# Nitsche-based enforcement of Dirichlet boundary conditions for the Hodge-Laplacian

Master's Thesis Computational Science and Engineering

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Supervised by Prof. Dr. R. Hiptmair, Prof. Dr. J. Schöberl, Dr. E. Zampa and W. Tonnon

ETH Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

SAM

## Author

LinkedIn

GitHub


**Camilo Tello BSc**


- **Bachelors:** Mechanical Engineering
- **Key Area Mathematics**
  - Comp. Math. Seminar (K. Sturm)
  - Sci. Comp. for FEM (J. Schöberl)
  - Computational Finance (A. Jüngel)
  - Non-Linear Coupled Field Problems (M. Rambausek)
- **Key Area Informatics**
  - Algorithmics (G. Raidl)
  - Heuristic Optimization Techniques (G. Raidl)
  - Adv. Regression and Classification (P. Filzmoser)
- **Current Occupation:** Research Assistant at ZHAW Institute for Computational Physics
- **Favorite Lectures:** NumPDE, Algorithmics

## Introduction

In this thesis, we develop a numerical method for the Hodge–Laplace equation, formulated within the framework of exterior calculus. Let $\Omega$ be a smooth domain in $\mathbb{R}^n$, Find $\omega \in \Lambda^k(\Omega)$ such that

$$\begin{aligned} (\delta \mathrm{d} + \mathrm{d}\delta)\omega &= f && \text{in } \Omega \\ \operatorname{tr}\omega &= 0 && \text{on } \Gamma \\ \operatorname{tr}\star\omega &= 0 && \text{on } \Gamma \end{aligned}$$

Under the boundary conditions described above, standard methods [Arnold, Ch. 4.5.2] become ill-posed. Consequently, we propose a Nitsche-type method for $\operatorname{tr}\omega = 0$, ensuring both well-posedness and consistency. [Ern, Ch. 37]

## Method & Implementation

Find $\omega \in \Lambda^k(\Omega)$ and $\sigma \in \Lambda^{k-1}(\Omega)$ such that

$$\begin{aligned} \langle \mathrm{d}\omega, \mathrm{d}\eta\rangle_\Omega - \langle \operatorname{tr}\star \mathrm{d}\omega, \operatorname{tr}\eta\rangle_\Gamma + \langle \mathrm{d}\sigma, \eta\rangle_\Omega & \\ -\langle \operatorname{tr}\star \mathrm{d}\eta, \operatorname{tr}\omega\rangle_\Gamma + \frac{C_\omega}{\mathrm{h}}\langle \operatorname{tr}\omega, \operatorname{tr}\eta\rangle_\Gamma &= \langle f, \eta\rangle_\Omega \\ \langle \sigma, \tau\rangle_\Omega - \langle \omega, \mathrm{d}\tau\rangle_\Omega &= 0 \end{aligned}$$

$$\forall \eta \in \Lambda^k(\Omega),\ \forall \tau \in \Lambda^{k-1}(\Omega)$$

Partial differential equations involving differential forms can be reformulated for established numerical methods by leveraging the isomorphisms that map differential forms to their corresponding vector proxies.

$$\begin{array}{ccccc} H\Lambda^0(\Omega) & \xrightarrow{d^0} & H\Lambda^1(\Omega) & \xrightarrow{d^1} & H\Lambda^2(\Omega) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\Omega,\mathbb{R}) & \xrightarrow{\nabla} & \boldsymbol{H}(\mathrm{curl},\Omega,\mathbb{R}^3) & \xrightarrow{\nabla\times} & \boldsymbol{H}(\mathrm{div},\Omega,\mathbb{R}^3) \\ \downarrow \Pi_h^0 & & \downarrow \Pi_h^1 & & \downarrow \Pi_h^2 \\ V_h^0 & \xrightarrow{d_h} & V_h^1 & \xrightarrow{d_h} & V_h^2 \end{array}$$

Now we can represent $k$-forms with their corresponding elements from classical vector spaces while preserving essential topological and geometrical properties. Find $\mathbf{u} \in \mathbf{H}(\mathrm{curl},\Omega)$ and $p \in H^1(\Omega)$ such that

$$\begin{aligned} \langle \operatorname{curl}\mathbf{u}_h, \operatorname{curl}\mathbf{v}_h\rangle_\Omega - \langle \gamma_n(\mathbf{u}_h), \gamma_\parallel(\mathbf{v}_h)\rangle_\Gamma + \langle \nabla p_h, \mathbf{v}_h\rangle_\Omega & \\ -\langle \gamma_n(\mathbf{v}_h), \gamma_\parallel(\mathbf{u}_h)\rangle_\Gamma + \frac{C_\omega}{\mathrm{h}}\langle \gamma_\parallel(\mathbf{u}_h), \gamma_\parallel(\mathbf{v}_h)\rangle_\Gamma &= \langle f, \mathbf{v}_h\rangle_\Omega \\ \langle p_h, q_h\rangle_\Omega - \langle \mathbf{u}_h, \nabla q_h\rangle_\Omega &= 0 \end{aligned}$$

$$\forall \mathbf{v} \in \mathbf{H}(\mathrm{curl},\Omega) \text{ and } \forall q \in H^1(\Omega)$$

where $\gamma_\parallel(\mathbf{v}_h) = \mathbf{n} \times (\mathbf{v}_h \times \mathbf{n})$ corresponds to $\operatorname{tr}\omega$ and $\gamma_n(\mathbf{v}_h) = \mathbf{n} \times (\nabla \times \mathbf{v}_h)$ to $\operatorname{tr}\star \mathrm{d}\omega$ [Boon, Ch. 3].

## Well Posedness

For coercive and continuous $a(\cdot,\cdot)$, $c(\cdot,\cdot)$ in their Kernel spaces and for inf-sup stable $b(\cdot,\cdot)$, perturbed saddle point problems are well-posed [Boffi, Thm. 5.5.1].

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle_{V'\times V} & \forall v \in V \\ b(u,q) - \lambda^2 c(p,q) = \langle g, q\rangle_{Q'\times Q} & \forall q \in Q \end{cases}$$

Our method can be categorized as such a problem by e.g. setting $b(\mathbf{u}_h, \mathbf{q}_h) = \langle \mathbf{u}_h, \nabla q_h\rangle_\Omega$. While for $Q_h$ the $H^1$ norm suffices, $V_h$ requires to be endowed with the following norm

$$\|\mathbf{u}_h\|_\#^2 = \|\mathbf{u}_h\|^2_{H(\mathrm{curl},\Omega)} + \sum_{F_h\in\Gamma} \frac{1}{h}\|\gamma_\parallel(\mathbf{u}_h)\|^2_{L^2(F_h)} + \sum_{F_h\in\Gamma} h\|\gamma_n(\mathbf{u}_h)\|^2_{L^2(F_h)}$$

**Theorem.** $\exists\, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_h > 0$ such that

$$\alpha_1\|\mathbf{v}_h\|_\#^2 \le a_h(\mathbf{v}_h, \mathbf{v}_h) \le \alpha_2\|\mathbf{v}_h\|_\#^2 \qquad \forall \mathbf{v}_h \in K_h$$

$$\beta_1\|q_h\|^2_{H^1(\Omega)} \le c(q_h, q_h) \le \beta_2\|q_h\|^2_{H^1(\Omega)} \qquad \forall q_h \in H_h^\perp$$

$$\inf_{q_h\in H_h^\perp}\ \sup_{\mathbf{v}_h\in V_h} \frac{b(\mathbf{v}_h, q_h)}{\|q_h\|_{H^1(\Omega)}\|\mathbf{v}_h\|_\#} =: \gamma_h$$

□.

## Consistency

We introduce the strong form of the Hodge Laplacian for 1-forms in 3D in the vector proxy setting. Let $(\mathbf{u}, p) \in [H^2(\Omega)]^3 \times H^1(\Omega)$ solve

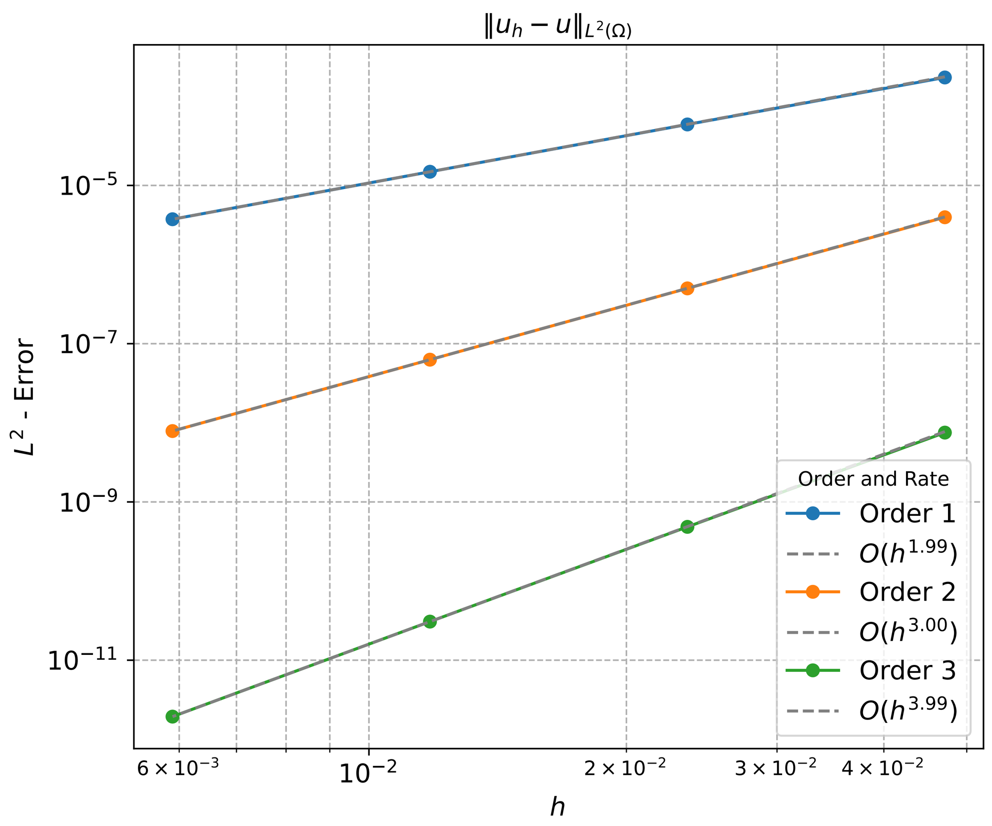
$$\begin{aligned} \nabla p - \nabla\times(\nabla\times\mathbf{u}) &= \mathbf{f} \\ \nabla\cdot\mathbf{u} + p &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma \end{aligned}$$

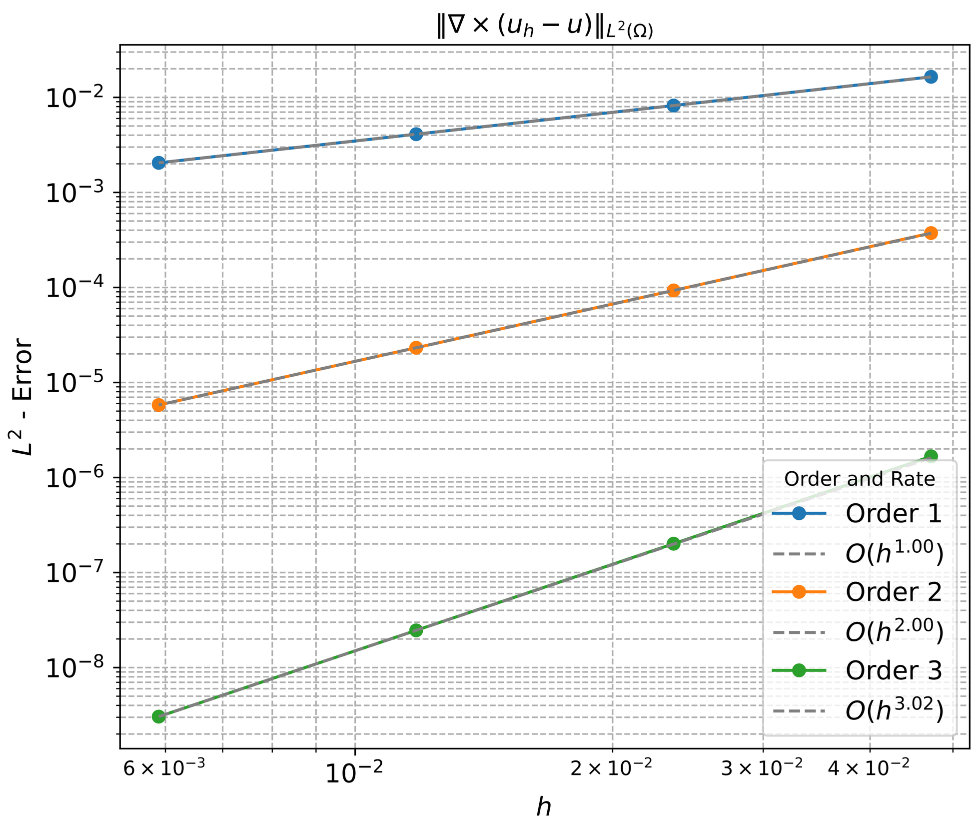
**Theorem.** $(\mathbf{u}, p)$ solve the Nitsche type mixed method for the Hodge Laplacian.

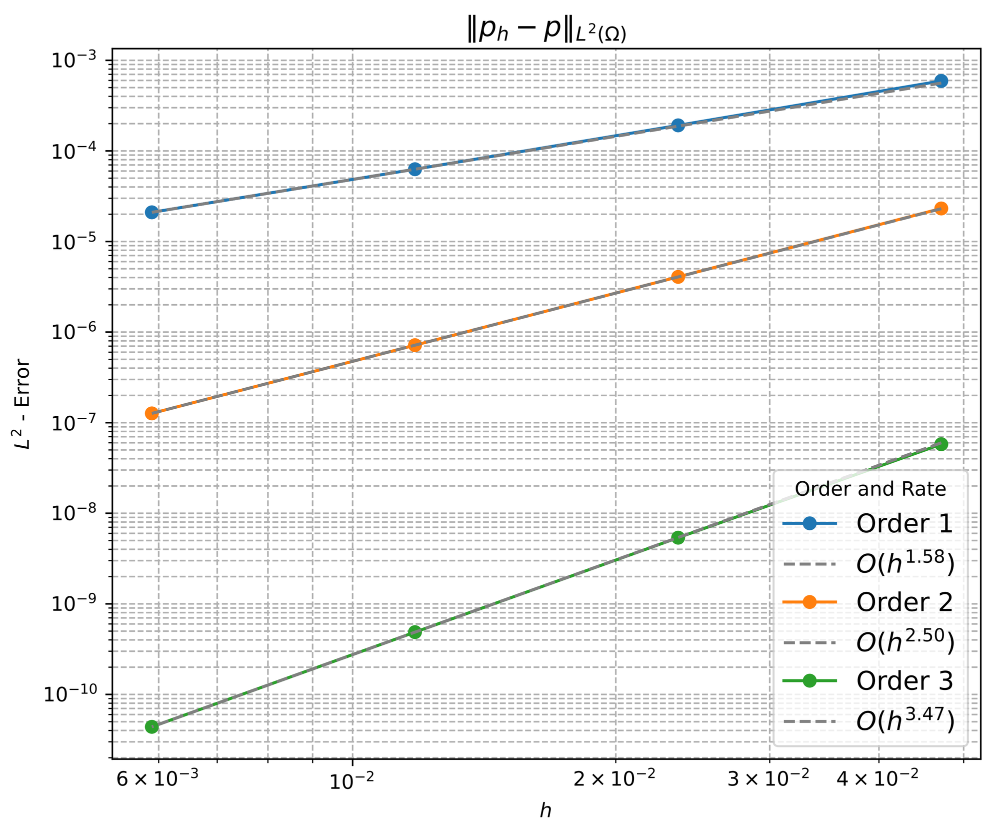
$$a_h(\mathbf{u}, \mathbf{v}_h) - b(p, \mathbf{v}_h) = \ell_h(\mathbf{v}_h)$$

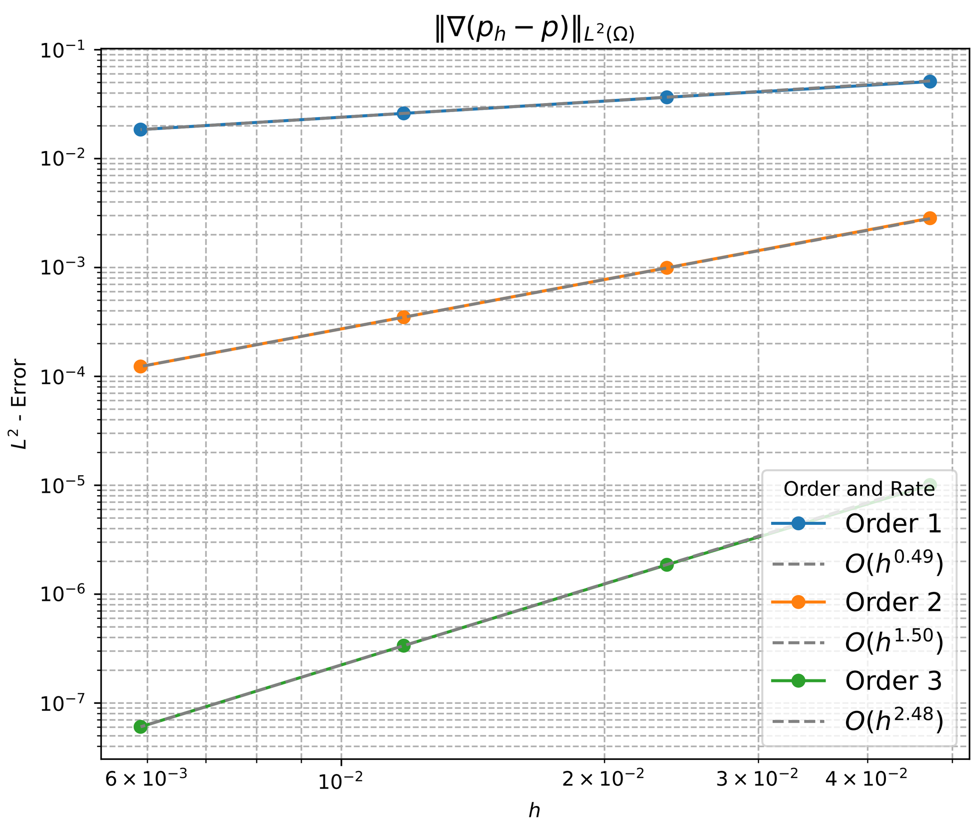
□.

## Experimental Results




## Conclusion & Outlook

The current analysis results are only valid for 1-forms in 3-D, the general result for k-forms in n-D is still open. The experimental results suggest a super convergence of the proposed method.

$$\begin{aligned} \|u - u_h\|^2_{L^2(\Omega)} &= \mathcal{O}(h^{p+1}), & \|p - p_h\|^2_{L^2(\Omega)} &= \mathcal{O}(h^{p-1/2}) \\ \|\nabla\times(u - u_h)\|^2_{L^2(\Omega)} &= \mathcal{O}(h^{p}), & \|\nabla(p - p_h)\|^2_{L^2(\Omega)} &= \mathcal{O}(h^{p-3/2}) \end{aligned}$$

The current a-priori error rate result obtained by the analysis is highly sub optimal, not coinciding with the observed behavior. More experiments need to be done and sharper bounds need to be proven. The implementation for the experimental results are done in NGSolve for 1-forms in 2D and 3D, with manufactured solutions that are products of trigonometric functions.

## References


[Arnold] Douglas N. Arnold. *Finite Element Exterior Calculus*. Society for Industrial and Applied Mathematics, 2018.

[Boffi] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications*. Springer Series in Computational Mathematics. Springer Berlin Heidelberg, 2013. ISBN: 9783642365195.

[Boon] Wietse M. Boon, Ralf Hiptmair, Wouter Tonnon, and Enrico Zampa. "H(curl)-based approximation of the Stokes problem with slip boundary conditions". In: *arXiv preprint* (2024). arXiv: 2407.13353 [math.NA].

[Ern] Alexandre Ern and Jean-Luc Guermond. *Finite Elements II: Galerkin Approximation, Elliptic and Mixed PDEs*. Springer, 2021.