A mass, energy, and helicity conserving dual-field discretization of the incompressible Navier-Stokes problem

Master's Thesis Computational Science and Engineering

Carried out at Seminar for Applied Mathematics ETH Zürich
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Introduction

Incompressible flows can be modeled as a continuum described by the well-known Navier-Stokes equations. This work presents a dual-field Finite Element (FEM) scheme that conserves mass, kinetic energy, and helicity, enabling the study of turbulence phenomena like energy and helicity cascades. Initially developed for periodic domains [2], we extend it to Dirichlet boundary conditions, with the periodic case presented for simplicity. The scheme solves for velocity u, v, vorticity ζ, ω, and pressure p, q twice each, yielding key quantities through simple computations with minimal additional cost.

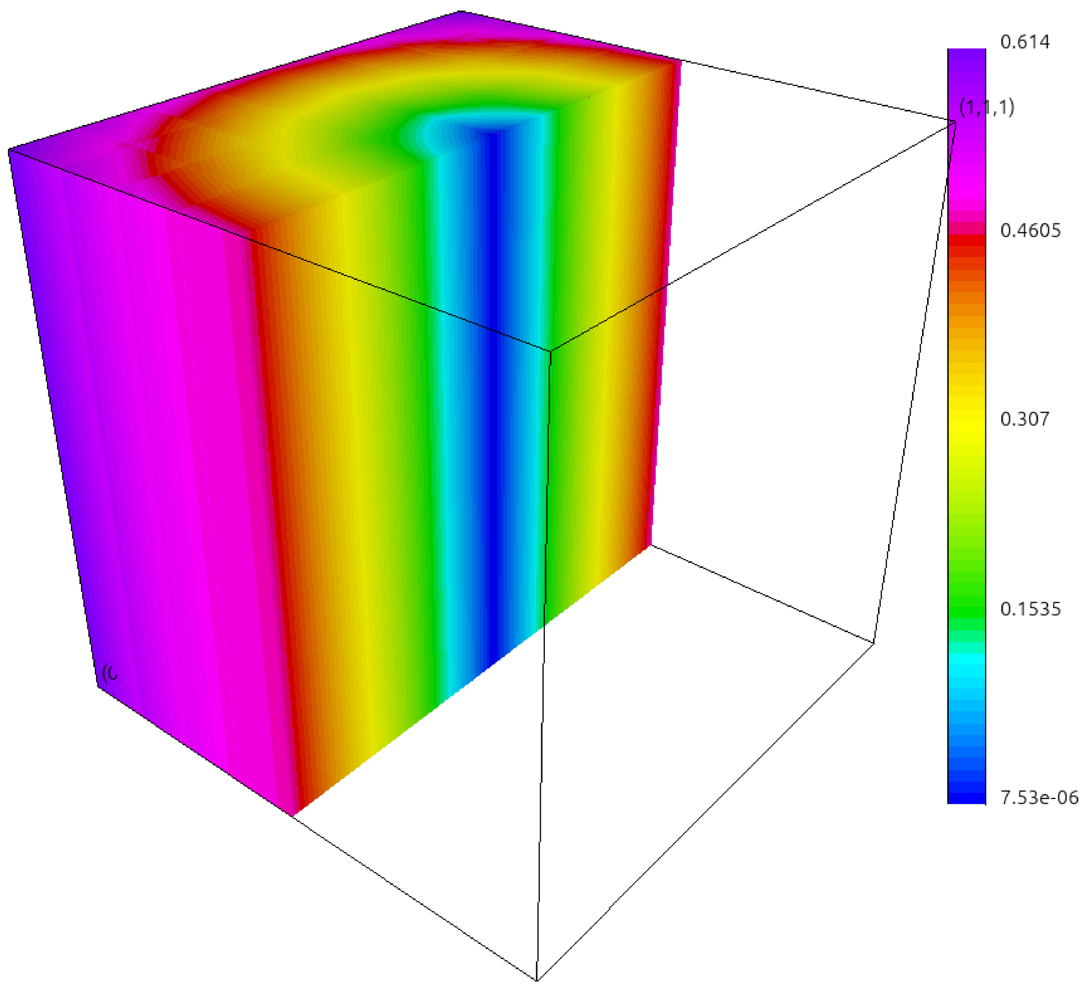


Figure: Simulated solution $u \in H(\mathrm{curl}, \Omega)$

Weak formulation

Given $f \in \left[L^2(\Omega)\right]^3$,
find $(u, \zeta, p) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}, \Omega) \times H^1(\Omega)$
and $(v, \omega, q) \in H(\mathrm{div}, \Omega) \times H(\mathrm{curl}, \Omega) \times L^2(\Omega)$
such that they satisfy the primal system

$$\left\langle \frac{\partial u}{\partial t}, \tilde{u} \right\rangle + \langle \omega \times u, \tilde{u} \rangle + \frac{1}{\mathrm{Re}} \langle \zeta, \nabla \times \tilde{u} \rangle + \langle \nabla p, \tilde{u} \rangle = \langle f, \tilde{u} \rangle \quad \forall \tilde{u} \in H(\mathrm{curl}, \Omega)$$
$$\left\langle \nabla \times u, \tilde{\zeta} \right\rangle - \left\langle \zeta, \tilde{\zeta} \right\rangle = 0 \qquad \forall \tilde{\zeta} \in H(\mathrm{div}, \Omega)$$
$$\langle u, \nabla \tilde{p} \rangle = 0 \qquad \forall \tilde{p} \in H^1(\Omega)$$

as well as the dual system

$$\left\langle \frac{\partial v}{\partial t}, \tilde{v} \right\rangle + \langle \zeta \times v, \tilde{v} \rangle + \frac{1}{\mathrm{Re}} \langle \nabla \times \omega, \tilde{v} \rangle - \langle q, \nabla \cdot \tilde{v} \rangle = \langle f, \tilde{v} \rangle \quad \forall \tilde{v} \in H(\mathrm{div}, \Omega)$$
$$\langle v, \nabla \times \tilde{\omega} \rangle - \langle \omega, \tilde{\omega} \rangle = 0 \qquad \forall \tilde{\omega} \in H(\mathrm{curl}, \Omega)$$
$$\langle \nabla \cdot v, \tilde{q} \rangle = 0 \qquad \forall \tilde{q} \in L^2(\Omega)$$

A coupling of the dual and primal system occurs through $\omega \in H(\mathrm{curl}, \Omega)$ and $\zeta \in H(\mathrm{div}, \Omega)$.

Time-discretization

Given $\left(\omega^{k-1}, v^{k-1}, f, \zeta^{k-\frac{1}{2}}\right)$, seek $\left(\omega^k, v^k, q^{k-\frac{1}{2}}\right)$ such that they satisfy

$$\left\langle \frac{v^k - v^{k-1}}{\Delta t}, \tilde{v} \right\rangle + \left\langle \zeta^{k-\frac{1}{2}} \times \frac{v^k + v^{k-1}}{2}, \tilde{v} \right\rangle + \frac{1}{\mathrm{Re}} \left\langle \nabla \times \frac{\omega^k + \omega^{k-1}}{2}, \tilde{v} \right\rangle - \left\langle q^{k-\frac{1}{2}}, \nabla \cdot \tilde{v} \right\rangle = \langle f, \tilde{v} \rangle$$
$$\left\langle v^k, \nabla \times \tilde{\omega} \right\rangle - \left\langle \omega^k, \tilde{\omega} \right\rangle = 0$$
$$\left\langle \nabla \cdot v^k, \tilde{q} \right\rangle = 0$$

And given $\left(u^{k-\frac{1}{2}}, \zeta^{k-\frac{1}{2}}, f, \omega^k\right)$, seek $\left(p^k, u^{k+\frac{1}{2}}, \zeta^{k+\frac{1}{2}}\right)$ such that they satisfy

$$\left\langle \frac{u^{k+\frac{1}{2}} - u^{k-\frac{1}{2}}}{\Delta t}, \tilde{u} \right\rangle + \left\langle \omega^k \times \frac{u^{k+\frac{1}{2}} + u^{k-\frac{1}{2}}}{2}, \tilde{u} \right\rangle + \frac{1}{\mathrm{Re}} \left\langle \frac{\zeta^{k+\frac{1}{2}} + \zeta^{k-\frac{1}{2}}}{2}, \nabla \times \tilde{u} \right\rangle - \left\langle \nabla p^k, \tilde{u} \right\rangle = \langle f, \tilde{u} \rangle$$
$$\left\langle \nabla \times u^{k+\frac{1}{2}}, \tilde{\zeta} \right\rangle - \left\langle \zeta^{k+\frac{1}{2}}, \tilde{\zeta} \right\rangle = 0$$
$$\left\langle u^{k+\frac{1}{2}}, \nabla \tilde{p} \right\rangle = 0$$

The scheme decouples the fields via the vorticities ω and ζ, significantly speeding up computation by solving only one system per time step.

Space discretization

- $X^{P1} \subset H^1(\Omega)$, *Lagrange Elements*
- $X^{ND} \subset H(\mathrm{curl}, \Omega)$, *Nédelec Elements*
- $X^{RT} \subset H(\mathrm{div}, \Omega)$, *Raviart-Thomas Elements*
- $X^{P0} \subset L^2(\Omega)$, *Discontinuous Elements*

The above spaces together with associated bounded projection operators form the following commuting diagram (*De Rham complex*):

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\nabla} & H(\mathrm{curl}, \Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}, \Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \downarrow I^{P1} & & \downarrow I^{ND} & & \downarrow I^{RT} & & \downarrow I^{P0} \\ X^{P1}(\Omega) & \xrightarrow{\nabla} & X^{ND}(\Omega) & \xrightarrow{\mathrm{curl}} & X^{RT0}(\Omega) & \xrightarrow{\mathrm{div}} & X^{P0}(\Omega) \end{array}$$

The two horizontal sequences of function spaces are *exact*, i.e. the image a differential operators coincides with the nullspace of the following one, e.g.

$$\mathrm{range}(\nabla) = \ker(\mathrm{curl}).$$

Properties

On the continuous level, solutions u, ζ, p, and v, ω, q conserve energy,

$$\mathcal{E}_u := \int_\Omega u \cdot u, \qquad \mathcal{E}_v := \int_\Omega v \cdot v,$$

and helicity

$$\mathcal{H}_u := \int_\Omega u \cdot \omega, \qquad \mathcal{H}_v := \int_\Omega v \cdot \zeta.$$

The *de Rham complex* ensures that symbolic equivalence transformations in the conservation proofs apply to the discrete formulation, see [2, 1]. The time-stepping scheme similarly conserves the above integrals. Further properties:

- The time-scheme linearizes the convective terms.
- The dual representation of each unknown gives two solutions whose difference can be used as an error indicator.

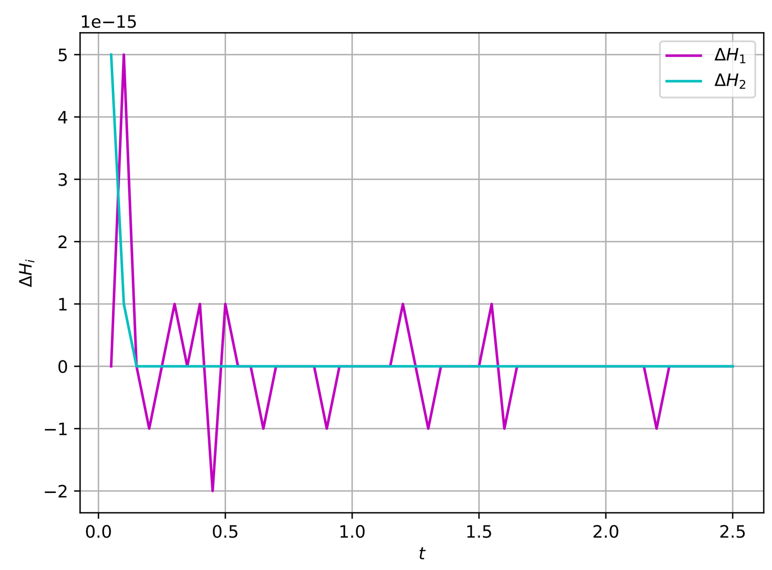
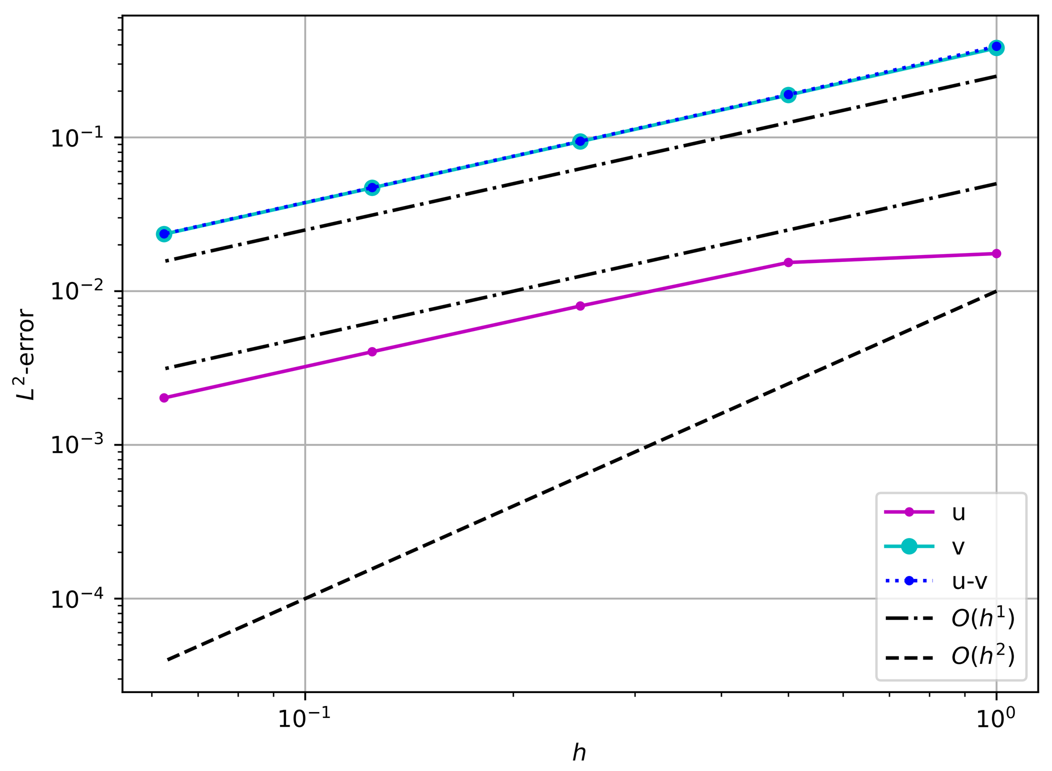
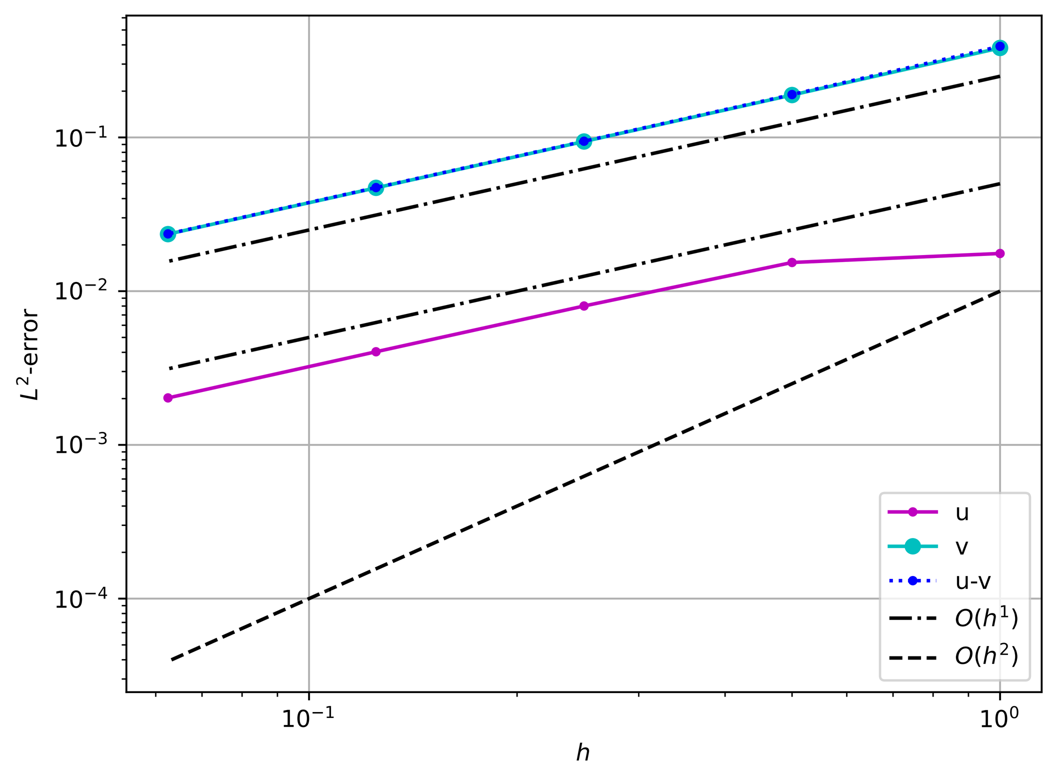


Figure: Helicity-conservation properties of Dirichlet problem with $\Delta t = 0.05$ and $\mathrm{Re} \to \infty$.

Experimental Results



$\mathrm{Re} = 1$



$\mathrm{Re} \to \infty$

Manufactured solutions

We use the *Taylor-Green vortex solution*

$$u^* : (t, x, y, z) \mapsto \begin{pmatrix} \cos(x)\sin(y)e^{-2t/\mathrm{Re}} \\ -\sin(x)\cos(y)e^{-2t/\mathrm{Re}} \\ 0 \end{pmatrix}$$

with vorticity field $\omega^* := \nabla \times u^*$, and forcing term

$$f^* := u^* \times \omega^* + \frac{1}{\mathrm{Re}} \nabla \times \omega^*.$$

References

[1] Markus Renoldner. *A mass, energy, and helicity conserving dual-field discretization of the incompressible Navier-Stokes problem*. Master Thesis. 2023. DOI: 10.34726/hss.2023.110820. URL: https://doi.org/10.34726/hss.2023.110820.

[2] M. Gerritsma Y. Zhang A. Palha and L. G. Rebholz. "A mass-, kinetic energy- and helicity-conserving mimetic dual-field discretization for three-dimensional incompressible Navier-Stokes equations, part I: Periodic domains". In: *Journal of Computational Physics* 451 (Feb. 2022), p. 110868. DOI: 10.1016/j.jcp.2021.110868.