasianalytic, there is a constant $C \ge 1$ such that

$$P_M(z) \le P_N(z) + C, \quad z \in \mathbb{C}. \tag{5.8}$$

Lemma 5.6. *Let M and N be weight sequences such that M is non-quasianalytic and $M \prec_{s\omega_1} N$. Then for all $\varepsilon > 0$ there exists $K > 0$ such that*

$$P_M(x + iy) \le \omega_N(x) + \varepsilon y + K, \quad x + iy \in \mathbb{C}.$$

Proof. Cf. [7, Lemma 2.2] with the obvious changes. □

Having this we prove the following mixed version of [22, Lemma 1.9].

Lemma 5.7. *Let $M^{(i)}$, $1 \le i \le 3$, be non-quasianalytic weight sequences with $M^{(1)} \prec_{s\omega_1} M^{(2)} \prec_{s\omega_1} M^{(3)}$. Then there exists $A > 0$ such that*

$$P_{M^{(1)}}(z + w) \le P_{M^{(3)}}(z) + A, \quad z, w \in \mathbb{C},\ |w| \le 1.$$

Proof. First observe that P_M has the following alternative form

$$P_M(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega_M(|y|t + x)}{t^2 + 1}\, dt, \quad (y \ne 0). \tag{5.9}$$

Now take $w = u + iv \in \mathbb{C}$ with $|w| \le 1$ and $z = x + iy \in \mathbb{C}$ with $y > 1$. Then $\operatorname{Im}(z + w) = y + v > 0$ and, by (5.7),

$$\begin{aligned} P_{M^{(1)}}(z + w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega_{M^{(1)}}((y + v)t + x + u)}{t^2 + 1}\, dt \\ &\le \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega_{M^{(2)}}(yt + x) + \omega_{M^{(2)}}(vt + u) + K}{t^2 + 1}\, dt \\ &\le P_{M^{(2)}}(z) + K \underbrace{\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega_{M^{(2)}}(|t| + 1) + 1}{t^2 + 1}\, dt}_{=:B>1}, \end{aligned}$$

since $K \ge 1$ and $\omega_{M^{(2)}}(vt+u) = \omega_{M^{(2)}}(|vt+u|) \le \omega_{M^{(2)}}(|v||t|+|u|) \le \omega_{M^{(2)}}(|t|+1)$. By (5.8), the choice $A = BK + C'$ establishes the claim for $y > 1$, and by symmetry for $y < -1$. If $|y| \le 1$, then Lemma 5.6 and (5.7) yield constants $K_i \ge 1$ such that

$$P_{M^{(1)}}(z + w) \le \omega_{M^{(2)}}(x + u) + K_1 \le \omega_{M^{(3)}}(x) + K_2,$$

and (5.6) finishes the proof. □

The effect of derivation closedness on P_M is captured in the next lemma.

Lemma 5.8. *Let $l \in \mathbb{N}$ and let $M^{(k)}$, for $1 \le k \le l + 1$, be weight sequences such that $dc(M^{(k)}, M^{(k+1)}) < \infty$ for all k. Then there exists $C > 0$ such that*

$$P_{M^{(l+1)}}(z) + \log(1 + |z|^l) \le P_{M^{(1)}}(Cz) + C, \quad z \in \mathbb{C}.$$

Proof. By Lemma 2.4, we have, for some $C > 0$,

$$\sigma(t) := \omega_{M^{(l+1)}}(t) + \log(1 + t^l) \le \omega_{M^{(1)}}(Ct) + C, \quad t \ge 0.$$

It is easy to see that σ is a pre-weight function. By monotonicity and additivity of P_ω in ω, and (5.6) applied to $\log(1 + t^l)$, we infer

$$P_{M^{(l+1)}}(z) + \log(1 + |z|^l) \le P_\sigma(z) \le P_{M^{(1)}}(Cz) + C$$

and are done. □

5.4. **Scheme of the proof of Theorem 5.1.** The key is the following proposition.

Proposition 5.9 ([6, Corollary 2.3])**.** *Let E, F, G be Fréchet–Schwartz spaces and let $T \in L(E, F)$ and $R \in L(G, F)$ have dense range. Assume that F' endowed with the initial topology with respect to $T^t : F' \to E'$ is bornological. Then the following conditions are equivalent:*

1. *$R(G) \subseteq T(E)$.*
2. *If $B \subseteq F'$ is such that $T^t(B)$ is bounded in E', then $R^t(B)$ is bounded in G'.*

$$E \xrightarrow{T} F \xleftarrow{R} G \qquad\qquad E' \xleftarrow{T^t} F' \xrightarrow{R^t} G'$$

Let $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ be weight matrices, $\mathfrak{N}$ derivation closed and non-quasianalytic. We will apply Proposition 5.9 to

$$\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) \xrightarrow{j_0^\infty} \Lambda^{(\mathfrak{N})} \xleftarrow{\text{incl}} \Lambda^{(\mathfrak{M})} \tag{5.10}$$

By Theorem 4.7 and Theorem 4.8, we have the following commuting diagram

$$\begin{array}{ccccc} \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})' & \xleftarrow{(j_0^\infty)^t} & (\Lambda^{(\mathfrak{N})})' & \xrightarrow{\text{incl}^t} & (\Lambda^{(\mathfrak{M})})' \\ {\scriptstyle\mathcal{F}}\big\downarrow & & \big\downarrow{\scriptstyle S} & & \big\downarrow{\scriptstyle S} \\ \mathcal{A}_{\Omega_{\mathfrak{N}}^+} & \xleftarrow{\text{incl}} & \mathcal{A}_{\Omega_{\mathfrak{N}}} & \xrightarrow{\text{incl}} & A_{\Omega_{\mathfrak{M}}} \end{array} \tag{5.11}$$

where the vertical arrows are isomorphisms. This will lead to

Proposition 5.10. *Let $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ be weight matrices, $\mathfrak{N}$ derivation closed and non-quasianalytic. Then the following conditions are equivalent:*

1. *$\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$.*
2. *If $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ is bounded in $\mathcal{A}_{\Omega_{\mathfrak{N}}^+}$, then B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.*

We will prove Proposition 5.10 in Section 5.5. In Section 5.6 we will make the connection between condition (2) and the right-hand side of (L), and thus complete the proof of Theorem 5.1.

5.5. **Proof of Proposition 5.10.** The proof is based on the following Phragmén–Lindelöf theorem; cf. [3, Theorem 6.5.4].

Theorem 5.11. *Let f be holomorphic in the upper half plane and continuous up to the boundary. Assume that the zeros of f have no finite limit point, and*

$$\liminf_{r\to\infty} \frac{\sup_{|z|=r} \log|f(z)|}{r} < \infty, \qquad \int_{-\infty}^{\infty} \frac{\max(0, \log|f(t)|)}{1+t^2}\, dt < \infty. \tag{5.12}$$

Then (writing $z = x + iy$)

$$\log |f(z)| \le \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t-x)^2 + y^2}\, dt + \frac{2y}{\pi} \lim_{r\to\infty} \frac{1}{r} \int_0^{\pi} \log |f(re^{i\theta})| \sin(\theta)\, d\theta.$$

Every function $f \not\equiv 0$ in $\mathcal{A}_{\Omega_{\mathfrak{N}}}$, or $\mathcal{A}_{\Omega_{\mathfrak{N}}^+}$, satisfies the assumptions of Theorem 5.11. Indeed, since f is entire, its zeros cannot have any finite limit point unless $f \equiv 0$. Since all $N \in \mathfrak{N}$ are non-quasianalytic and so $\omega_N(t) = o(t)$, see Section 2.3, also the conditions (5.12) are clear.

A direct application of this result yields the following corollary.

Corollary 5.12. *Let N be a non-quasianalytic weight sequence. Let $f \in \mathcal{H}(\mathbb{C})$ be such that, for some positive integer k,*

$$\log |f(z)| = o(|z|) \quad \text{as } |z| \to \infty \qquad \text{and} \qquad \log |f(x)| \le \omega_N(kx),\ x \in \mathbb{R}.$$

Then

$$|f(z)| \le e^{P_N(kz)}, \quad z \in \mathbb{C}.$$

Proof of Proposition 5.10. We have to verify the assumptions of Proposition 5.9 with the choices of (5.10).

We saw in the proof of Theorem 4.8 that $\Lambda^{(\mathfrak{N})}$ and $\Lambda^{(\mathfrak{M})}$ are Fréchet–Schwartz spaces. For $\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$, this is a consequence of the compactness of the inclusions $\mathcal{E}^{N^{(\frac{1}{k+1})}, \frac{1}{k+1}}(K) \hookrightarrow \mathcal{E}^{N^{(\frac{1}{k})}, \frac{1}{k}}(K)$ for compact intervals $K \subseteq \mathbb{R}$ (cf. [11, §22, Satz 3.1]); here $\mathcal{E}^{M,a}(K)$ denotes the normed space of all functions $f \in C^\infty(K)$ such that $\|f\|^M_{K,a} < \infty$.

Both maps in (5.10) have dense range, since the finite sequences are dense in $\Lambda^{(\mathfrak{N})}$.

Next we prove that $(\Lambda^{(\mathfrak{N})})'$ endowed with the initial topology with respect to $(j^\infty_0)^t : (\Lambda^{(\mathfrak{N})})' \to \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})'$ is bornological. By (5.11), this amounts to showing that

$$\mathcal{A}_{\Omega_{\mathfrak{N}}} \text{ endowed with the trace topology of } \mathcal{A}_{\Omega^+_{\mathfrak{N}}} \text{ is bornological.} \tag{5.13}$$

To prove (5.13), we set

$$\omega^{(k)}(z) := \omega_{N^{(\frac{1}{k})}}(z), \quad A_k := A_{k|\operatorname{Im} z| + \omega^{(k)}(kz)}.$$

For every $k \in \mathbb{N}_{\ge 1}$, there exists $l > k$ such that $A_k \hookrightarrow A_l$ is compact; cf. Section 4.2. Thus, by [6, Proposition 2.6(2) $\Leftrightarrow$ (3)], (5.13) holds if and only if

$$\bigcup_{k\in\mathbb{N}_{\ge 1}} \overline{Y}_k^{A_k} = \overline{\mathcal{A}_{\Omega_{\mathfrak{N}}}}^{\mathcal{A}_{\Omega^+_{\mathfrak{N}}}}, \qquad \text{where } Y_k := \mathcal{A}_{\Omega_{\mathfrak{N}}} \cap A_k. \tag{5.14}$$

The inclusion $\bigcup_{k\in\mathbb{N}_{\ge 1}} \overline{Y}_k^{A_k} \subseteq \overline{\mathcal{A}_{\Omega_{\mathfrak{N}}}}^{\mathcal{A}_{\Omega^+_{\mathfrak{N}}}}$ is clear and we are left to prove the converse. To this end, we will show the two inclusions

$$\overline{\mathcal{A}_{\Omega_{\mathfrak{N}}}}^{\mathcal{A}_{\Omega^+_{\mathfrak{N}}}} \subseteq \mathcal{A}_{\mathcal{P}_{\mathfrak{N}}} \subseteq \bigcup_{k\in\mathbb{N}_{\ge 1}} \overline{Y_k}^{A_k}, \tag{5.15}$$

where we put $P^{(k)} := P_{\omega^{(k)}}$ and $\mathcal{P}_{\mathfrak{N}} := \big(z \mapsto P^{(k)}(kz)\big)_k$.

Let us start with the first inclusion in (5.15). By (5.6) and Lemma 5.6 (and Corollary 2.7), for each $k \in \mathbb{N}_{\ge 1}$ there exist $l \in \mathbb{N}_{\ge 1}$ and $A > 0$ such that

$$\omega^{(k)}(z) \le P^{(k)}(z) \le \omega^{(l)}(z) + |\operatorname{Im} z| + A, \quad z \in \mathbb{C}. \tag{5.16}$$

This shows $\mathcal{A}_{\Omega_{\mathfrak{N}}} \subseteq \mathcal{A}_{\mathcal{P}_{\mathfrak{N}}} \subseteq \mathcal{A}_{\Omega^+_{\mathfrak{N}}}$. Thus it suffices to show that $\mathcal{A}_{\mathcal{P}_{\mathfrak{N}}}$ is closed in $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$, i.e., $\mathcal{A}_{\mathcal{P}_{\mathfrak{N}}} \cap A_k$ is closed in A_k for all k (cf. [11, §25, Satz 1.2]). So let $f \in \overline{\mathcal{A}_{\mathcal{P}_{\mathfrak{N}}} \cap A_k}^{A_k}$. Then there is a sequence $f_j \in \mathcal{A}_{\mathcal{P}_{\mathfrak{N}}} \cap A_k$ converging to f in A_k. Clearly, there exists $C > 0$ such that

$$|f_j(z)| \le C e^{k|\operatorname{Im} z| + \omega^{(k)}(kz)}, \quad z \in \mathbb{C},\ j \in \mathbb{N}.$$

Since $f_j \in \mathcal{A}_{\mathcal{P}_{\mathfrak{N}}}$, there exist $C_j > 0$ and k_j such that

$$|f_j(z)| \le C_j e^{P^{(k_j)}(k_j z)}, \quad z \in \mathbb{C},$$

consequently, $\log |f_j(z)| = o(|z|)$ as $|z| \to \infty$. Now Corollary 5.12 implies that all f_j are contained and uniformly bounded in some step of $\mathcal{A}_{\mathcal{P}_{\mathfrak{N}}}$. This shows that $f \in \mathcal{A}_{\mathcal{P}_{\mathfrak{N}}}$, and we are done.

It remains to prove the second inclusion in (5.15). To this end, we use [39, Theorem 1] which states the following: *Let $(\varphi_j)_j$ be an increasing sequence of subharmonic functions on $\mathbb{C}$ converging to some subharmonic function φ, and assume that $e^{-\varphi_1}$ is locally integrable on $\mathbb{C}$. Then any function in A^2_φ can be approximated in $L^2_{\varphi(z)+\log(1+|z|^2)}$ by a sequence in $\bigcup_{k \in \mathbb{N}_{\ge 1}} A^2_{\varphi_k(z)+\log(1+|z|^2)}$.*

Let $f \in \mathcal{A}_{\mathcal{P}_{\mathfrak{N}}}$. By Lemma 4.1, there exists $k \in \mathbb{N}_{\ge 1}$ such that $f \in A^2_\varphi$, where $\varphi(z) := 2P^{(k)}(kz) + \log(1 + |z|^4)$. For this k, we introduce the function ω_j by

$$\omega_j(t) := 2\omega^{(k)}(kt),\ |t| \le j, \quad \omega_j(t) := a_j \log |t| + b_j,\ |t| \ge j,$$

where $a_j, b_j \in \mathbb{R}$ are chosen such that ω_j is continuous, increasing, and $t \mapsto \omega_j(e^t)$ is convex. Then $\big(\varphi_j(z) := P_{\omega_j}(z) + \log(1 + |z|^4)\big)_j$ is an increasing sequence of subharmonic functions converging to φ; cf. Remark 5.4. Thus there is a sequence $(f_j)_j$ such that $f_j \in A^2_{\varphi_j(z)+\log(1+|z|^2)}$ and $f_j \to f$ in $A^2_{\varphi(z)+\log(1+|z|^2)}$. By (5.16) and Lemma 2.4, there exist $s \in \mathbb{N}_{\ge 1}$ and $K \ge 1$ such that

$$\begin{aligned} \varphi(z) + \log(1 + |z|^2) &\le 2\omega^{(l)}(kz) + 2k|\operatorname{Im} z| + \log(1 + |z|^4) + \log(1 + |z|^2) + 2A \\ &\le 2\omega^{(s)}(sz) + 2s|\operatorname{Im} z| + K \end{aligned}$$

for all $z \in \mathbb{C}$ so that $A^2_{\varphi(z)+\log(1+|z|^2)} \hookrightarrow A^2_{2\omega^{(s)}(sz)+2s|\operatorname{Im} z|}$. By Lemma 5.7 and (5.16), there exist $t \in \mathbb{N}_{\ge 1}$ and $L \ge 1$ such that

$$2\omega^{(s)}(s(z + u)) + 2s|\operatorname{Im}(z + u)| \le 2\omega^{(t)}(tz) + 2t|\operatorname{Im} z| + L, \quad z, u \in \mathbb{C},\ |u| \le 1.$$

Then Lemma 4.1 implies $A^2_{\varphi(z)+\log(1+|z|^2)} \hookrightarrow A_{\omega^{(t)}(tz)+t|\operatorname{Im} z|} = A_t$. Thus $f_j \to f$ in A_t. Since $P_{\omega_j}(z) = O(\log |z|)$ as $|z| \to \infty$, all f_j are actually polynomials and hence contained in Y_t. So also the second inclusion in (5.15) is proved. □

5.6. **Proof of Theorem 5.1.** Let $\mathfrak{M}(\preccurlyeq)\mathfrak{N}$ be weight matrices, $\mathfrak{N}$ derivation closed and non-quasianalytic; cf. Lemma 5.3. We write $\omega^{(k)}(z) := \omega_{N^{(\frac{1}{k})}}(z)$.

We need the following lemma.

Lemma 5.13. *Let $a_j \ge 1$ be a sequence tending to ∞ and k_0 a positive integer. There exist a sequence of polynomials $(p_j)_j$ and $k \in \mathbb{N}_{\ge k_0}$ such that $p_j(ia_j) = e^{P^{(k_0)}(ia_j)}$ and*

$$|p_j(z)| \le C e^{P^{(k)}(Dz)}, \quad z \in \mathbb{C},\ j \ge 1, \tag{5.17}$$

with uniform constants $C, D > 0$.

Proof. We follow closely the arguments in the proof of [22, Proposition 2.3]. Let $k_1 \le k_2$ be chosen such that $N^{(\frac{1}{k_0})} \prec_{s\omega_1} N^{(\frac{1}{k_1})} \prec_{s\omega_1} N^{(\frac{1}{k_2})}$; cf. Corollary 2.7. We can find positive numbers A_j, B_j, and R_j such that

$$\omega_j(t) := \begin{cases} \omega^{(k_2)}(t), & |t| \le R_j, \\ A_j \log|t| + B_j, & |t| > R_j, \end{cases}$$

is continuous, increasing, $t \mapsto \omega_j(e^t)$ is convex, $\omega_j \le \omega^{(k_2)}$, and

$$\sup_{|z-ia_j|\le 1} |P^{(k_2)}(z) - P_{\omega_j}(z)| \le \frac{1}{j}, \quad \text{for all } j; \tag{5.18}$$

cf. Remark 5.4. Let $\varphi : \mathbb{C} \to [0,1]$ be a C^∞-function with support contained in the unit disc and $\varphi(z) = 1$ for $|z| \le \frac{1}{2}$. As in [22], we set

$$u_j(z) := \Big(1 - \frac{z}{ia_j}\Big)^{-1} e^{P^{(k_0)}(ia_j)}\, \overline{\partial}\varphi(z - ia_j).$$

By Lemma 5.7, there is $A > 0$ such that, for all j,

$$P^{(k_0)}(ia_j) \le P^{(k_2)}(z) + A, \quad |z - ia_j| \le 1. \tag{5.19}$$

Thus, there exists $M \ge 1$ such that for all j we have

$$\begin{aligned} &\int_{\mathbb{C}} |u_j(z)|^2 e^{-2P_{\omega_j}(z) - \log(1+|z|^2)} d\lambda(z) \\ &\quad = \int_{|z-ia_j|\le 1} |u_j(z)|^2 e^{-2P_{\omega_j}(z) - \log(1+|z|^2)} d\lambda(z) \le M. \end{aligned}$$

Since $\overline{\partial} u_j = 0$, we infer from [13, Theorem 4.4.2] the existence of $v_j \in C^\infty(\mathbb{C})$ with $\overline{\partial} v_j = u_j$ such that

$$\int_{\mathbb{C}} |v_j(z)|^2 e^{-2P_{\omega_j}(z) - 3\log(1+|z|^2)}\, d\lambda(z) \le M.$$

Then

$$p_j(z) := \varphi(z - ia_j) e^{P^{(k_0)}(ia_j)} - \Big(1 - \frac{z}{ia_j}\Big) v_j(z)$$

is entire and $p_j(ia_j) = e^{P^{(k_0)}(ia_j)}$.

We claim that there exists $M' > 0$ such that, for all j,

$$\int_{\mathbb{C}} |p_j(z)|^2 e^{-2P_{\omega_j}(z) - 4\log(1+|z|^2)} d\lambda(z) \le M'. \tag{5.20}$$

Indeed, by (5.19) and (5.18),

$$\begin{aligned} &\int_{\mathbb{C}} |\varphi(z - ia_j)|^2 e^{2P^{(k_0)}(ia_j)} e^{-2P_{\omega_j}(z) - 4\log(1+|z|^2)} d\lambda(z) \\ &\le e^{2A} \int_{|z-ia_j|\le 1} e^{2(P^{(k_2)}(z) - P_{\omega_j}(z)) - 4\log(1+|z|^2)} d\lambda(z) \\ &\le e^{2A} \int_{|z-ia_j|\le 1} e^{\frac{2}{j} - 4\log(1+|z|^2)} d\lambda(z), \end{aligned}$$

which is bounded in j. And, since $|1 - \frac{z}{ia_j}|^2 \le 2(1 + |z|^2)$,

$$\int_{\mathbb{C}} \Big|1 - \frac{z}{ia_j}\Big|^2 |v_j(z)|^2 e^{-2P_{\omega_j}(z) - 4\log(1+|z|^2)} d\lambda(z)$$

$$\leq 2 \int_{\mathbb{C}} |v_j(z)|^2 e^{-2P_{\omega_j}(z) - 3\log(1+|z|^2)} d\lambda(z) \leq 2M.$$

This yields (5.20). Since $2P_{\omega_j} + 4\log(1+|z|^2) = O(\log(1+|z|^2))$ as $|z| \to \infty$, we infer that p_j is actually a polynomial.

Let us show (5.17). Recall that $P_{\omega_j} \leq P^{(k_2)}$. By Lemma 5.8, we find $k_3 \in \mathbb{N}_{\geq 1}$ and $K > 0$ such that

$$P^{(k_2)}(z) + \log(1+|z|^2) \leq P^{(k_3)}(k_3 z) + K, \quad z \in \mathbb{C}.$$

Together with (5.20) this yields that $(p_j)_j$ is bounded in $A^2_{2P^{(k_3)}(k_3 z)}$. Take integers $k_4 \leq k_5$ such that $N^{(\frac{1}{k_3})} \prec_{s\omega_1} N^{(\frac{1}{k_4})} \prec_{s\omega_1} N^{(\frac{1}{k_5})}$. By Lemma 5.7, there is $K_1 > 0$ such that

$$2P^{(k_3)}(k_3(z+w)) \leq 2P^{(k_5)}(k_5 z) + K_1, \quad z, w \in \mathbb{C},\ |w| \leq 1.$$

Combining this with Lemma 4.1, we find that $(p_j)_j$ is bounded in $A_{P^{(k_5)}(k_5 z)}$, which shows (5.17) and thus finishes the proof. □

Proof of Theorem 5.1. By Proposition 5.10, we need to show that the following conditions are equivalent:

(1) $\forall y > 0\ \exists x > 0 :\ M^{(x)} \prec_L N^{(y)}$.
(2) If $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ is bounded in $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$, then B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.

(1) ⇒ (2) Let $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ be bounded in $\mathcal{A}^+_{\Omega_{\mathfrak{N}}}$. So there exist $C > 0$ and $k \in \mathbb{N}_{\geq 1}$ such that

$$|f(z)| \leq C e^{k|\operatorname{Im} z| + \omega^{(k)}(kz)}, \quad z \in \mathbb{C},\ f \in B,$$

where we again use the notation $\omega^{(k)}(z) := \omega_{N^{(\frac{1}{k})}}(z)$. Since $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$, we have

$$|f(z)| \leq C_f e^{\omega^{(k_f)}(k_f z)}, \quad z \in \mathbb{C},$$

which yields $\log|f(z)| = o(|z|)$ as $|z| \to \infty$. Then Corollary 5.12 implies

$$|f(z)| \leq C e^{P^{(k)}(kz)}, \quad z \in \mathbb{C},\ f \in B,$$

where $P^{(k)} := P_{\omega^{(k)}}$. By (1) (and the arguments after (5.6)), there exist $l \in \mathbb{N}_{\geq 1}$ and $K > 0$ such that

$$P^{(k)}(kz) \leq P^{(k)}(ik|z|) \leq \omega_{M^{(\frac{1}{l})}}(lz) + K, \quad z \in \mathbb{C}.$$

This shows that B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.

(2) ⇒ (1) We argue by contradiction. Suppose that there exist $k_0 \in \mathbb{N}_{\geq 1}$ and a sequence of real numbers $a_j \geq 1$ tending to infinity such that for all j

$$P^{(k_0)}(ia_j) \geq \omega_{M^{(\frac{1}{j})}}(ja_j) + j. \tag{5.21}$$

By Lemma 5.13, there is a sequence of polynomials $(p_j)_j$ and $k \in \mathbb{N}_{\geq k_0}$ such that $p_j(ia_j) = e^{P^{(k_0)}(ia_j)}$. This gives the desired contradiction: The sequence $(p_j)_j$ is contained in $\mathcal{A}_{\Omega_{\mathfrak{N}}}$, since $\log(|z|) = o(\omega^{(k)}(z))$ as $|z| \to \infty$. By (5.17) and Lemma 5.6 (in view of Corollary 2.7), $(p_j)_j$ is bounded in $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$. But, by (5.21), for every fixed $l \in \mathbb{N}_{\geq 1}$ and $j \geq l$, we have

$$p_j(ia_j) = \exp\left(P^{(k_0)}(ia_j)\right) \geq e^j \exp\left(\omega_{M^{(\frac{1}{j})}}(ja_j)\right) \geq e^j \exp\left(\omega_{M^{(\frac{1}{l})}}(la_j)\right).$$

Thus $(p_j)_j$ is unbounded in every step of the inductive limit defining $\mathcal{A}_{\Omega_{\mathfrak{M}}}$ and hence in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$, the limit being regular due to Corollary 4.4. □

5.7. **Theorem 5.1 without derivation closedness.** If we do not require derivation closedness in Theorem 5.1 for $\mathfrak{N}$, we still can infer some information on the image of the Borel map, but for a (in general) smaller class. Let us be more precise. For a weight matrix $\mathfrak{N} = (N^{(\frac{1}{k})})_{k\in\mathbb{N}_{\ge 1}}$, we may consider the matrix $\mathfrak{N}_{(dc)} = (N^{(\frac{1}{k})}_{(dc)})_{k\in\mathbb{N}_{\ge 1}}$ consisting of "shifted" sequences:

$$(N^{(\frac{1}{k})}_{(dc)})_j := N^{(\frac{1}{k})}_{j-k} \text{ for } j \ge k, \quad (N^{(\frac{1}{k})}_{(dc)})_j := 1 \text{ for } j < k.$$

Then $\mathfrak{N}_{(dc)}$ is easily seen to be derivation closed, and $N^{(\frac{1}{k})}_{(dc)} \le N^{(\frac{1}{k})}$ for all k. (For a single weight sequence N, we may still perform this construction with $N^{(\frac{1}{k})} := N$ for all k which leads to a derivation closed matrix $\mathfrak{N}_{(dc)}$ such that $\mathcal{E}^{(\mathfrak{N}_{(dc)})} \subseteq \mathcal{E}^{(N)}$.)

We get the following version of Theorem 5.1.

Theorem 5.14. *Let* $\mathfrak{M}, \mathfrak{N}$ *be weight matrices. Then*

$\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty \mathcal{E}^{(\mathfrak{N}_{(dc)})}(\mathbb{R})$

$\iff \quad \forall y > 0\ \forall n \in \mathbb{N}\ \exists x, C > 0\ \forall t \ge 0 : \ P_{N^{(y)}}(it) + \log(1 + t^n) \le \omega_{M^{(x)}}(Ct) + C.$

Proof. Observe that

$$\omega_{N^{(\frac{1}{n})}}(t) + \log(1 + t^n) - C \le \omega_{N^{(\frac{1}{n})}_{(dc)}}(t) \le \omega_{N^{(\frac{1}{n})}}(t) + \log(1 + t^n) + C.$$

These inequalities transfer also to the respective harmonic extensions. And this immediately yields the result via an application of Theorem 5.1. □

6. Comparison and conclusions

Let us apply our results to Denjoy–Carleman and Braun–Meise–Taylor classes and compare them with the known classical extension results.

6.1. **Denjoy–Carleman classes.** Taking $\mathfrak{M} = (M)$ and $\mathfrak{N} = (N)$ in Theorem 3.1, we recover the Beurling result of [37] (see also [17]).

Remark 6.1. It might be irritating that $\mathfrak{N} = (N)$ clearly fails (3.2), which was used in the proof of Theorem 3.1 in a crucial way, but Lemma 2.5 associates with N an equivalent weight matrix with the desired properties (which consists of infinitely many different weight sequences that are however all equivalent to N).

Let us now discuss Theorem 5.1 in this special setting. Let M, N be weight sequences, and N in addition derivation closed, such that $(m_k)^{1/k}$ and $(n_k)^{1/k}$ tend to ∞. Let $\mathfrak{M} = (M^{(x)})_{x>0}$, $\mathfrak{N} = (N^{(y)})_{y>0}$ be the weight matrices equivalent to M, N, respectively, provided by Lemma 2.5. Thus, for each $k \in \mathbb{N}_{\ge 1}$ there are constants $A_k, B_k > 0$ such that, for all $t \ge 0$,

$$\omega_M(2^k t) - \log(B_k) \le \omega_{M^{(\frac{1}{k})}}(t) \le \omega_M(2^k t) - \log(A_k),$$
$$\omega_N(2^k t) - \log(B_k) \le \omega_{N^{(\frac{1}{k})}}(t) \le \omega_N(2^k t) - \log(A_k),$$

and consequently,

$$P_N(2^k t) - \log(B_k) \le P_{N^{(\frac{1}{k})}}(t) \le P_N(2^k t) - \log(A_k).$$

This now shows that the right-hand side of (L) reduces to

$$\exists C > 0\ \forall t \ge 0 : \ P_N(it) \le \omega_M(Ct) + C,$$

i.e., $M \prec_L N$.

In this case, Theorem 5.1 specializes to a version of [20, Theorem 2.3] (see also the remarks before Corollary 2.4 in said paper). Incorporating the Roumieu case (see [37] and [17]) and the implications of Theorem 3.1, we conclude

Theorem 6.2. *Let M, N weight sequences, N derivation closed, with $(m_k)^{1/k} \to \infty$ and $(n_k)^{1/k} \to \infty$. Then the following are equivalent:*

(1) $\Lambda^{(M)} \subseteq j_0^\infty \mathcal{E}^{(N)}(\mathbb{R})$.
(2) $\Lambda^{\{M\}} \subseteq j_0^\infty \mathcal{E}^{\{N\}}(\mathbb{R})$.
(3) $M \prec_L N$.
(4) $M \prec_{SV} N$.

If M has moderate growth, then the conditions are also equivalent to

(5) *There is $C > 0$ such that $\kappa_N(s) = O(\omega_M(Cs))$ as $s \to \infty$.*

In fact, that (3) implies (5) follows from (5.3). And, for (5) $\Rightarrow$ (3) note that moderate growth of M is equivalent to

$$\exists H \geq 1\ \forall t \geq 0: \quad 2\omega_M(t) \leq \omega_M(Ht) + H,$$

see [19, Proposition 3.6], which allows to "move constant factors in front of ω_M to its argument".

Finally, we want to make the connection to the condition $M \prec_{\gamma_1} N$ defined by

$$\sup_{j \geq 1} \frac{\mu_j}{j} \sum_{k \geq j} \frac{1}{\nu_k} < +\infty.$$

Note that $M \prec_{\gamma_1} M$ is the condition (γ_1) in [25] and $(M.3)$ in [19]. If M is a weight sequence, then $M \prec_{\gamma_1} M$ and $M \prec_{SV} M$ are equivalent (see [37, Theorem 3.6] and [17, Theorem 5.2]), but in the mixed setting they fall apart, in general. For weight sequences M, N such that $M \leq CN$ for some $C \geq 1$ we have that $M \prec_{\gamma_1} N$ implies $M \prec_{SV} N$. If additionally M has moderate growth, then $M \prec_{\gamma_1} N$ if and only if $M \prec_{SV} N$ since these conditions persist if M (or N) is replaced by an equivalent weight sequence and since M has moderate growth if and only if $\mu_k \leq C_1 (M_k)^{1/k}$ (see e.g. [29, Lemma 2.2]). Thus, under this additional requirement on M, if $M_k \leq CN_k$, then also $\mu_k \leq C_2 \nu_k$. Invoking [31, Lemma 5.7], we see that, under these circumstances, $M \prec_{\gamma_1} N$ implies $\kappa_N(s) = O(\omega_M(s))$ as $s \to \infty$ as well.

So we have the following supplement.

Supplement 6.3. *In the setting of Theorem 6.2, if M has moderate growth and $M \leq CN$, then the conditions (1)–(5) are further equivalent to each of the following conditions:*

(6) $M \prec_{\gamma_1} N$.
(7) $\kappa_N(s) = O(\omega_M(s))$ *as* $s \to \infty$.

Clearly, (5) and (7) are equivalent if ω_M is a weight function, but in general it is just a pre-weight function.

6.2. **Braun–Meise–Taylor classes.** Let $\Sigma = (S^{(x)})_{x>0}$ and $\Omega = (W^{(x)})_{x>0}$ be the matrices associated with the weight functions σ and ω, respectively. By Lemma 2.2, the basic assumptions in Theorem 5.1 hold for the choices $\mathfrak{M} = \Sigma$, $\mathfrak{N} = \Omega$, provided that $\omega(t) = o(t)$ and $\sigma(t) = o(t)$ as $t \to \infty$. By (2.5),

$$\Lambda^{(\sigma)} \cong \Lambda^{(\Sigma)}, \quad \mathcal{E}^{(\Omega)}(\mathbb{R}) \cong \mathcal{E}^{(\omega)}(\mathbb{R}).$$

In this case, the right-hand side of (L), i.e., for all $y > 0$ there is $x > 0$ with $S^{(x)} \prec_L W^{(y)}$ which amounts to

$$P_{W^{(y)}}(is) \le \omega_{S^{(x)}}(Cs) + C, \quad s \ge 0, \tag{6.1}$$

is equivalent to (5.4), i.e., $\kappa_\omega(t) = O(\sigma(t))$ as $t \to \infty$. Indeed, by Lemma 2.2, we have $\omega \sim \omega_{W^{(x)}}, \sigma \sim \omega_{S^{(x)}}$ and thus, by definition, $\kappa_\omega \sim \kappa_{W^{(x)}}$ and $P_\omega \sim P_{W^{(x)}}$ for all $x > 0$, whence one implication follows from (5.3). Conversely, let $y > 0$ be given. Then, for all $x > 0$,

$$\begin{aligned} P_{W^{(y)}}(is) &\overset{(5.3)}{\le} \frac{4}{\pi}\kappa_{W^{(y)}}(s) \overset{(2.6)}{\le} \frac{4}{y\pi}\kappa_\omega(s) \le \frac{4C}{y\pi}(\sigma(s)+1) \\ &\overset{(2.6)}{\le} \frac{4C}{y\pi}(2x\omega_{S^{(x)}}(s) + D_x + 1), \end{aligned}$$

and (6.1) follows if we put $x := \frac{y\pi}{8C}$.

In this case, Theorem 5.1 specializes to [6, Theorem 3.6]. Incorporating also the Roumieu part [6, Theorem 3.7] (see also [24, Section 5]) and the implications of Theorem 3.1, we find

Theorem 6.4. *Let ω, σ be weight functions satisfying $\omega(t) = o(t)$, $\sigma(t) = o(t)$ as $t \to \infty$ and let $\Omega = (W^{(x)})_{x>0}$, $\Sigma = (S^{(x)})_{x>0}$ be the associated weight matrices. Then the following conditions are equivalent:*

1. $\Lambda^{(\sigma)} \subseteq j_0^\infty \mathcal{E}^{(\omega)}(\mathbb{R})$.
2. $\Lambda^{\{\sigma\}} \subseteq j_0^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$.
3. $\kappa_\omega(t) = O(\sigma(t))$ *as* $t \to \infty$.
4. *For all* $y > 0$ *there is* $x > 0$ *such that* $S^{(x)} \prec_{SV} W^{(y)}$.
5. *For all* $y > 0$ *there is* $x > 0$ *such that* $S^{(x)} \prec_L W^{(y)}$.
6. *There are* $x, y > 0$ *such that* $\kappa_{W^{(y)}}(t) = O(\omega_{S^{(x)}}(t))$ *as* $t \to \infty$.

Note that any of the six conditions implies that ω is non-quasianalytic; cf. Lemma 5.3.

Appendix A. Density of entire functions

The following lemma is probably well-known, but we include a proof for the convenience of the reader. The proof closely follows the arguments in [15, Proposition 3.2] and [12, Proposition 3.2].

Lemma A.1. *Let M be a weight sequence with $(m_j)^{1/j} \to \infty$. Let $f \in \mathfrak{E}^{(M)}(\mathbb{R})$, and $I_k := [-k, k]$. Then there exists a sequence of entire functions f_j such that $\|f - f_j\|^M_{I_k,m,r} \to 0$ for all $m \in \mathbb{N}$ and $r > 0$; see (4.4) for the definition of the seminorm.*

Proof. Let $\chi \in C^\infty(\mathbb{R})$ be a function with compact support and $0 \le \chi \le 1$. Suppose that χ is 1 on $[-k-1, k+1]$ and 0 outside $[-k-2, k+2]$. Set $E_j(z) := \sqrt{\frac{j}{\pi}} e^{-jz^2}$ and $f_j := E_j * \chi f$. Then f_j is entire. It is easily seen by induction on p that

$$f_j^{(p)}(x) = E_j * \chi f^{(p)}(x) + \sum_{\nu=1}^{p} E_j^{(p-\nu)} * \chi' f^{(\nu-1)}(x).$$

This yields

$$|(f-f_j)^{(p)}(x)| \leq |f^{(p)}(x) - E_j * \chi f^{(p)}(x)| + \sum_{\nu=1}^{p} |E_j^{(p-\nu)} * \chi' f^{(\nu-1)}(x)|.$$

For $|x| \leq k$, we find (see [12, (3.5)])

$$|f^{(p)}(x) - E_j * \chi f^{(p)}(x)| \leq \frac{C_0}{\sqrt{j}} \Big(\sup_{|\xi| \leq k+1} |f^{(p+1)}(\xi)| + 2 \sup_{|\xi|\leq k+2} |f^{(p)}(\xi)| \Big),$$

for some absolute constant C_0. Moreover, (see [12, (3.6) and (3.7)])

$$\begin{aligned} |E_j^{(p-\nu)} * \chi' f^{(\nu-1)}(x)| &\leq D \sup_{|\xi|\leq k+2} |f^{(\nu-1)}(\xi)| \sup_{|y|\geq 1} |E_j^{(p-\nu)}(y)| \\ &\leq D \sup_{|\xi|\leq k+2} |f^{(\nu-1)}(\xi)| (C_1(p-\nu))^{p-\nu} \sqrt{\frac{j}{\pi}} e^{-C_2 j} \end{aligned}$$

for some constant D depending on χ and absolute constants $C_1, C_2 > 0$. Since $(m_p)^{1/p} \to \infty$, there is a constant C_3 such that $(C_1(p+l))^{p+l} \leq C_3(\frac{r}{2})^p M_p$ for all $p \in \mathbb{N}$ and $0 \leq l \leq m$. Altogether, we find, for $|x| \leq k$, $p \in \mathbb{N}$, and $0 \leq l \leq m$,

$$\begin{aligned} |(f-f_j)^{(p+l)}(x)| &\leq \frac{3C_0}{\sqrt{j}} \|f\|^M_{I_{k+2},m+1,r} r^p M_p \\ &+ D\sqrt{\frac{j}{\pi}} e^{-C_2 j} \|f\|^M_{I_{k+2},m,r/2} \Big(C_3 \sum_{\nu=1}^{p} (\tfrac{r}{2})^{p-1} M_{\nu-1} M_{p-\nu} + C_4 \sum_{\nu=p+1}^{p+l} (\tfrac{r}{2})^p M_p \Big) \\ &\leq \|f\|^M_{I_{k+2},m+1,r} r^p M_p \Big(\frac{3C_0}{\sqrt{j}} + DC_5 \sqrt{\frac{j}{\pi}} e^{-C_2 j} \Big) \end{aligned}$$

for absolute constants C_4, C_5. This implies $\|f - f_j\|^M_{I_k,m,r} \to 0$ as $j \to \infty$. □

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