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Convergence of adaptive BEM driven by functional error estimates

Slides

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TU WIEN ASC INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING

NumPDEs Workgroup on Numerics of PDEs

FWF Österreichischer Wissenschaftsfonds

Outline

Introduction

Model problem

Laplace equation

- $\Delta u^\star = 0 \quad$ in $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$
- $u^\star = g \quad$ on $\Gamma := \partial\Omega$

Fundamental solution

- $G(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } d = 2 \\ \frac{1}{4\pi} \frac{1}{|x|} & \text{if } d = 3 \end{cases}$

Single Layer potential

- $(\widetilde{V}\phi)(x) := (G * \phi)(x) = \int_\Gamma G(x - y)\phi(y)\,\mathrm{d}y$
- $V\phi := (\widetilde{V}\phi)|_\Gamma$

Boundary integral equation (BIE)

Properties of the single layer potential

- $\widetilde{V}: H^{-1/2}(\Gamma) \to H^1(\Omega) \implies V: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$
- $\Delta(\widetilde{V}\phi) = 0$ for all $\phi \in H^{-1/2}(\Gamma)$
- V is elliptic on $H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*$, i.e., $\|\phi\|^2_{H^{-1/2}(\Gamma)} \le C_{\mathrm{ell}} \langle V\phi\,,\,\phi\rangle_{H^{1/2}\times H^{-1/2}}$

BEM ansatz

- $u^\star = \widetilde{V}\phi^\star$ with unknown $\phi^\star \in H^{-1/2}(\Gamma)$
- BIE: Solve $V\phi^\star = g$
- Lax–Milgram: Unique solvability of BIE

- Approximation $\phi_h \approx \phi^\star$ leads to $u_h := \widetilde{V}\phi_h \approx u^\star$

Why BEM?

Advantages

- Mesh on the boundary only = dimension reduction
- $\Delta u_h = 0$ independent of discretization, i.e., approximations are harmonic
- Exterior problems

Challenge

- Solve for $\phi^\star$ instead of $u^\star$
- $\phi^\star$ has no immediate physical relevance

$\Longrightarrow$ Control $\|\nabla(u^\star - u_h)\|_{L^2(\Omega)}$ instead of $\|\phi^\star - \phi_h\|_{H^{-1/2}(\Gamma)}$

Functional estimates

Functional error identity

Theorem

- $u, v \in H^1(\Omega)$ harmonic, i.e. $\Delta u = \Delta v = 0$

$$\Longrightarrow \max_{\substack{\boldsymbol{\tau} \in H(\mathrm{div},\Omega) \\ \mathrm{div}\,\boldsymbol{\tau} = 0}} \left[2\langle (u-v)|_\Gamma \, , \, \boldsymbol{\tau}|_\Gamma \cdot n|_\Gamma \rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2 \right] = \|\nabla(u-v)\|_\Omega^2 = \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = (u-v)|_\Gamma}} \|\nabla w\|_\Omega^2$$

- Lower bound: Variational argument: $\|x\|_{\mathcal{H}}^2 = \max_{y \in \mathcal{H}} \left[2\langle x \, , \, y \rangle_{\mathcal{H}} - \|y\|_{\mathcal{H}}^2 \right]$

- Upper bound: Energy minimization property of harmonic functions (Dirichlet principle)

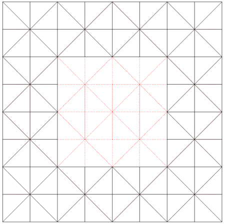
Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Computable bounds

- **Goal:** Find easily computable functions $\boldsymbol{\tau}_h$ and w_h depending on mesh $\mathcal{T}_h^\Gamma$ s.t.

$$2\langle g - u_h|_\Gamma \,,\, \boldsymbol{\tau}_h|_\Gamma \cdot n|_\Gamma \rangle_\Gamma - \|\boldsymbol{\tau}_h\|_\Omega^2 \leq \|\nabla(u^\star - u_h)\|_\Omega^2 \leq \|\nabla w_h\|_\Omega^2$$

- **Idea:** Solve auxiliary problems on strip domain ω_h along the boundary
  - Mesh $\mathcal{T}_h^\omega$ on ω_h
  - Number of dofs in ω_h should be comparable to number of dofs on Γ
  - ω_h varies within adaptive algorithm



Computable lower bound

Auxiliary problem

- Raviart-Thomas space $\mathcal{RT}^q_*(\mathcal{T}_h^\omega) \subset \{\boldsymbol{\sigma} \in H(\operatorname{div}, \omega_h) \mid \boldsymbol{\sigma} \cdot n = 0 \text{ on } \partial\omega_h \setminus \Gamma\}$
- Piecewise polynomials $\mathcal{P}^q(\mathcal{T}_h^\omega) \subset L^2(\omega_h)$

- Compute FEM-solution $(\boldsymbol{\tau}_h, p_h) \in \mathcal{RT}^q_*(\mathcal{T}_h^\omega) \times \mathcal{P}^q(\mathcal{T}_h^\omega)$ such that

$$\begin{aligned} \langle \boldsymbol{\tau}_h \,,\, \boldsymbol{\sigma}_h \rangle_{\omega_h} + \langle \operatorname{div} \boldsymbol{\sigma}_h \,,\, p_h \rangle_{\omega_h} &= \langle g - u_h|_\Gamma \,,\, \boldsymbol{\sigma}_h|_\Gamma \cdot n|_\Gamma \rangle_\Gamma & \forall \boldsymbol{\sigma}_h \in \mathcal{RT}^q_*(\mathcal{T}_h^\omega) \\ \langle \operatorname{div} \boldsymbol{\tau}_h \,,\, q_h \rangle_{\omega_h} &= 0 & \forall q_h \in \mathcal{P}^q(\mathcal{T}_h^\omega) \end{aligned}$$

- Note: $2\langle g - u_h|_\Gamma \,,\, \boldsymbol{\tau}_h|_\Gamma \cdot n|_\Gamma \rangle_\Gamma - \|\boldsymbol{\tau}_h\|_\Omega^2 = \|\boldsymbol{\tau}_h\|_{\omega_h}^2$ and $\operatorname{div} \boldsymbol{\tau}_h = 0$

$\Longrightarrow \quad \|\boldsymbol{\tau}_h\|_\Omega \leq \|\nabla(u^\star - u_h)\|_\Omega$

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Computable upper bound I

- Problem: Discrete functions w_h cannot satisfy $w_h|_\Gamma = g - u_h|_\Gamma$ exactly
- Solution: Allow for data oscillations
  - Employ $H^1(\Gamma)$-stable projection $J_h\colon H^1(\Gamma) \mapsto \mathcal{S}^q(\mathcal{T}_h^\omega|_\Gamma)$
  - Assume additional regularity $g \in H^1(\Gamma)$:

$$C_{\mathsf{osc}}^{-1} \|(1-J_h)(g-u_h|_\Gamma)\|_{H^{1/2}(\Gamma)} \leq \|h^{1/2}\nabla_\Gamma (1-J_h)(g-u_h|_\Gamma)\|_{L^2(\Gamma)} =: \mathrm{osc}_h$$

$$\Longrightarrow \quad \min_{\substack{w\in H^1(\Omega)\\ w|_\Gamma = g-u_h|_\Gamma}} \|\nabla w\|_\Omega \leq \min_{\substack{w\in H^1(\Omega)\\ w|_\Gamma = J_h(g-u_h|_\Gamma)}} \|\nabla w\|_\Omega + C_{\mathsf{osc}}\mathrm{osc}_h$$

Aurada, Feischl, Führer, Karkulik, Praetorius: *Applied Numerical Mathematics*, 95 (2015)

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Computable upper bound II

Auxiliary problem

- FEM space $\mathcal{S}^q_*(\mathcal{T}^\omega_h) \subset \{w \in H^1(\omega_h) \mid w = 0 \text{ on } \partial\omega_h \setminus \Gamma\}$
- Compute FEM-solution $w_h \in \mathcal{S}^q_*(\mathcal{T}^\omega_h)$ such that

$$\begin{aligned} \langle \nabla w_h \,,\, \nabla v_h \rangle_{\omega_h} &= 0 && \forall v_h \in \mathcal{S}^q_0(\mathcal{T}^\omega_h) \\ w_h|_\Gamma &= J_h(g - u_h|_\Gamma) \end{aligned}$$

- Set $\eta_h := \|\nabla w_h\|_\Omega$

$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \le \eta_h + C_{\mathsf{osc}}\mathrm{osc}_h$

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Galerkin BEM and adaptive algorithm

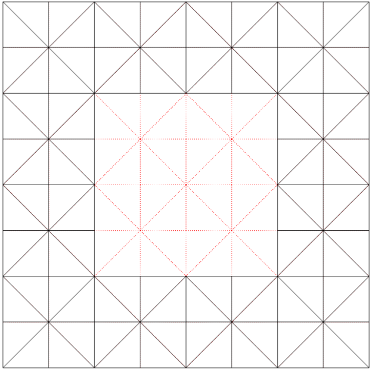
Galerkin discretization

- Mesh $\mathcal{T}_h$ of Ω
- Boundary mesh $\mathcal{T}_h^\Gamma$ induced by $\mathcal{T}_h$
- Galerkin BEM : Find $\phi_h^\star \in \mathcal{P}^p(\mathcal{T}_h^\Gamma)$ s.t.

$$\langle V\phi_h^\star\,,\,\psi_h\rangle_\Gamma = \langle g\,,\,\psi_h\rangle_\Gamma \qquad \forall \psi_h \in \mathcal{P}^p(\mathcal{T}_h^\Gamma)$$

Additional assumptions

- Scott-Zhang interpolation operator J_h
- Strip domain ω_h : k-patch of Γ w.r.t. $\mathcal{T}_h$



Adaptive algorithm

- Input: Initial mesh $\mathcal{T}_0$, marking parameter $\theta \in (0,1]$, tolerance $\varepsilon > 0$

Iterate until tolerance is met

1. Extract boundary mesh $\mathcal{T}_\ell^\Gamma$, strip domain ω_ℓ and strip mesh $\mathcal{T}_\ell^\omega$ from $\mathcal{T}_\ell$
2. Compute $\phi_\ell^\star$ by Galerkin BEM
3. Compute discretized residual $J_\ell(g - \widetilde{V}\phi_\ell^\star)$ and data oscillations $\mathrm{osc}_\ell(\partial T \cap \Gamma)$
4. Compute FEM-solution $w_\ell \in \mathcal{S}_*^q(\mathcal{T}_\ell^\omega)$
5. Compute error indicators $\eta_\ell(T)$ and $\mathrm{osc}_\ell(\partial T \cap \Gamma)$ for all $T \in \mathcal{T}_\ell$
6. Choose minimal $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ s.t. $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$
7. Refine at least all elements in $\mathcal{M}_\ell$

Estimator convergence

Theorem

There holds

$$\|\nabla(u^\star - u_\ell^\star)\|_\Omega \leq \eta_\ell + C_{\mathsf{osc}}\mathrm{osc}_\ell \xrightarrow{\ell\to\infty} 0.$$

- A priori convergence of Galerkin schemes: $\phi_\ell^\star \to \phi_\infty$ in $H^{-1/2}(\Gamma)$
  $\implies$ $u_\ell^\star \to u_\infty$ in $H^1(\Omega)$
- Challenge: Show that $\phi_\infty = \phi^\star$ and $u_\infty = u^\star$
- $\mathrm{osc}_\ell \to 0$
- Use elliptic regularity to show that $\eta_\ell \to 0$ (since ω_ℓ varies)

Numerical experiments

Setting

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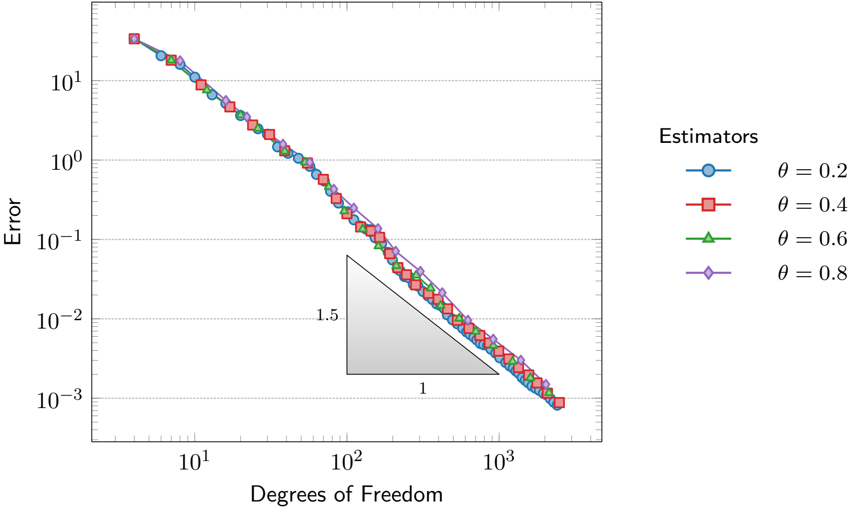
- Lowest order BEM: $p = 0$
- Lowest order auxiliary FEM-problems: Consider $\mathcal{RT}^0(\mathcal{T}_h^\omega)$, $\mathcal{P}^0(\mathcal{T}_h^\omega)$ and $\mathcal{S}^1(\mathcal{T}_h^\omega)$
- Strip domain: 2-patch of Γ, i.e.,

$$\omega_h := \{T \in \mathcal{T}_h \,|\, \text{there exists } T' \in \mathcal{T}_h \text{ s.t. } T \cap T' \neq \emptyset \neq T' \cap \Gamma\}$$

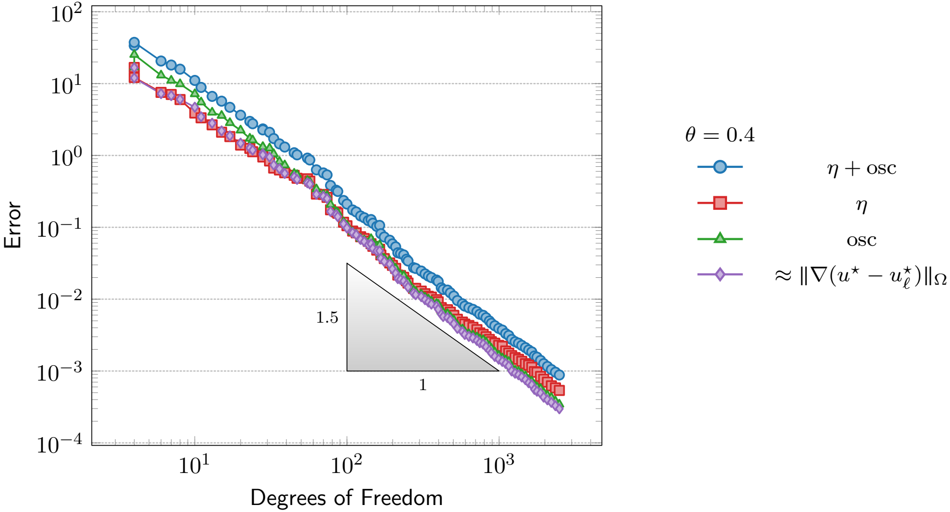
Square Domain

- $\Omega = (0, 1/2)^2$
- $u^\star(x, y) = \sinh(2\pi x)\cos(2\pi y)$

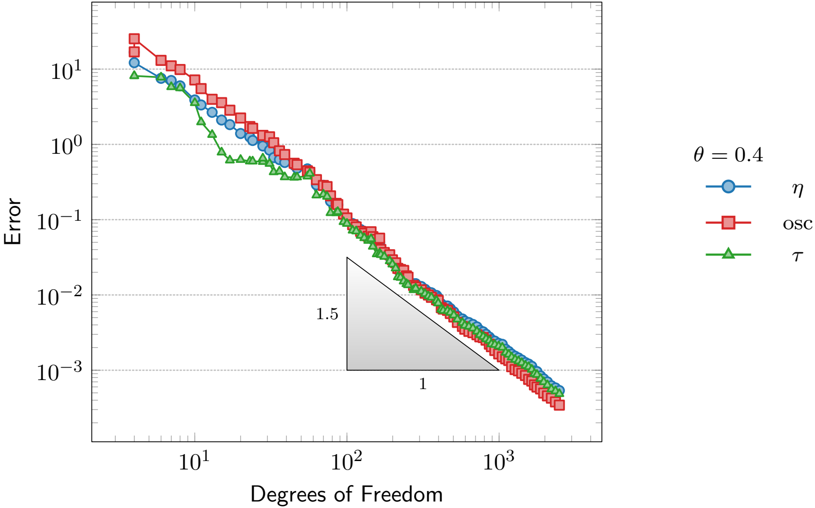
Square Domain



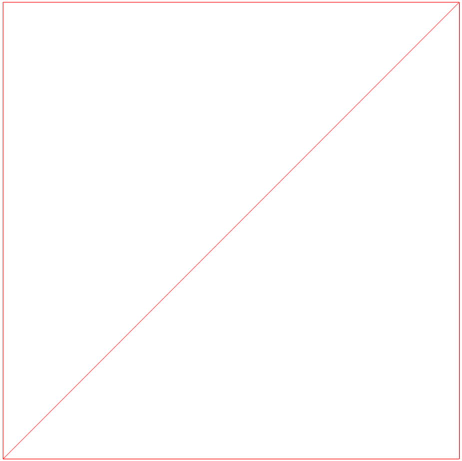
Square Domain



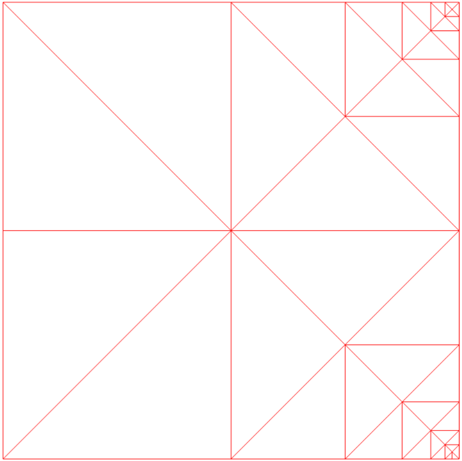
Square Domain



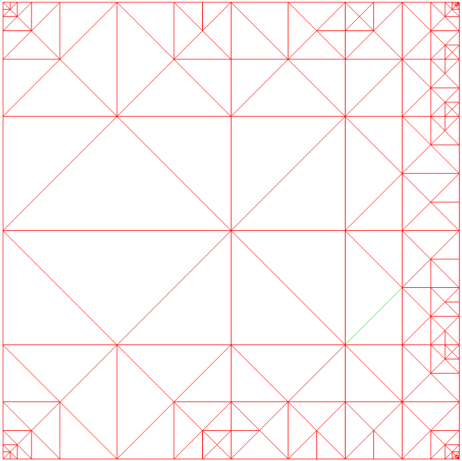
Boundary Layer, $\ell = 0$



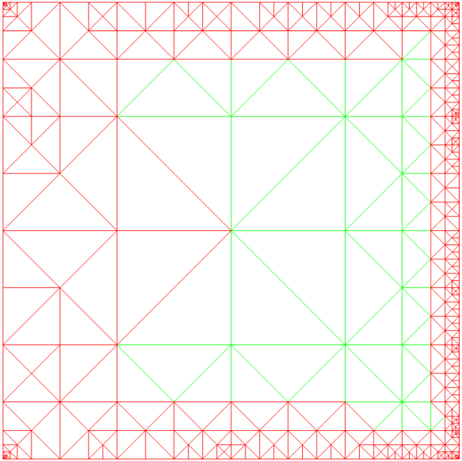
Boundary Layer, $\ell = 20$



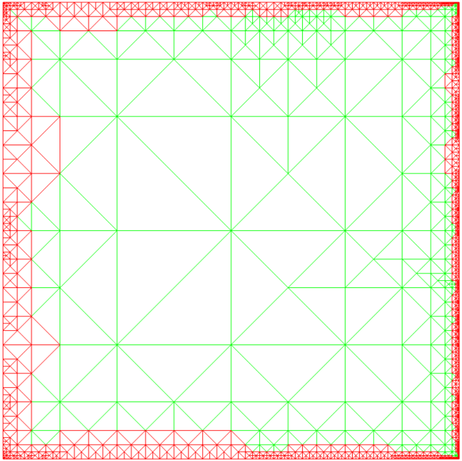
Boundary Layer, $\ell = 40$



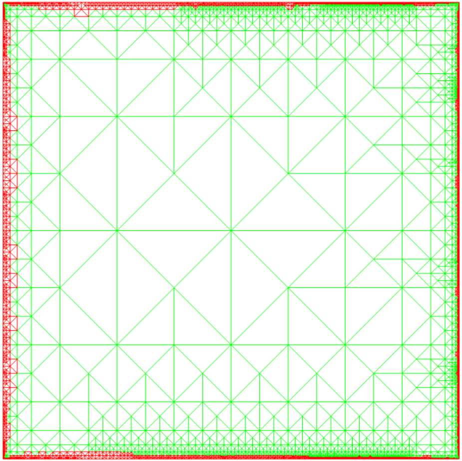
Boundary Layer, $\ell = 60$



Boundary Layer, $\ell = 100$



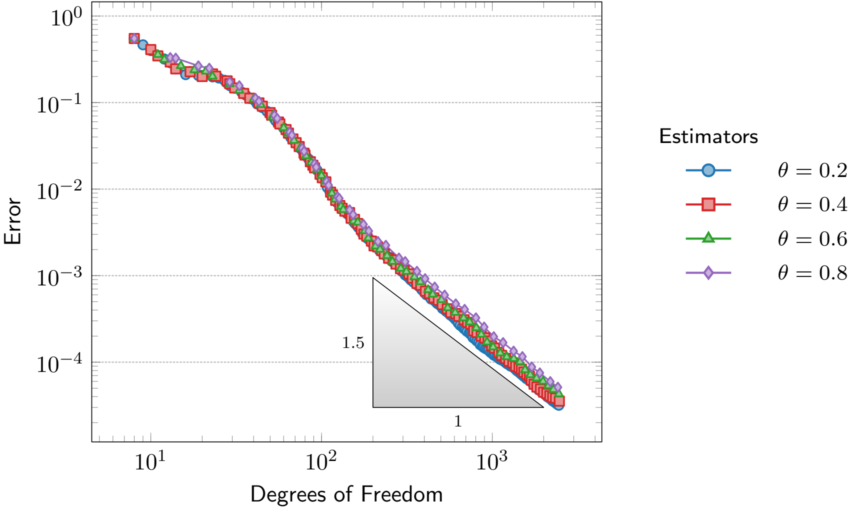
Boundary Layer, $\ell = 150$



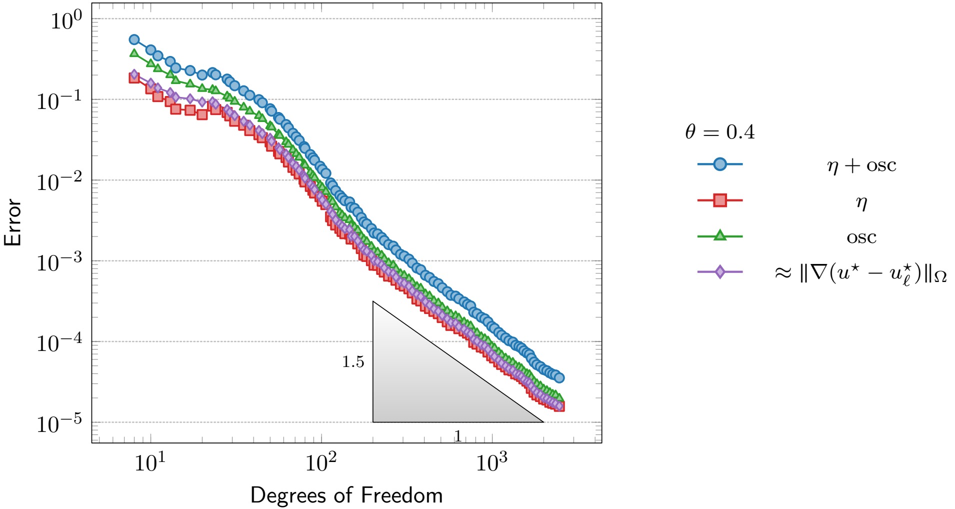
L-shaped Domain

- Ω is the rotated and shrinked L-shaped domain
- Reentrant corner at $(0,0)$
- $u^\star(r,\theta) = r^{2/3}\cos(2\theta/3)$ in polar coordinates

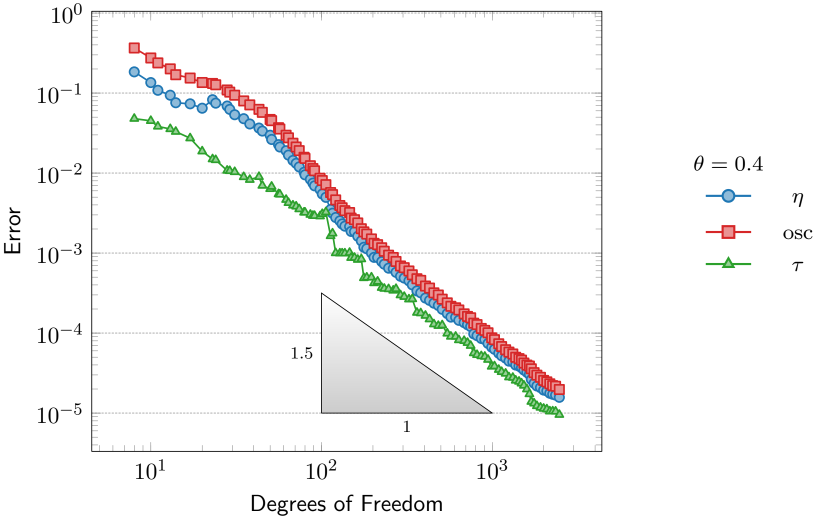
L-shaped Domain



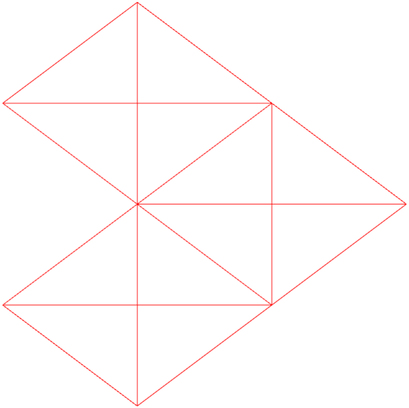
L-shaped Domain



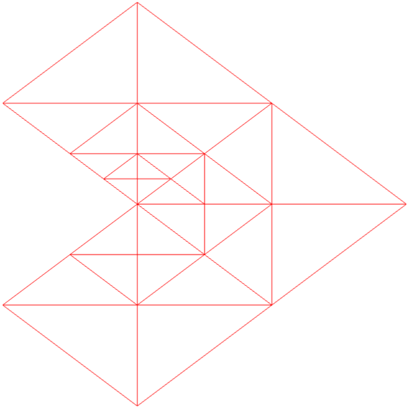
L-shaped Domain



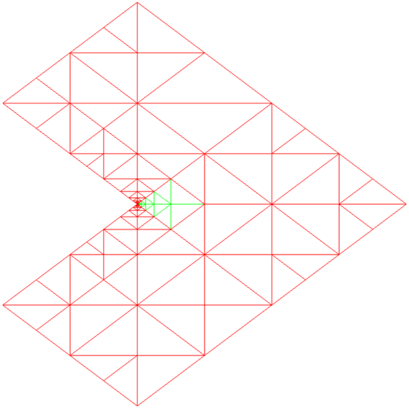
Boundary Layer, $\ell = 0$



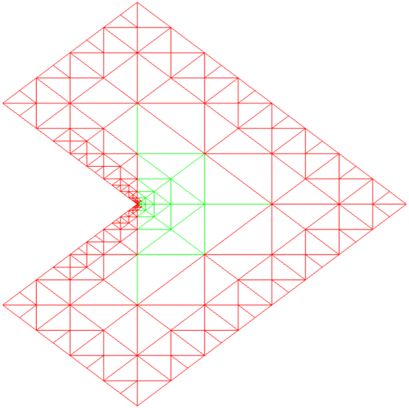
Boundary Layer, $\ell = 5$



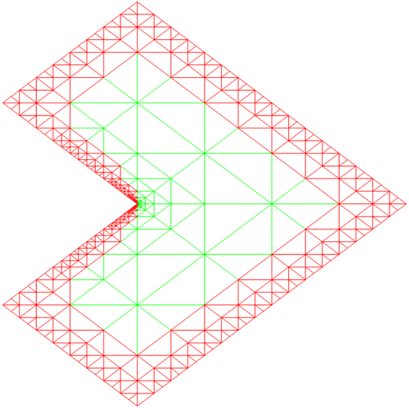
Boundary Layer, $\ell = 20$



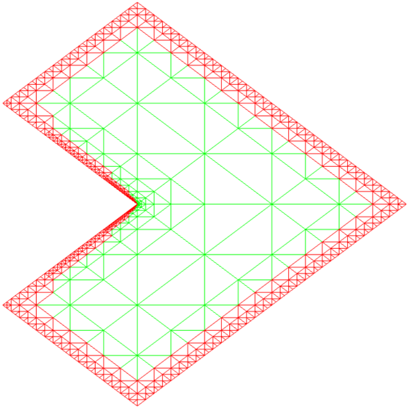
Boundary Layer, $\ell = 30$



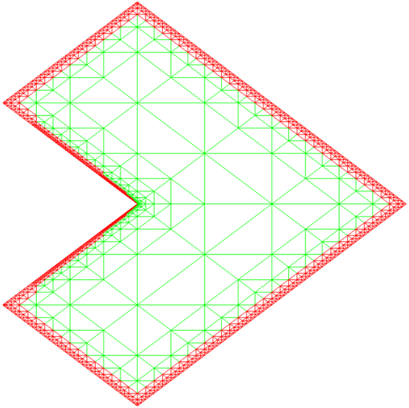
Boundary Layer, $\ell = 40$



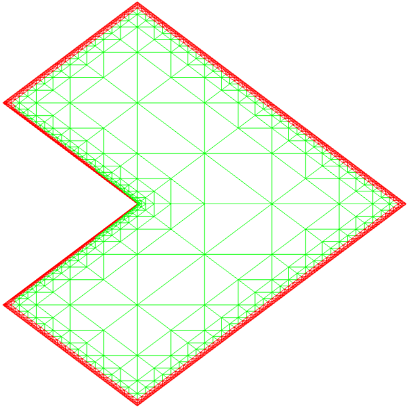
Boundary Layer, $\ell = 50$



Boundary Layer, $\ell = 60$



Boundary Layer, $\ell = 70$



Summary

- Control $\|\nabla(u^\star - u_h)\|_\Omega$ instead of $\|\phi^\star - \phi_h\|_{H^{-1/2}(\Gamma)}$
- Functional error estimates for BEM with known constants 1: $\tau_h \le \|\nabla(u^\star - u_h)\|_\Omega \le \eta_h$
- Independent of approximation $\phi_h \approx \phi^\star$
- Adaptive algorithm guarantees convergence

Outlook

- Extensions
  - Exterior Domains
  - Direct Ansatz
  - Non-vanishing volume term, Poisson problem
- Goals
  - Iterative solver
  - Matrix compression
  - $3D$-experiments

Thank you for your attention!

Kurz, Pauly, Praetorius, Repin, Sebastian
Functional a posteriori error estimates for boundary element methods
Numerische Mathematik, 147 (2021)

Freiszlinger, Pauly, Praetorius
Convergence of adaptive boundary element methods driven by functional a posteriori error estimates
(2024+)

Slides

Strip domain

- Difficulty: Strip domain ω_ℓ varies with each step of algorithm
  - Usual tools for convergence analysis do not apply
  - Not clear if $(w_\ell)_{\ell\in\mathbb{N}}$ bounded in $H^1(\Omega)$
- Solution: Connect the norms on ω_ℓ and Γ

Theorem

For $1 < r < \infty$ *and* $g \in W^{1/r',r}(\Gamma)$, *there exists* $v \in W^{1,r}(\omega_\ell)$ *s.t.*

$$\begin{aligned} v|_\Gamma &= g \\ v|_{\partial\omega_\ell\setminus\Gamma} &= 0 \\ \|v\|_{L^r(\omega_\ell)} &\lesssim \|g\|_{L^r(\Gamma)} \\ \|\nabla v\|_{L^r(\omega_\ell)} &\lesssim \|h_\ell^{-1/r'} g\|_{L^r(\Gamma)} + |g|_{W^{1/r',r}(\Gamma)} \end{aligned}$$

Elliptic regularity

- Goal: Show that $(w_\ell)_{\ell\in\mathbb{N}}$ is bounded in $W^{1,r}(\Omega)$ for some $r > 2$
- Immediate consequence: $\|\nabla w_\ell\|^2_{L^2(T)} \le |T|^{1-2/r}\|\nabla w_\ell\|^2_{L^r(T)} \xrightarrow{|T|\to 0} 0$

Theorem

There exists $r_0 > 2$ s.t. for all $r \in [2, r_0)$ there is $-1/2 \le s_r < 1/2$ and $p_r \ge 1$ s.t.

$$\|\nabla w_\ell\|_{L^r(\omega_\ell)} \lesssim \|\phi^\star - \phi^\star_\ell\|^{p_r}_{H^{s_r}(\Gamma)} < +\infty.$$

- For $r = 2$, one can choose $s_r = -1/2$, $p_r = 1$

$\Longrightarrow$ $\|\nabla w_\ell\|_{L^2(\omega_\ell)} \lesssim \|\phi^\star - \phi^\star_\ell\|_{H^{1/2}(\Gamma)}$ (= Efficiency)