Genericity Iterations and $L(\mathbb{R}) \models AD$

Master's Thesis

Lena Wallner, October 2023

supervised by Assoc. Prof. Dr. Sandra Müller

TECHNISCHE UNIVERSITÄT WIEN

universität wien

Overview

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Genericity Iteration

Theorem (Woodin)

Let M be a countable model of ZFC and $a \in \mathbb{R}$. Assume that

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will see in the proof

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Then there is

will see in the proof

- a countable iteration $i : M \to M^*$ and

this will be an elementary embedding

- $h \subseteq Col(\omega, i(\delta))$ generic over M^*

such that $a \in M^*[h]$.

Genericity Iteration: Sketch of Proof

- "δ is Woodin in M" is witnessed by a set of extenders in M
- build an iteration tree using those extenders

Iteration Trees

Stage 0

$E \in M$

Iteration Trees

Stage 1

$Ult(M,E)$

$E \in M$

Iteration Trees

Stage 1

$E_1 \in M_1 := Ult(M, E)$

$E \in M$

Iteration Trees

Stage 2

$$
\begin{tikzcd}
E_1 \in M_1 := \mathrm{Ult}(M,E) & & \mathrm{Ult}(M,E_1) \\
& E \in M \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Iteration Trees

stage 2

$Ult(M_1, E_1)$

$\uparrow$

$E_1 \in M_1 := Ult(M, E)$

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Iteration Trees

Stage 2

$$E_2 \in M_2 := \mathrm{Ult}(M_1, E_1)$$

$$E_1 \in M_1 := \mathrm{Ult}(M, E)$$

$$E \in M$$

Iteration Trees

Stage 3

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\begin{tikzcd}[column sep=small]
E_2 \in M_2 := Ult(M_1, E_1) & & & \\
& E_1 \in M_1 := Ult(M, E) \arrow[ul] & & Ult(M, E_2) =: M_3 \ni E_3 \\
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Iteration Trees

Stage ω

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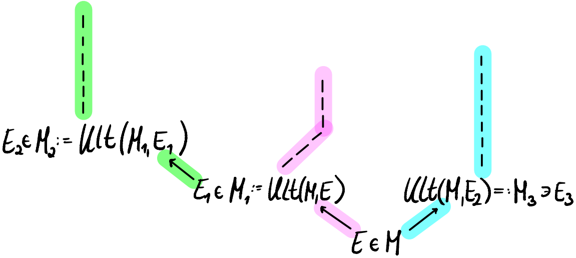
Iteration Trees

Stage ω

find cofinal branch



Iteration Trees

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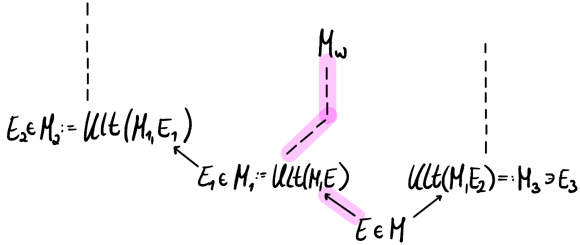
M_ω is the direct limit along this branch



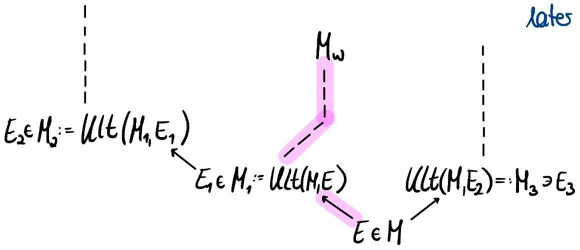
Iteration Trees

Stage ω

find cofinal branch

M_ω is the direct limit along this branch

(ω_1+1)-iterability assures that we find a branch with wellfounded limit and such that there are also branches with wellfounded limits at every later limit step



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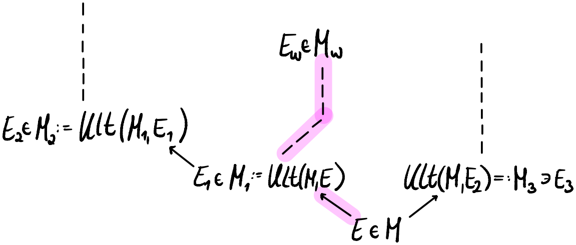
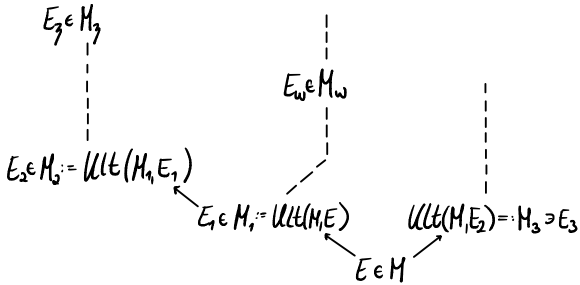


Stage $\xi+1$

find $\alpha \leq \xi$



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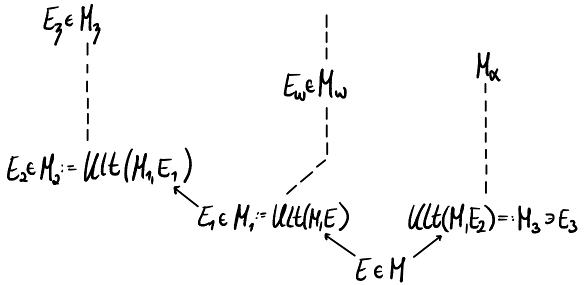
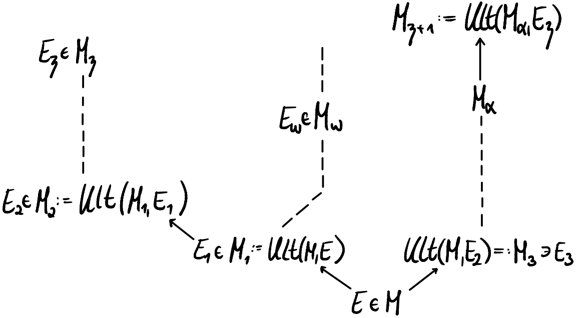


Iteration Trees

Stage $\zeta+1$

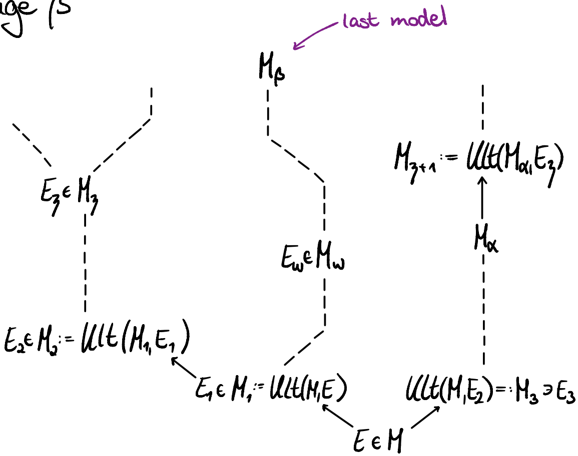
find $\alpha \leq \zeta$

apply E_ζ to M_α



Iteration Trees

Construction stops at some countable stage β



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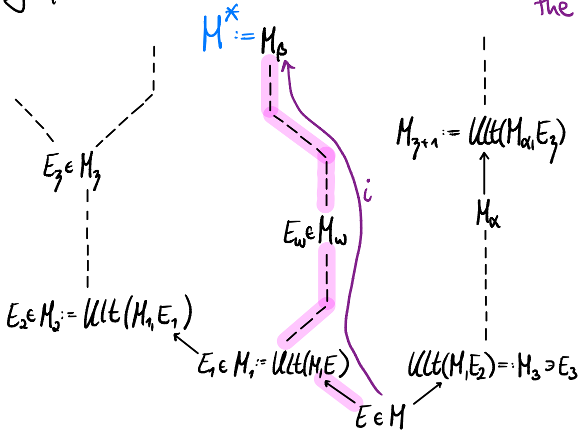

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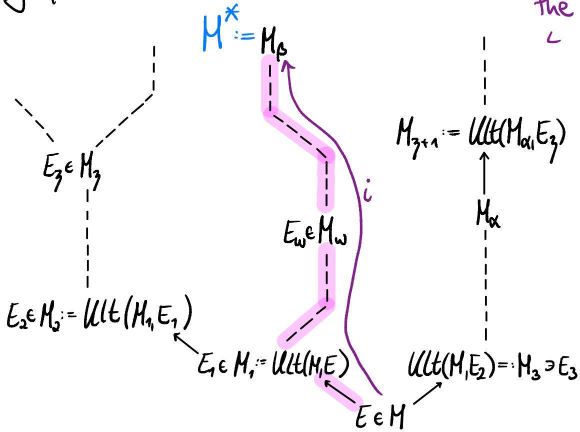

<u>Idea for genericity</u>

choose the extenders s.t. in every step, one obstacle to genericity is removed

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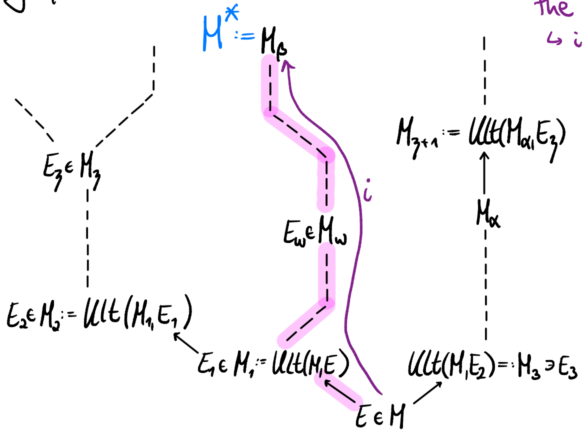

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Theorem (Neeman, Woodin)

Let M be a countable model of ZFC such that

- M has ω-many Woodin cardinals $\delta_0 < \delta_1 < \delta_2 < \ldots$,
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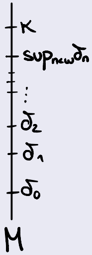

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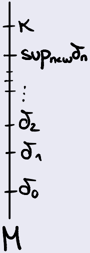

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smallest model of ZF
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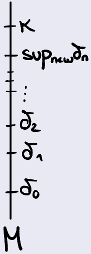

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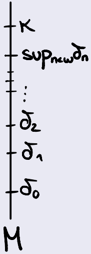

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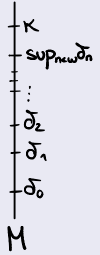

Consistency

Why is $L(\mathbb{R}) \models AD$ interesting?

- $L(\mathbb{R})$ is the smallest model of ZF containing all the reals and ordinals.
- AD = *Axiom of Determinacy* came up in early 60's,

  implies regularity properties for sets of reals
- there was no proof for the consistency of AD
- a proof was found after 25 years of development in inner model theory
- Theorem shows that AD is consistent (relative to ZFC and large cardinals)

Sketch of Proof - Preparation

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$$\boxed{\text{We show } L(\mathbb{R})^P \models AD. \text{ Then } L(\mathbb{R}) \models AD.}$$

Sketch of Proof - Plan

Plan

Find an iteration $k : M \to N$ such that

1. there is $H = h_0 \times h_1 \times h_2 \times \cdots \subseteq k(\mathbb{Q})$ which is generic over N,
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Then

$$AD \text{ is a } \Pi_1(\mathbb{R})\text{-statement}$$

$$+ \text{ a version of the Derived Model Theorem}$$

$$\Rightarrow L(\mathbb{R})^P \models AD.$$

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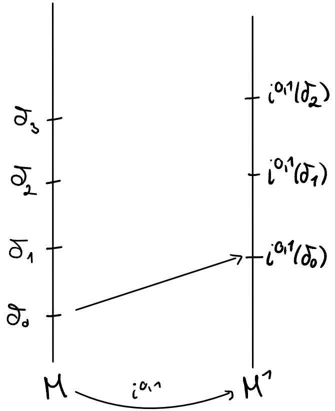

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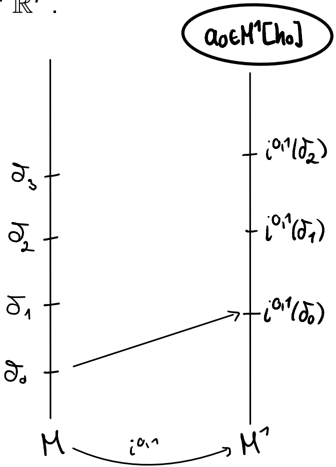

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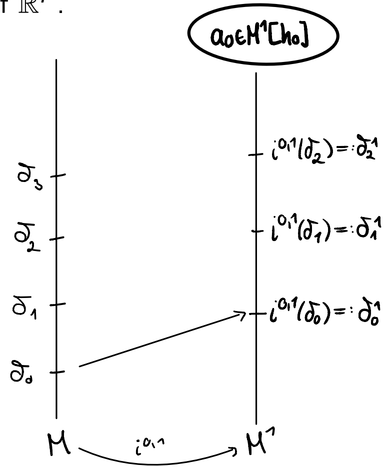

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- countable iteration $i^{1,2} : M^1 \to M^2$ and
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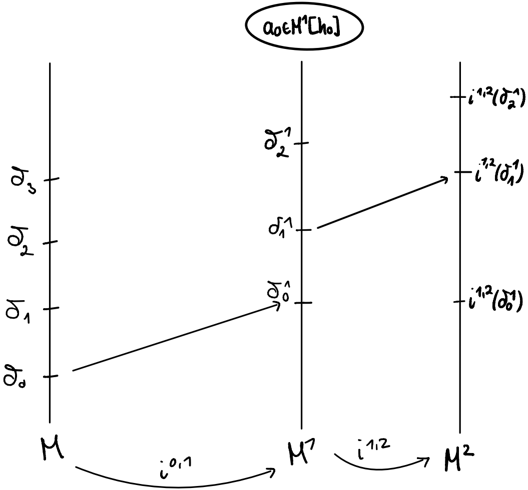

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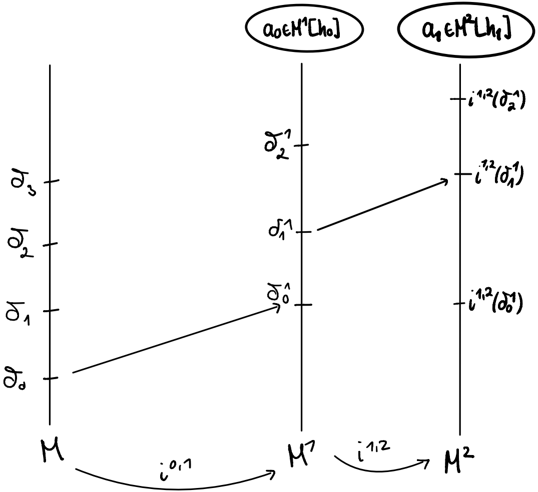

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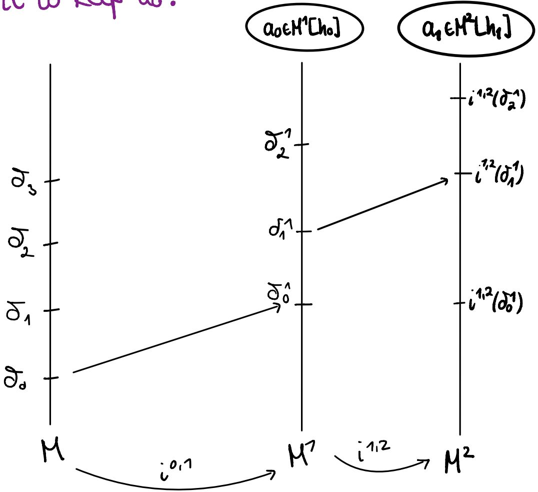

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But we want to keep a_0!

Repeat this for a_1:

Use improved Genericity Iteration for M^1, a_1 and δ_1^1.

We get

with $crit(i^{1,2}) > \delta_0^1$

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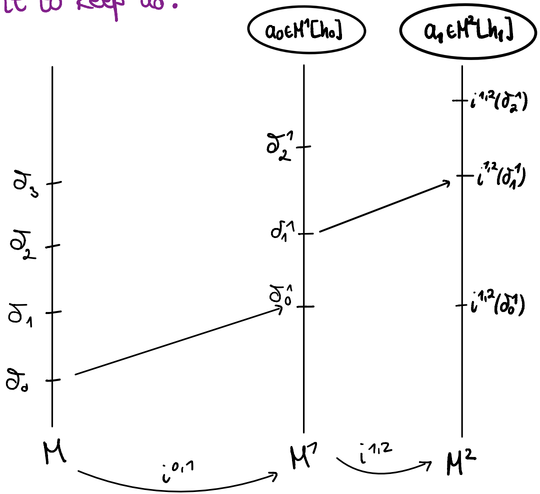

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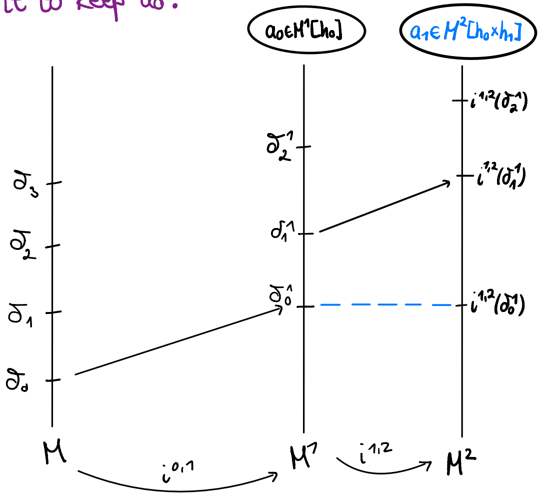

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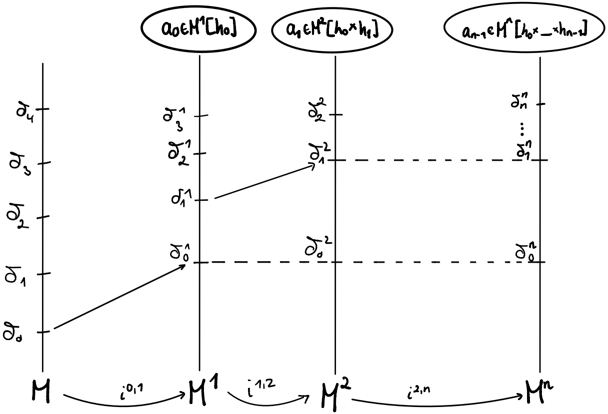
Repeat this for every $n < \omega$



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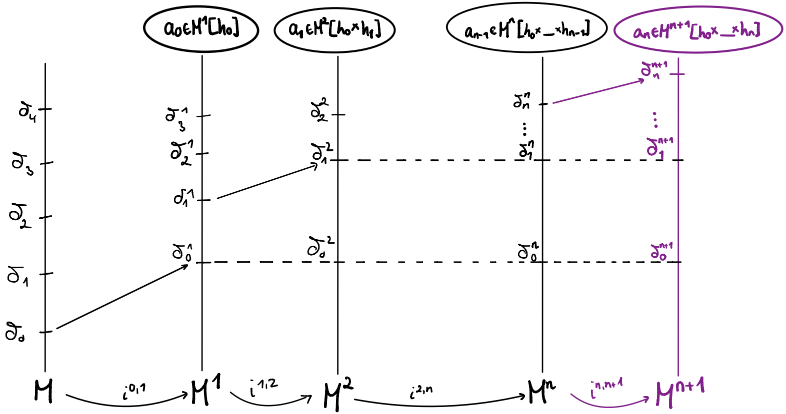

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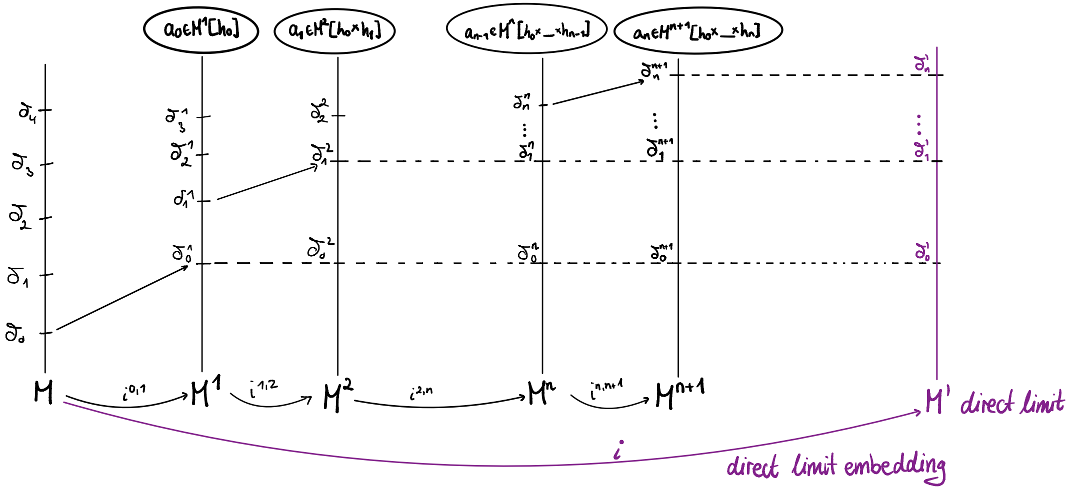
Repeat this for every $n < \omega$ and build the direct limit M'.



Step 1: Recall Plan

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Find an iteration $k : M \to N$ such that   Set $N := M'$ and $H := h_0 \times h_1 \times$ ___ .

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ordinals of M' belong to $P[\langle a_n \mid n<\omega \rangle]$

$\Rightarrow P \cap On \not\subseteq M' \cap On$ ↯

$\Rightarrow M'$ is not the model that we are looking for!

Step 1: Incorporate one real after the other

M' and $H := h_0 \times h_1 \times h_2 \times \ldots$ satisfy

1. H is M'-generic for $i(\mathbb{Q})$ (need more bookkeeping to arrange that)
2. $\bigcup_{n<\omega} \mathbb{R}^{M'[h_0 \times \ldots h_{n-1}]} = \mathbb{R}^P$
3. the ordinals of M' belong to $P[\langle a_n \mid n < \omega \rangle] \Rightarrow P \cap On \not\subseteq M' \cap On$.

We stretch M' to obtain 3 whilst keeping 1 and 2.

Step 2: $P \cap On \subseteq N \cap On$

Build a linear iteration on M' using $i(\kappa) =: \kappa'$ of length $\xi := P \cap On$.

Step 2: $P \cap On \subseteq N \cap On$

$L(\mathbb{R}) \models AD$

Theorem (Neeman, Woodin)

Let M be a countable model of ZFC such that

- M has ω-many Woodin cardinals $\delta_0 < \delta_1 < \delta_2 < \ldots$,
- M has a measurable cardinal $\kappa > \sup_{n<\omega} \delta_n$ and
- M is (a little bit more than) $(\omega_1 + 1)$-iterable.

Then $L(\mathbb{R}) \models AD$.

Main References

- Itay Neeman: *Determinacy in $L(\mathbb{R})$*, In: Handbook of Set Theory, 2010
- John Steel: *An Introduction to Iterated Ultrapowers*, 2015
- Ilijas Farah: *The extender algebra and Σ^2_1-absoluteness*, 2016