A mass conserving mixed stress-strain rate finite element method for non-newtonian fluid simulations

Master's Thesis Computational Science and Engineering

Carried out at the institute of Analysis and Scientific Computing TU Vienna
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Introduction

In this work [1], we extend the mass conserving mixed stress element ($\mathcal{MCS}$) introduced in [2] by additionally approximating the strain rate tensor ε. This extension is referred to as the mass-conserving mixed stress-strain rate element ($\mathcal{MCS}$-$\mathcal{S}$) and enables the inclusion of arbitrary non-Newtonian constitutive relations given in implicit form

$$\mathcal{G}(\boldsymbol{\tau}, \varepsilon) = 0.$$

Constitutive Relations

Newtonian fluids are characterized by a linear relationship between the deviatoric stress tensor $\boldsymbol{\tau}$ and the strain rate tensor ε given by the constitutive relation

$$\mathcal{G}_n(\boldsymbol{\tau}, \varepsilon) := \boldsymbol{\tau} - 2\mu\varepsilon = 0,$$

where μ is the dynamic viscosity. In real-world applications, however, the constitutive relation is a non-linear function of the strain rate tensor ε and possibly other quantities such as the temperature T or the pressure p. These fluids are referred to as non-Newtonian fluids, and two commonly used constitutive relations are outlined below:

- **Ostwald-de-Waele (Power-law)**

  This constitutive relation is based on two empirical parameters $K > 0, 1 < r < 2$ and reads as

  $$\mathcal{G}_o(\boldsymbol{\tau}, \varepsilon) := \boldsymbol{\tau} - 2K(2|\varepsilon|)^{r-2}\varepsilon.$$

  Above model finds applications in the modelling of shear-thinning fluids such as blood, polymer melts, and lubricating grease, where an increasing shear rate leads to a decreasing viscosity.

- **Bingham**

  This constitutive relation belongs to the class of viscoplastic fluids, which are characterized by an internal stiffness preventing any deformation until a certain stress threshold τ_y is exceeded. The relation reads as

  $$\mathcal{G}_b(\boldsymbol{\tau}, \varepsilon) := \begin{cases} \boldsymbol{\tau} - \left(2\mu + \frac{\tau_y}{|\varepsilon|}\right)\varepsilon & |\boldsymbol{\tau}| > \tau_y, \\ \varepsilon = \mathbf{0} & |\boldsymbol{\tau}| \le \tau_y, \end{cases}$$

  but due to the non-differentiability at $\varepsilon = \mathbf{0}$, we consider the regularized version

  $$\mathcal{G}_b^\kappa(\boldsymbol{\tau}, \varepsilon) := \boldsymbol{\tau} - \left(2\mu + \frac{2\tau_y}{\sqrt{\kappa^2 + 4|\varepsilon|^2}}\right)\varepsilon,$$

  with limits

  $$\lim_{\kappa \to 0} \mathcal{G}_b^\kappa = \mathcal{G}_b, \qquad \lim_{\kappa \to \infty} \mathcal{G}_b^\kappa = \mathcal{G}_n.$$

Governing Equations

The incompressible, stationary Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$, are given by

$$\begin{aligned} \mathcal{G}(\boldsymbol{\tau}, \varepsilon) &= 0 \quad \text{in} \quad \Omega, \\ -\operatorname{div}(\boldsymbol{\tau}) + \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}(\boldsymbol{u}) &= 0 \quad \text{in} \quad \Omega. \end{aligned}$$

Here, $\boldsymbol{u}$ is the velocity field, p is the mechanical pressure, $\boldsymbol{\tau}$ is the deviatoric stress tensor, ε is the strain rate tensor, and $\boldsymbol{f}$ is a given body force.

Discrete Variational Formulation

Find $((\varepsilon_h, \boldsymbol{u}_h, \omega_h), (\boldsymbol{\tau}_h, p_h)) \in Y_h \times X_h$ such that

$$\begin{aligned} (\mathcal{G}(\boldsymbol{\tau}_h, \varepsilon_h), \xi_h)_\Omega &= 0 & \forall \xi_h \in \Xi_h, \\ (\varepsilon_h + \omega_h, \rho_h)_\Omega + b_{2h}(\rho_h, \boldsymbol{u}_h) &= 0 & \forall \rho_h \in \Sigma_h^\oplus, \\ b_{2h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) + (\operatorname{div}(\boldsymbol{v}_h), p_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega & \forall \boldsymbol{v}_h \in V_h, \\ (\zeta_h, \boldsymbol{\tau}_h)_\Omega &= 0 & \forall \zeta_h \in W_h, \\ (\operatorname{div}(\boldsymbol{u}_h), q_h)_\Omega &= 0 & \forall q_h \in Q_h, \end{aligned}$$

with $Y_h := \Xi_h \times V_h \times W_h$ and $X_h := \Sigma_h^\oplus \times Q_h$. The discrete spaces and the slightly non-conforming bilinear form $b_{2h}(\boldsymbol{\tau}_h, \boldsymbol{u}_h)$ are defined in [1], which also provides a comprehensive proof of the well-posedness of the discrete variational formulation in the linear Newtonian context.

The choice of discrete spaces $\boldsymbol{u}_h \in H(div, \Omega)$ and $p_h \in L_0^2(\Omega)$, together with $div(V_h) \subset Q_h$, ensures that the discrete velocity field $\boldsymbol{u}_h$ is pointwise divergence-free and therefore retains the mass conservation property of the ($\mathcal{MCS}$) element.

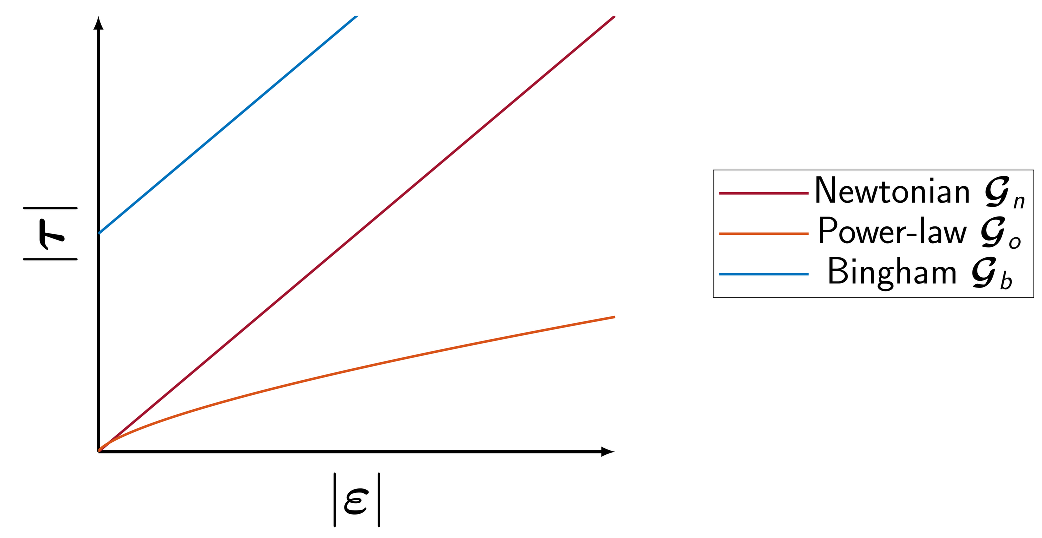

Figure: Representation of non-Newtonian constitutive relations

Numerical Experiments

We consider a flow in a rectangular channel $\Omega = (0, 1) \times (-1, 1)$ induced by a prescribed body-force $\boldsymbol{f}$. In this setting there exists an analytic solution and we compare the L^2-error in the velocity $\boldsymbol{u}$, deviatoric stress tensor $\boldsymbol{\tau}$ and strain rate tensor ε of the ($\mathcal{MCS}$-$\mathcal{S}$) against the Taylor-Hood ($\mathcal{TH}$) and the Scott-Vogelius ($\mathcal{SV}$) element.

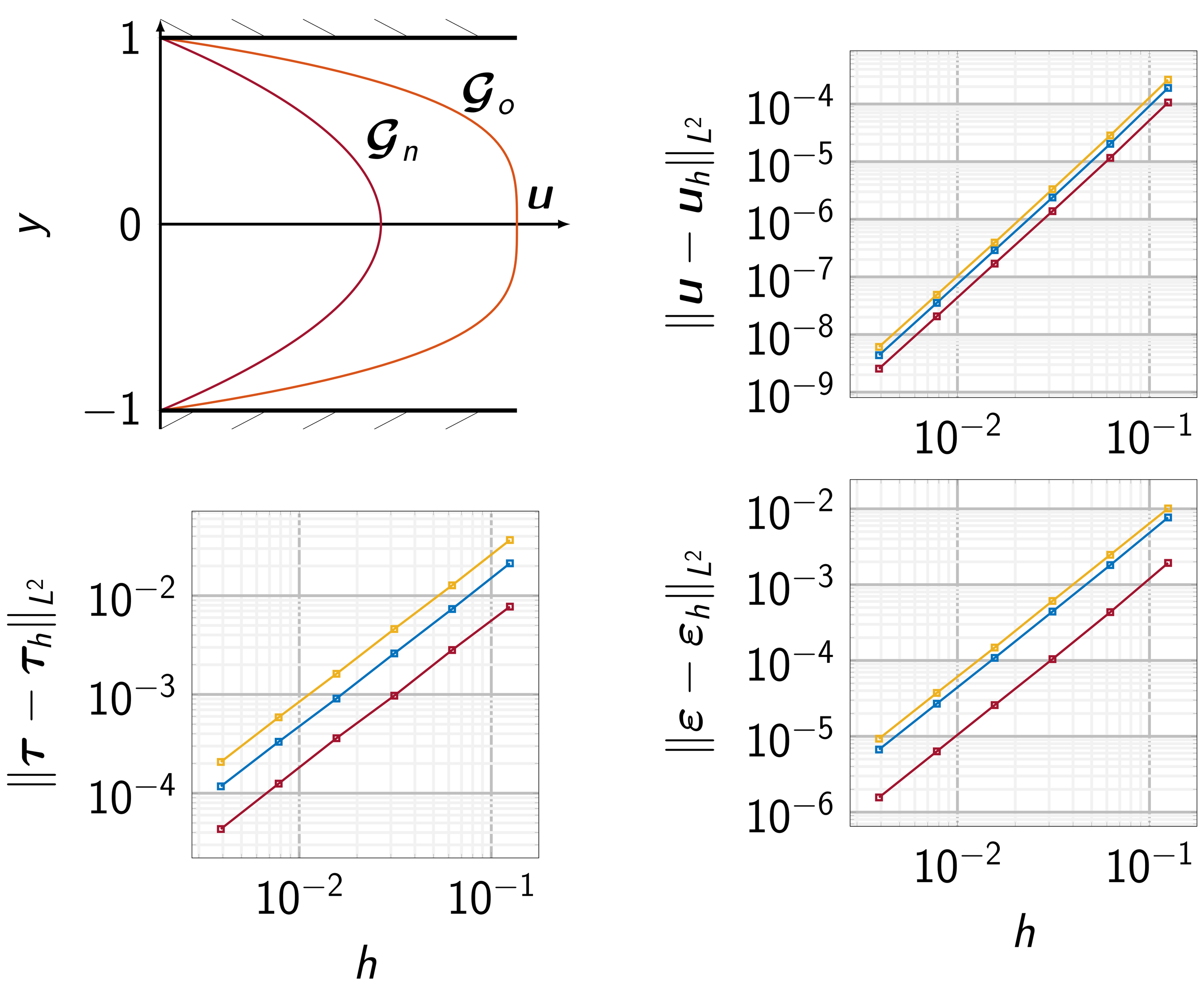

Figure: Ostwald-de-Waele results for a power-law exponent $r = 1.4$

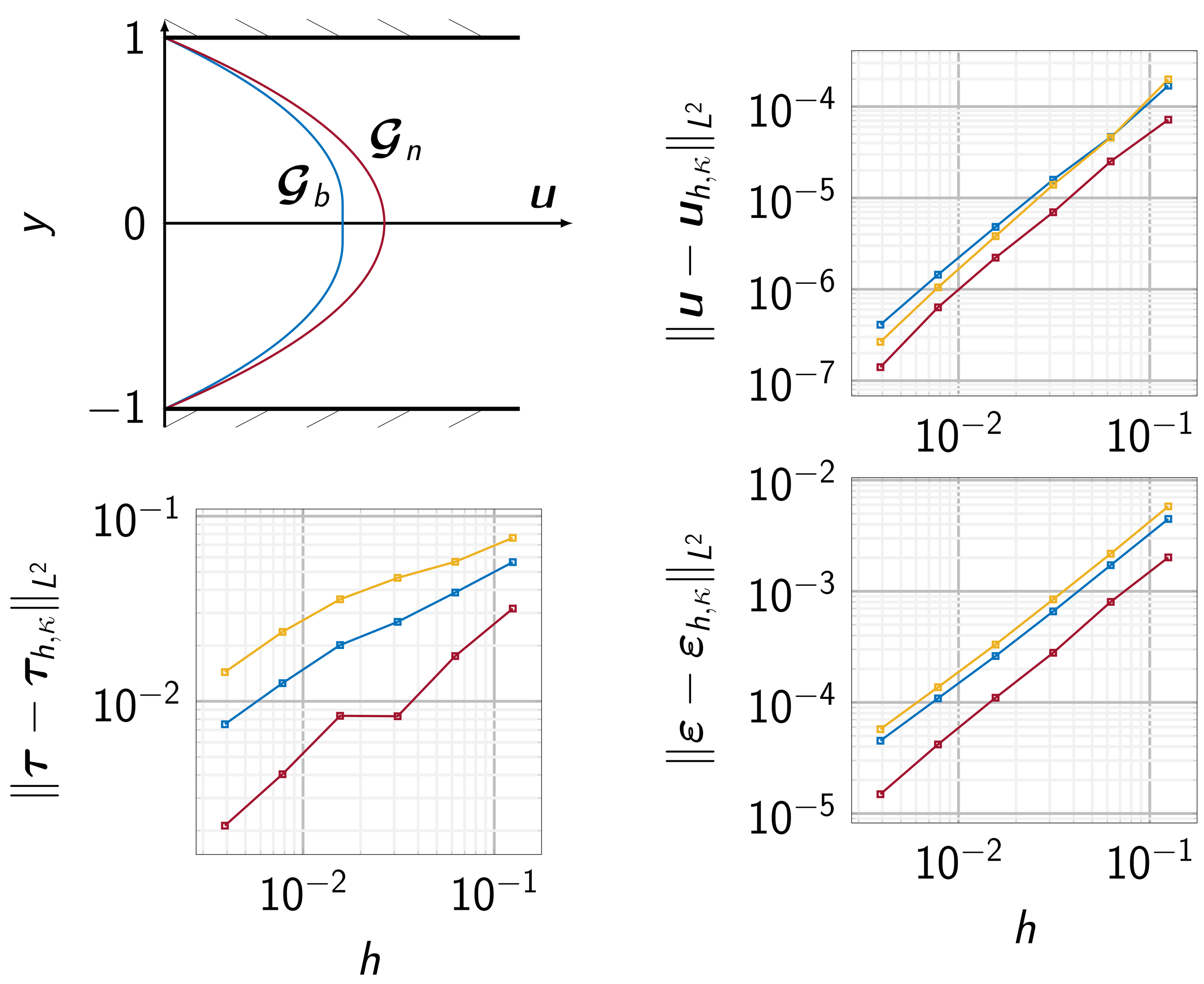

Figure: Bingham results for a yield stress $\tau_y = 0.2$

References

[1] Jan Ellmenreich. "A Mass Conserving Mixed Stress-Strain Rate Finite Element Method for Non-Newtonian Fluid Simulations". Thesis. TU Wien, 2021. 84 pp. DOI: 10.34726/hss.2021.95386. URL: https://repositum.tuwien.at/handle/20.500.12708/19043 (visited on 06/17/2023).

[2] Jay Gopalakrishnan, Philip L. Lederer, and Joachim Schöberl. *A Mass Conserving Mixed Stress Formulation for Stokes Flow with Weakly Imposed Stress Symmetry*. Mar. 1, 2019. arXiv: 1901.04648 [math]. URL: http://arxiv.org/abs/1901.04648 (visited on 04/24/2023). Pre-published.